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A Paradigmatic Approach to Investigate Restricted Totient Graphs and their Indices

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Abstract

Let n be a positive integer and let $R = \{r_1, r_2, \cdots, r_{\varphi(n)}\}$ be the set of relatively prime residues of n. We call n a totient number if the sum of the relatively prime residues of n is $2^k n,\ k \geq 1$. A graph G admits a totient labeling with an injective function $\xi : V \to N$ if there exists an induced function $\xi^* : E \to N$ defined by $\xi^*(xy) = \xi(x)\xi(y)$ so that $\xi(x)\xi(y)$ is a totient number for all $xy \in E$. A graph G admits a restricted totient labeling with an injective mapping $\xi : V \to \{1, 2, \cdots, |V|\}$, if there exists an induced function $\xi^* : E \to N$ defined as $\xi^*(xy) = \xi(x)\xi(y)$ with $x \neq y$ such that $\xi(x)\xi(y)$ is a totient number for each $xy \in E$. A least positive integer z is said to be a totient index of G if there exists a totient labeling ξ of G such that the cardinality of range of the induced function ξ^* is z. The objective of this paper is to investigate totient and restricted totient labelings of various classes of graphs. Moreover , we determine the totient index of various classes of graphs by using the notion of totient labeling.

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1 Introduction

The set of integers plays a vital role in both additive and multiplicative number theory. The set of positive integers has always been considered as the back bone of computing science for various algorithms. Therefore, finding new classes of integers along with their complete characterizations in number theory is very fascinating and challenging.
Graph theory has a catalytic role in computer science, electrical engineering, chemical engineering, biological science and many other fields. There are real life problems which can not be addressed easily unless we use graph theory. For example, in the early times the seven bridges problem was only handled with graphs. It was Euler who discovered that such type of walk was not possible by means of a graph. Graph labeling is crucial in graph theory. Many shortest distance problems were solved using the notion of graph labeling. Applications of labeled graphs are in coding theory, X-ray crystallography, communications networks, circuit design, among others.
Initially, labeling of a graph was investigated by Rosa [8] in 1967. In 2018, Bu, Wang and Yang [4] introduced a list of injective Coloring of Planar Graphs. Hoa and Rosenfeld [5] investigated a new labeling of C_{2n} and showed that K_4+M_{6n} decomposes $K_{6n}+4$. Enumeration of squares of 2^k using rooted trees were discussed in [6]. Bloom and Golomb [1] investigated the application of numbered undirected graphs. Khalid and Ali [3] introduced and investigated three new classes of integers by using the notion of Euler's totient function. In [2,7], Khalid and Ali investigated the labeling of various classes of graphs by means of super totient numbers.
Some results of [3], given below, will be used in the sequel.

Definition 1.1. [3]. An integer $n > 0$, is said to be totient number if the sum of co-prime residues of n is $2^k n,\ k \geq 1$; that is,

$$\sum_{d<n,(d,n)=1} d = 2^k n,\ k \geq 1.$$

Proposition 1.2. [3]. *An integer $n > 0$ is totient if and only if $\varphi(n) = 2^{k+1}, k \geq 1$.*

Theorem 1.3. [3]. *A positive integer n is a totient number if and only if $n = 2^k p_1 p_2 \cdots p_m$, $n \neq 2$, $n \neq 2^2$, $n \neq 2 \cdot 3$, where each p_i, $1 \leq i \leq m,\ k \geq 1$, are Fermat's prime numbers.*

Definition 1.4. A mapping ξ from a vertex set to N admits a totient labeling if there exists an induced function ξ^* from the set of edges to the set of positive integers defined by $\xi^*(xy) = \xi(x)\xi(y)$ assigning a totient number to each edge of G. The totient graph on a vertex set $\{2, 4, 6, 8\}$ is shown in Figure 1.

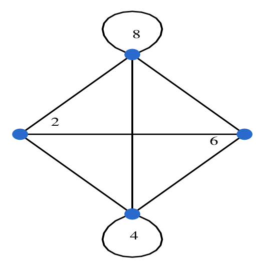

Figure 1: Totient graph on vertex set $\{2, 4, 6, 8\}$.

