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Sufficient conditions for class C_α

Olubunmi A. Fadipe-Joseph[1], **M. O. Oluwayemi**[2]

[1]Department of Mathematics
University of Ilorin
Ilorin, Nigeria

[2]Department of Mathematics
Landmark University
Omu-Aran, Nigeria

email: famelov@unilorin.edu.ng, oluwayemimatthew@gmail.com

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Abstract

In this work, a certain class C_α is defined and its coefficient bounds established. Moreover, initial coefficients of the class C_α related to sigmoid functions and its Fekete-Szego functional are obtained. We also find an upper bound for the second Hankel determinant $|H_2(1)|$ for this class.

1 Preliminaries and results

Let A denote the class of functions $f(z)$ of the form:

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \dots \tag{1.1}$$

(with the normalized conditions $f(0) = 0$ and $f'(0) = 1$).

The class $S^*(\gamma)$ (or $C(\gamma)$) of starlike (or convex) functions of order $\boldsymbol{\gamma}$, $0 \le \gamma < 1$ is defined by

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$$Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \tag{1.2}$$

or

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \gamma. \tag{1.3}$$

In particular, $S^*(0) = S^*$ is a starlike function and $C(0) = C$ is a convex function. Let P be the set of all analytic functions of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.4}$$

satisfying $Re\,(p(z)) > 0$, $z \in U$. Then, $|c_n| \leq 2$ for every $n = 1, 2, 3, \ldots$.

Various work on coefficient bounds estimate for univalent functions have been done in recent time. The work of [3], [4], [5], [7] and [8] in this direction are worth mentioning.

In [1], the authors established that the modified sigmoid function belongs to the class of Caratheodory function. The series of the function was given as follows:

$$\Phi(z) = \frac{2}{1+e^{-z}} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + \ldots. \tag{1.5}$$

See also [2] for details.

In this work, coefficient bounds for a class of univalent functions is investigated.

Lemma 1. [5] Let $p \in P$. Then the power series for $p(z)$ converges in D to a function in P if and only if, for $n \in N$, the Toeplitz determinants

$$D_n = \begin{bmatrix} 2 & c_1 & c_2 & \ldots & c_n \\ \overline{c_1} & 2 & c_1 & \ldots & c_{n-1} \\ \vdots & & & & \vdots \\ \overline{c_n} & \overline{c_{n-1}} & \overline{c_{n-2}} & \ldots & 2 \end{bmatrix} \tag{1.6}$$

are all nonnegative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k \rho_0(e^{itk} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$ for $k \neq j$, where $p_0(z) = (1+z)(1-z)$; in this case, $D_n > 0$ for $n < m-1$ and $D_n = 0$ for $n \leq m$. For $n = 2$

$$D_2 = \begin{bmatrix} 2 & c_1 & c_2 \\ \overline{c_1} & 2 & c_1 \\ \overline{c_2} & c_1 & 2 \end{bmatrix} = 8 + 2c_1^2 Rec_2 - 2|c_2|^2 - 4c_1^2 \geq 0$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{1.7}$$

for some x, $|x| \leq 1$. When $n = 3$,

$$D_3 = \begin{bmatrix} 2 & c_1 & c_2 & c_3 \\ c_1 & 2 & c_1 & c_2 \\ \overline{c_2} & c_1 & 2 & c_1 \\ \overline{c_3} & \overline{c_2} & c_1 & 2 \end{bmatrix} \geq 0$$

which is equivalent to

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y, \tag{1.8}$$

for some y, with $|y| \leq 1$ and some x with $|x| \leq 1$. When $n = 4$,

$$8c_4 = c_1^4 + (4 - c_1^2)x[c_1^2(x^2 - 3x + 3) + 4x] - 4(4 - c_1^2)(1 - |x|^2)[c_1(x - 1)y + \overline{x}y^2 - (1 - |y|^2)z, \tag{1.9}$$

for some x, y, z, with $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$.

Lemma 2. [6] Let $p \in P$. Then,

$$|c_2 - \nu c_1^2| \leq = \begin{cases} -4\nu + 2, & \nu < 0 \\ 2, & 0 \leq \nu \leq 1, . \\ 4\nu - 2 & \nu > 1. \end{cases}$$

Lemma 3. [6] Let $p \in P$. Then,

$$|c_2 - \nu c_1^2| \leq 2 \max(1, |2\nu - 1|)$$

The inequalities are sharp.