Definition 1.5. A graph G admits restricted totient labeling with an injective mapping $\xi : V \to \{1, 2, \cdots, |V|\}$, if there exists an induced function $\xi^* : E \to N$, defined by $\xi^*(xy) = \xi(x)\xi(y)$ with $x \neq y$ such that $\xi(x)\xi(y)$ is a totient number for each edge of G. The restricted totient graph on vertex set $\{1, 2, 3, 4, 5\}$ is shown in Figure 2.

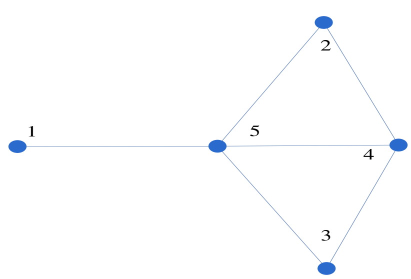

Figure 2: Restricted totient graph.

2 Restricted Totient Labeling of Graphs

Let $\Im_n$ be a restricted totient graph on the vertex set $\{v_1, v_2, \cdots, v_n\}$ such that there is an edge between v_i and v_j if and only if v_iv_j is a totient number with $i \neq j$. The following proposition describes the structure of $\Im_n$, for each positive integer n.

Proposition 2.1. *If n is a positive integer, then $\Im_n$ defined as:*

$$P = \{2^\alpha|\alpha \geq 3\} \cup \{2^\alpha \cdot p_1 : \alpha \geq 2\} \cup \{2^\alpha \cdot p_2, 2^\alpha \cdot p_3, 2^\alpha \cdot p_4, 2^\alpha \cdot p_5|\alpha \geq 0\}$$

$$\cup\{2^\alpha \cdot \prod_{i=1}^{t} p_i^{\alpha_i}|\alpha \geq 0, \alpha_i = 0, \ or\ 1, \ if\ \alpha_1 = 1\ remaining\ \alpha_i = 0, \ then\ \alpha \geq 2\},$$

$$Q = \{\prod_{j=1}^{k} q_j^{\beta_j}|\beta_j \geq 0, \ and\ atleast\ one\ q_j > 2\ not\ a\ Fermat's\ prime\} \cup$$

$$\{2^\alpha \cdot \prod_{i=1}^{t} p_i^{\alpha_i}|\alpha \geq 0, \alpha_i > 1\}, \quad where \ \ t \in \{1,2,3,4,5\}.$$

$$Then, \ \Im_n = K_n \setminus (\{v_p v_q : v_p \in P, \ v_q \in Q\} \cup \{v_{q_i} v_{q_j} : v_{q_i}, v_{q_j} \in Q\}),$$

where, $p_1 = 3, p_2 = 5, p_3 = 17, p_4 = 257, p_5 = 65537$ and K_n the notation of complete graph on n vertices.

The restricted totient graph up to 5 vertices 5 is shown in Figure 3.

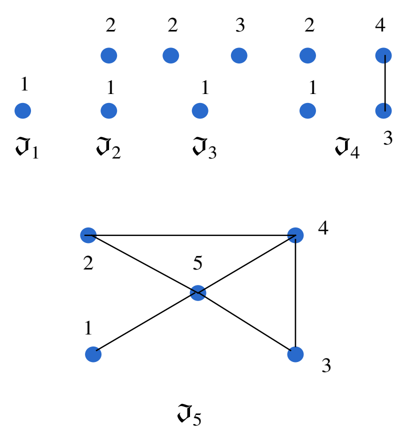

Figure 3: Restricted totient graphs upto vertices 5.

The next theorem shows the existence of a restricted totient graph.

Theorem 2.2. *A simple graph G on h vertices is a restricted totient graph if and only if it is isomorphic to a subgraph of $\Im_n$.*

Corollary 2.3. *The degree of v_1 in $\Im_n$ is*

$$\lfloor \log_2(n) \rfloor + \lfloor \log_2(\frac{n}{2 \cdot 3}) \rfloor + \lfloor \log_2(\frac{2n}{5}) \rfloor + \lfloor \log_2(\frac{2n}{17}) \rfloor + \lfloor \log_2(\frac{2n}{253}) \rfloor$$
$$+ \lfloor \log_2(\frac{2n}{65537}) \rfloor + \lfloor \log_2(\frac{n}{r}) \rfloor + \lfloor \log_2(\frac{2n}{s}) \rfloor - 2,$$

where, $r \in \{2^{\alpha} \cdot \prod_{i=1}^{5} p_i^{\alpha_i} | \alpha \geq 0, \alpha_i = 0, or 1,\ if\ \alpha_1 = 1\ remaining\ \alpha_i = 0,\ then\ \alpha \geq 2\} \setminus (\{2^{\alpha} \cdot p_1 : \alpha \geq 2\}),\ and\ s \in \{\prod_{i=1}^{5} p_i^{\alpha_i} | \alpha_i = 0, or 1\} \setminus \{p_1\}.$

Obviously, if we remove some vertices from a graph G, then the remaining graph will always be a subgraph of G. Thus, we can have the following result.

Corollary 2.4. *If G is a restricted totient graph, then the spanning subgraph S of G is also a restricted totient graph.*

Theorem 2.5. *A complete graph K_n is a restricted totient graph if and only if $n = 1$.*

Proof. As K_n is a simple graph and each vertex has degree $n-1$ by Corollary 2.3, the vertex v_1 has degree $n-1$ if and only if $n = 1$. □

Theorem 2.6. *A complete bipartite graph $K_{m,n}$ is a restricted totient graph if and only if $m = 1, n \in \{4, 5\}$.*

Proof. Let $A = \{a_1\}$ and $B = \{b_1, b_2, \cdots, b_5\}$ be two partite subsets of a vertex set of a complete bipartite graph $K_{1,5}$. An injective mapping on a set of vertices defined as $\xi(a_1) = 5$, and $\xi(b_1) = 1, \xi(b_2) = 2, \xi(b_3) = 3, \xi(b_4) = 4, \xi(b_5) = 6$ clearly induces the function $\xi^*(a_i b_j) = \xi(a_i)\xi(b_j), i \in \{1\}, j \in \{1, 2, 3, 4, 5\}$ which assigns a totient number. Moreover, if we take an additional vertex in A or B; that is the integer 7 appears on a vertex which is not connected to any other vertex. In other words, we make a restricted totient $K_{m,n}$ if and only if $m = 1, n \in \{4, 5\}$. □

Proposition 2.7. *A path graph P_n admits a restricted totient labeling if and only if $n \in \{5, 6\}$.*

Proof. Let P_n be a path graph on 6 vertices. An injective mapping on the vertex set is $\xi(a_1) = 1, \xi(a_2) = 5, \xi(a_3) = 2, \xi(a_4) = 4, \xi(a_5) = 3, \xi(a_6) = 6$. The induced function $\xi^*(a_i a_{i+1}) = \xi(a_i)\xi(a_{i+1}),\ \ i \in \{1, 2, 3, 4, 5\}$ assigns a totient number for each edge of P_6. Moreover, if the vertex a_6 is removed from P_6, then the remaining P_5 admits a restricted totient labeling. □

Proposition 2.8. *A general tree admits a restricted totient labeling if and only if the cardinality of vertex set is 6.*

3 Totient Index of K_n and $K_{m,n}$ Graphs

In this section, we introduce the notion of the restricted totient index number and investigate the totient index number of some well-known classes of graphs.

Definition 3.1. *The totient index of a graph G is the least positive integer s such that there exists a totient labeling ξ of G with the cardinality of the range of the induced function ξ^* of ξ being s. Throughout this section, we label the totient index as $H_\odot$.*
To give the results on totient index of various classes of graphs, the following lemma is important.