Definition. Let $0 < \alpha \leq 1$ and $f(z)$ defined by (1.1). The univalent function $f(z) \in A$ belongs to class C_α in the unit disk if

$$Re \left\{ f \in S : \quad \frac{\{(zf'(z))'\}^\alpha}{f'(z)} > 0, \quad z \in U \right\}. \tag{1.10}$$

We have $C_1 \equiv C$; that is, for $\alpha = 1$, (1.10) reduces to (1.3) for $\gamma = 0$. The family of convex functions in univalent functions is currently an active area of research by function theorists.

Theorem 1. Let $f \in C_\alpha$ and $\Phi \in P$ given by (1.5). Then,

$$|a_2| \leq \frac{1}{4(2\alpha - 1)},$$

$$|a_3| \leq \frac{2\alpha^2 - 4\alpha + 1}{12(2\alpha - 1)^2(3\alpha - 1)}.$$

Moreover,

$$|a_3 - \lambda a_2^2| \leq \left| \frac{8\alpha^2 - (16 + 9\lambda)\alpha + 3\lambda + 4}{48(2\alpha - 1)^2(3\alpha - 1)} \right|.$$

Proof Let $f \in C_\alpha$. Then there exists $\Phi \in P$ such that $\frac{(f'(z)+zf''(z))^\alpha}{f'(z)} \prec \Phi(z)$. Thus,

$$(f'(z) + zf''(z))^\alpha = f'(z)\Phi(z) \tag{1.11}$$

$$(f'(z) + zf''(z))^\alpha = \left(1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \cdots + 2a_2z + 6a_3z^2 + 12a_4z^3 + \ldots\right)^\alpha$$

$$\left(1 + 4a_2z + 9a_3z^2 + 16a_4z^3 + 25a_5z^4 + \ldots\right)^\alpha = \left[1 + (4a_2z + 9a_3z^2)\right]^\alpha$$

so that

$$(f'(z) + zf''(z))^\alpha$$

$$= 1+4\alpha a_2z+[9\alpha a_3+\alpha(\alpha-1)8a_2^2]z^2+[\alpha(\alpha-1)36a_2a_3+\alpha(\alpha-1)(\alpha-2)\frac{32}{3}a_2^3]z^3+\ldots. \tag{1.12}$$

Simplifying, we have

$$f'(z)\Phi(z) = 1 + \left(\frac{1}{2} + 2a_2\right) z + (a_2 + 3a_3)\, z^2 + \left(4a_4 + \frac{3}{2}a_3 - \frac{1}{24}\right) z^3 + \ldots. \tag{1.13}$$

Substituting (1.13) and (1.12) into (1.11) upon expansion and comparing the coefficients, we get

$$4\alpha a_2 = 2a_2 + \frac{1}{2} \Rightarrow (4\alpha - 2)a_2 = \frac{1}{2}; \quad |a_2| = \frac{1}{4(2\alpha - 1)} \tag{1.14}$$

$$9\alpha a_3 + 8\alpha(\alpha-1)a_2^2 = a_2 + 3a_3 \Rightarrow (9\alpha-3)a_3 = a_2 - 8\alpha(\alpha-1)a_2^2$$

$$|a_3| \leq \frac{2\alpha^2-4\alpha+1}{12(3\alpha-1)(2\alpha-1)^2}. \tag{1.15}$$

From (1.14) and (1.15), there exists $\lambda \in R$ such that

$$|a_3-\lambda a_2^2| = \left|\frac{2\alpha^2-4\alpha+1}{12(3\alpha-1)(2\alpha-1)^2} - \lambda\left(\frac{1}{4(2\alpha-1)}\right)^2\right| = \left|\frac{2\alpha^2-4\alpha+1}{12(3\alpha-1)(2\alpha-1)^2} - \frac{\lambda}{16(2\alpha-1)^2}\right|$$

which implies that

$$|a_3-\lambda a_2^2| \leq \frac{8\alpha^2-(16+9\lambda)\alpha+3\lambda+4}{48(2\alpha-1)^2(3\alpha-1)}.$$

Corollary 1. Let $f \in C_\alpha$, $\lambda = 1$ and $\Phi \in P$ be given by (1.5). Then,

$$|a_3 - a_2^2| \leq \left|\frac{8\alpha^2-25\alpha+7}{48(2\alpha-1)^2(3\alpha-1)}\right|.$$

Corollary 2. Suppose $f \in C_\alpha$, $\lambda = 0$ and $\Phi \in P$ be given by (1.5). Then,

$$|a_3| \leq \frac{2\alpha^2-4\alpha+1}{12(2\alpha-1)^2(3\alpha-1)}$$

which yields (1.15).

Corollary 3. Let $f \in C_1$, $\lambda = 0$ and $\Phi \in P$ be given by (1.5). Then, $|a_3| \leq \frac{1}{24}$.

Theorem 2. Let $f \in C_\alpha$ and $p \in P$ be given by (1.4). Then,

$$|a_2| \leq \frac{1}{2}\left\{\frac{c_1}{(2\alpha-1)}\right\}, \quad |a_3| \leq \frac{1}{3}\left\{\frac{c_2}{3\alpha-1} - \frac{(2\alpha^2-4\alpha+1)\,c_1^2}{(2\alpha-1)^2(3\alpha-1)}\right\} \text{ and}$$

$$|a_4| \leq \frac{1}{4}\left\{\frac{c_3}{(4\alpha-1)} - \frac{(6\alpha^2-11\alpha+2)c_1c_2}{(4\alpha-1)(3\alpha-1)(2\alpha-1)} + \frac{(24\alpha^4-80\alpha^3+84\alpha^2-28\alpha+3)c_1^3}{3(4\alpha-1)(3\alpha-1)(2\alpha-1)^3}\right\}.$$

Moreover,

$$|a_3 - a_2^2| \leq \frac{8\alpha^2-8\alpha+2}{3(2\alpha-1)^2(3\alpha-1)}.$$

Proof: Let $f \in C_\alpha$. Then there exists $p \in P$ such that $\frac{(f'(z)+zf''(z))^\alpha}{f'(z)} \prec p(z)$ such that

$$(f'(z) + zf''(z))^\alpha = f'(z)p(z). \tag{1.16}$$

Eq. (1.12) gives the left hand side of (1.16) and using (1.1) and (1.4) in the right hand side of (1.16),

$$f'(z)p(z) = 1+(c_1 + 2a_2)\, z+(c_2 + 2c_1a_2 + 3a_3)\, z^2+(c_3 + 2c_2a_2 + 3c_1a_3 + 4a_4)\, z^3+\ldots. \tag{1.17}$$

Substituting (1.12) and (1.17) into (1.16) upon expansion and comparing the coefficients, we have the following results

$$4\alpha a_2 = 2a_2 + c_1 \Rightarrow (4\alpha - 2)a_2 = c_1; \quad a_2 = \frac{c_1}{2(2\alpha - 1)} \tag{1.18}$$

$$9\alpha a_3 + 8\alpha(\alpha - 1)a_2^2 = c_2 + 2c_1a_2 + 3a_3; \quad 9\alpha a_3 - 3a_3 = c_2 + 2c_1a_2 - 8\alpha(\alpha - 1)a_2^2$$

$$a_3 = \frac{1}{3}\left\{\frac{c_2}{3\alpha - 1} - \frac{(2\alpha^2 - 4\alpha + 1)\, c_1^2}{(2\alpha - 1)^2(3\alpha - 1)}\right\} \tag{1.19}$$

$$16\alpha a_4 + 36\alpha(\alpha - 1)a_2a_3 + \frac{32}{3}\alpha(\alpha - 1)(\alpha - 2)a_2^3 = c_3 + 2c_2a_2 + 3c_1a_3 + 4a_4$$

$$a_4 = \frac{1}{4}\left\{\frac{c_3}{(4\alpha - 1)} - \frac{(6\alpha^2 - 11\alpha + 2)c_1c_2}{(2\alpha - 1)(3\alpha - 1)(4\alpha - 1)} + \frac{(24\alpha^4 - 80\alpha^3 + 84\alpha^2 - 28\alpha + 3)c_1^3}{3(2\alpha - 1)^3(3\alpha - 1)(4\alpha - 1)}\right\}. \tag{1.20}$$

From (1.18) and (1.19),

$$|a_3 - a_2^2|$$

$$= \left|\frac{4(2\alpha - 1)^2c_2 - 4(2\alpha^2 - 4\alpha + 1)c_1^2 - 3(3\alpha - 1)c_1^2}{12(2\alpha - 1)^2(3\alpha - 1)}\right|.$$

Using (1.7) of Lemma 1 in the above equation,

$$|a_3 - a_2^2| = \left|\frac{(8\alpha^2 - 8\alpha + 2)[c_1^2 + x(4 - c_1^2)] - (8\alpha^2 - 7\alpha + 1)c_1^2}{12(2\alpha - 1)^2(3\alpha - 1)}\right|. \tag{1.21}$$