Lemma 3.2. *Let G be any graph.*
(a): If the maximum degree of G is Δ, then $H_\odot$ is at least Δ.
(b): If $H_\odot = s$ and H is the subgraph of G, then $H_\odot$ of H is at most s.
(c): If $H_\odot = s$, then there exists a totient labeling ξ of G such that the cardinality of the range of induced function ξ^* of ξ is s and the range of ξ is a subset of $\{2^i : i \geq 3\}$.

Proof. (a) Let u be a vertex of G with degree Δ. Let $a_1, a_2, \cdots, a_\Delta$ be the adjacent vertices of a. Let ξ be any totient labeling of G. Then $\xi^*(aa_i) = \xi(a)\xi(a_i)$ are different for each $1 \leq i \leq \Delta$, because ξ is an injective function. Thus the cardinality of images of ξ^* is at least Δ.
(b) Let $H_\odot = s$ and H be a subgraph of G. If ξ is the totient labeling for G, then it is easy to note that ξ will serve as totient labeling for subgraph H of G. Thus the cardinality of images of ξ^* is at most s.
(c) : Let G be a graph with vertex set $V = \{a_1, a_2, \cdots, a_n\}$ and $H_\odot = s$. Let g be a totient labeling of G with the cardinality of range of g^* is s. Let $\{b_1, b_2, \cdots, b_t\}$ be the set of primes with $g^* = b_1^{b_{i1}} b_2^{b_{i2}} \cdots b_t^{u_{it}}$ $\forall\ 1 \leq i \leq n$, where u_{ij} be nonnegative integers. Let $\xi : V \to N$ be defined as: $\xi(a_i) = 2^{u_{i1}+u_{i2}c+\cdots+u_{it}c^{t-1}}$, where $c = \max\{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq t\} + 1$. It can easily be seen that ξ is an injective function and its induced function ξ^* has range s. $\square$

Definition 3.3. The least positive integer s is known as the sum index of a graph G if there exists an injective function $\xi : V \to N$ whose induced function $\xi^+ : E \to N$ is defined by $\xi^+(ab) = \xi(a) + \xi(b)$ for all $ab \in E$ with the cardinality of its range as s. We use the symbol $H_\oplus$ for the Sum Index of G.

The following theorem gives the relationship between hyper totient index and sum index. A simple consequence of Lemma 3.2(c) is given in the following theorem.

Theorem 3.4. *The totient index of* G *is equal to the sum index of* G; *that is,* $H_{\odot} = H_{\oplus}$.

Theorem 3.5. *Let* K_n *be a complete graph. Then* $H_{\odot} = 2n - 3$.

Proof. Let $V = \{a_1, a_2, \cdots, a_n\}$ be the vertex set for K_n. We define an injective function for vertices of K_n as:
$\xi(a_i) = i$ for all $1 \leq i \leq n$. The range of the induced function ξ^+ of ξ is $\{\xi^+(a_1a_2), \xi^+(a_1a_3), \cdots \xi^+(a_1a_h),$
$\xi^+(a_2a_3), \xi^+(a_2a_4), \cdots \xi^+(a_2a_h), \cdots \xi^+(a_{h-1}a_h)\}$. The cardinality of this set is $2n - 3$. □

Theorem 3.6. *Let* $K_{m,n}$ *be a complete graph. Then* $H_{\odot} = m + n - 1$.

Proof. Let $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_n\}$ be two partite set of vertices of complete bipartite graph $K_{m,n}$. Define an injective function for vertices labeling of $K_{m,n}$ as:
$\xi(a_i) = i$ and $\xi(b_j) = j + m$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. The range of the induced function $\xi^+ : E \to N$ is
$\{\xi^+(a_1b_1), \xi^+(a_1b_2), \cdots, \xi^+(a_1b_n), \xi^+(a_2b_2), \xi^+(a_2b_3), \cdots \xi^+(a_2b_n), \cdots, \xi^+(a_mb_n)\}$.
The cardinality of this set is $m + n - 1$. □

4 Totient Index of Graphs

In this section, we find the totient index of k-ary tree and caterpillar graphs. The following result gives the upper bound of the totient index of a k-ary tree.