Since $t_1 = t \in [0,2]$, replacing $|x|$ by μ in the above relation, we get

$$|a_3 - a_2^2| = \left| \frac{(8\alpha^2 - 8\alpha + 2)[c_1^2 + x(4 - c_1^2)] - (8\alpha^2 - 7\alpha + 1)c_1^2}{12(2\alpha-1)^2(3\alpha-1)} \right|$$

$$= \left| \frac{[8\alpha^2 - 8\alpha + 2 - 8\alpha^2 + 7\alpha - 1]c^2}{12(2\alpha-1)^2(3\alpha-1)} + \frac{(8\alpha^2 - 8\alpha + 2)(4 - c^2)\mu}{12(2\alpha-1)^2(3\alpha-1)} \right|$$

$$|a_3 - a_2^2| \le \left(\frac{1}{12(2\alpha-1)^2(3\alpha-1)} \right) \left| (1-\alpha)c^2 + (8\alpha^2 - 8\alpha + 2)(4 - c^2)\mu \right|$$

$$F(c,\mu) = \left(\frac{1}{12(2\alpha-1)^2(3\alpha-1)} \right) \left| (1-\alpha)c^2 + (8\alpha^2 - 8\alpha + 2)(4 - c^2)\mu \right|, \tag{1.22}$$

where $0 \le \mu = |x| \le 1$ and $0 \le c \le 2$.

$$F(c,\mu) = F(c,1) = Q(c) = \left(\frac{1}{12(2\alpha-1)^2(3\alpha-1)} \right) \left| (1-\alpha)c^2 + (8\alpha^2 - 8\alpha + 2)(4 - c^2) \right|$$

$$\frac{\partial F}{\partial \mu} = \left(\frac{1}{12(2\alpha-1)^2(3\alpha-1)} \right) \left| (8\alpha^2 - 8\alpha + 2)(4 - c^2) \right|. \tag{1.23}$$

Let $\mu \in (0,1)$. Fix $c \in (0,2)$. For $0 \le \alpha < 1$ and $0 \le \beta \le 1$, $F'(c,\mu) > 0$ using (1.23). Thus, $F(c,\mu)$ is increasing with no maximum value in the interior of the interval. Moreover, for a fixed $c \in [0,2]$, we have

$$\max_{0 \le \mu \le 1} F(c,\mu) = F(c,1) = Q(c). \tag{1.24}$$

In view of (1.24), simplifying (1.22), we get

$$Q(c) = \left(\frac{1}{12(2\alpha-1)^2(3\alpha-1)} \right) \left| (8\alpha^2 - 8\alpha + 2)(4 - c^2) \right| \tag{1.25}$$

$$Q'(c) = \left(\frac{1}{3(2\alpha-1)^2(3\alpha-1)} \right) \left| (4\alpha^2 - 4\alpha + 1)(c - 4) \right| \tag{1.26}$$

$$Q''(c) = \left(\frac{-4}{3(2\alpha-1)^2(3\alpha-1)}\right)\left|4\alpha^2-4\alpha+1\right| \tag{1.27}$$

$$Q_{max} = Q(0) = \frac{8(4\alpha^2-4\alpha+1)}{12(2\alpha-1)^2(3\alpha-1)} = \frac{2(4\alpha^2-4\alpha+1)}{3(2\alpha-1)^2(3\alpha-1)}. \tag{1.28}$$

Thus,

$$|a_3-a_2^2| \leq \frac{8(4\alpha^2-4\alpha+1)}{12(2\alpha-1)^2(3\alpha-1)} = \frac{2(4\alpha^2-4\alpha+1)}{3(2\alpha-1)^2(3\alpha-1)}$$

as required.

Corollary 4. Let $f \in C_1$. Then $|a_3-a_2^2| \leq \frac{1}{3}$.

Theorem 3. Let $f \in C_\alpha$ and $p \in P$. Then, there exists a complex number λ such that

$$|a_3-\lambda a_2^2| \leq \begin{cases} \frac{8\alpha-2+3\lambda(3\alpha-1)}{3(3\alpha-1)(2\alpha-1)^2}, & \lambda < \sigma_1, \\ \frac{2}{3(3\alpha-1)}, & \sigma_1 \leq \lambda \leq \sigma_2, , \\ \frac{2-8\alpha+3\lambda(3\alpha-1)}{3(3\alpha-1)(2\alpha-1)^2} & \lambda > \sigma_2. \end{cases}$$

where $\sigma_1 = \frac{8\alpha^2-16\alpha+4}{9\alpha-3}$ and $\sigma_2 = \frac{8\alpha^2}{9\alpha-3}$.

Proof Let $f \in C_\alpha$ and $p \in P$. Then, there exists $\lambda \in R$ such that from (1.18) and (1.19), we have that

$$|a_3-\lambda a_2^2| = \frac{1}{3(3\alpha-1)}\left|c_2-\nu c_1^2\right|, \tag{1.29}$$

where $\nu = \frac{8\alpha^2-16\alpha+4+9\alpha\lambda-3\lambda}{4(2\alpha-1)^2}$. Therefore, from (1.29) and Lemma 2, we have

$$\left(\frac{1}{3(3\alpha-1)}\right)[-4\nu+2] = \frac{1}{3(3\alpha-1)}\left\{-4\left(\frac{8\alpha^2-16\alpha+4+9\alpha\lambda-3\lambda}{4(2\alpha-1)^2}\right)+2\right\}; \quad \nu < 0$$

$$= \frac{-8\alpha^2+16\alpha-4+9\alpha\lambda-3\lambda+2(4\alpha^2-4\alpha+1)}{3(3\alpha-1)(2\alpha-1)^2}$$

$$\Rightarrow \nu = \frac{8\alpha^2-16\alpha+4+9\alpha\lambda-3\lambda}{4(2\alpha-1)^2} < 0.$$

Therefore, $8\alpha^2-16\alpha+4+9\alpha\lambda-3\lambda < 0$; $(9\alpha-3)\lambda < -(8\alpha^2-16\alpha+4)$

$$\lambda < \frac{16\alpha - 8\alpha^2 - 4}{9\alpha - 3}. \tag{1.30}$$

For $\frac{2}{3(3\alpha-1)}$, consider the inequality $0 \leq \nu \leq 1$.

If $0 \leq \nu$, then $0 \leq 8\alpha^2 - 16\alpha + 4 + 9\alpha\lambda - 3\lambda$ which implies that $(9\alpha-3)\lambda \leq 16\alpha - 8\alpha^2 - 4$ and thus,

$$\lambda \geq \frac{16\alpha - 8\alpha^2 - 4}{9\alpha - 3}. \tag{1.31}$$

If $\nu \leq 1$, then $8\alpha^2 - 16\alpha + 4 + 9\alpha\lambda - 3\lambda \leq 4(2\alpha-1)^2 = 16\alpha^2 - 16\alpha + 4$. Therefore, $\lambda(9\alpha - 3) \leq 8\alpha^2$ and

$$\frac{8\alpha^2}{9\alpha - 3} \geq \lambda. \tag{1.32}$$

Therefore, from (1.31) and (1.32), we have

$$\frac{16\alpha - 8\alpha^2 - 4}{9\alpha + 3} \leq \lambda \leq \frac{8\alpha^2}{9\alpha - 3}. \tag{1.33}$$

Finally,

$$\frac{4\nu - 2}{3(3\alpha-1)} = \frac{1}{3(3\alpha-1)} \left\{ 4\left(\frac{8\alpha^2 - 16\alpha + 4 + 9\alpha\lambda - 3\lambda}{4(2\alpha-1)^2} \right) - 2 \right\}; \quad \nu > 1$$

$$8\alpha^2 - 16\alpha + 4 + 9\alpha\lambda - 3\lambda > 4(2\alpha-1)^2 = 16\alpha^2 - 16\alpha + 4$$
$$(9\alpha - 3)\lambda > 16\alpha^2 - 16\alpha + 4 - 8\alpha^2 + 16\alpha - 4$$

$$\lambda > \frac{8\alpha^2}{9\alpha + 3}. \tag{1.34}$$

The result follows from (1.30), (1.33) and (1.34).

Theorem 4. Let $f \in C_\alpha$ and $p \in P$. Then, there exists $\lambda \in R$ such that

$$|a_3 - \lambda a_2^2| \leq \frac{2}{3(3\alpha-1)} \max\left\{ 1, \left| \frac{8\alpha + 3\lambda(1-3\alpha) - 2}{2(2\alpha-1)^2} \right| \right\}.$$

Proof Let $f \in C_\alpha$ and $p \in P$ defined by (1.4). Then, from (1.30),

$$|a_3 - \lambda a_2^2| = \frac{1}{3(3\alpha-1)} \left| c_2 - \nu c_1^2 \right|,$$

where

$$\nu = \frac{8\alpha^2 - 16\alpha + 4 + 9\alpha\lambda - 3\lambda}{4(2\alpha-1)^2}. \tag{1.35}$$

The result follows from Lemma 3.

Corollary 5. Let $f \in C_1$. Then, $|a_3 - \lambda a_2^2| \leq \frac{1}{3} \max\{1, |3 - 3\lambda|\}$.

Corollary 6. Let $f \in C_1$ and $\lambda = 1$. Then, $|a_3 - a_2^2| \leq \frac{1}{3}$
which agrees with Corollary 4.

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