Theorem 4.1. *Let* h *be the height of the* K*-ary tree with (*$k \geq 3$*). Then the totient index of the tree is at most* kh; *that is,* $H_{\odot} \leq kh$.

Proof. Let h be the height of a k-ary tree with $k \geq 3$ and let V be its set of vertices. Let a be a rooted vertex and let b is not a leaf of the tree. There are $b_1, b_2, b_3, \cdots, b_k$ children of b. Similarly, in general, $a_{i,j,1}, a_{i,j,2}, a_{i,j,3}, \cdots, a_{i,j,k}$ are children of vertices $a_{i,j}$. The injective function $h : V \to Z$ is defined as

$$h(v) = \begin{cases} 0, & \text{if } v = a, \\ \sum\limits_{j=1}^{t} (-1)^{t-j} 2^{(j-1)k+i_j}, & \text{if } v = a_{i_1,i_2,i_3,\cdots,i_t}, \quad 1 \leq i_1, i_2, i_3, \cdots, i_t \leq t. \end{cases}$$

where t is a level of the k-ary tree.
Let $S = max\{h(a) \mid a \in V\} + 1$. There is a function $\zeta : V \to N$ defined as

$$\zeta(a) = h(a) + S, \quad a \in V.$$

Clearly, ζ is also injective because h is injective. Moreover, the induced function ζ^+ is defined as

$$\zeta^+(e_i) = \begin{cases} 2^i, & \text{if } e_i = aa_i, \quad 1 \le i \le k, \\ 2^{tk+i_{t+1}}, & \text{if } e_i = a_{i_1,i_2,\cdots,i_t}a_{i_1,i_2,\cdots,i_{t+1}}, \quad \forall \;\; 1 \le t \le h-1 \\ & \qquad \text{and } 1 \le i_1, i_2, \cdots, i_{t+1} \le t. \end{cases}$$

The image of ζ^+ is $\{2^t \mid 1 \le t \le hk\}$. □

Since a tree with maximum degree Δ, can be embedded in a Δ- ary tree with height $\lceil D/2 \rceil$. Then, by Lemma 3.2(b) and Theorem 4.1, we have the following proposition.

Proposition 4.2. *Let Δ and D be the respective maximum degree and diameter of a general tree T. Then the totient index of T is at most $\lceil D/2 \rceil \Delta$; that is, $H_{\odot} \le \lceil D/2 \rceil \Delta$.*

Theorem 4.3. *If Δ is the maximum degree of a caterpillar graph, then its totient index is Δ; that is, $H_{\odot} = \Delta$.*

Proof. Let $V = \{a_1, a_2, a_3, \cdots, a_m\}$ be the vertex set of a caterpillar graph. Suppose $S = \{s_{rt} \mid 1 \le t \le deg(a_r) - 2\}$ is an adjacent set of vertices with a_r expect a_{r-1} and a_{r+1}, when $deg(a_r) \ge 3$, $2 \le r \le m-1$. The injective function $h : V \to N$ is defined as:

$$h(a) = \begin{cases} r/2, & \text{if } a = a_r \text{ and } r \text{ is an even}, \\ m - (r-1)/2, & \text{if } a = a_r \text{ and } r \text{ is an odd}, \\ (t+1)m - f(a_i), & \text{if } a = s_{rt}. \end{cases}$$

The induced function $h^+ : E \to N$ is:

$$h^+(e_r) = \begin{cases} m, & \text{if } \;\; e_r = a_r a_{r+1}, \;\; r \text{ is an even}, \\ m+1, & \text{if } \;\; e_r = a_r a_{r+1}, \;\; r \text{ is an odd}, \\ (t+1)m, & \text{if } \;\; e_r = a_r s_{rt}, \quad 2 \le r \le m-1, \;\; 1 \le t \le deg(a_r) - 2. \end{cases}$$

The image of h^+ is $\{m, m+1, 2m, 3m, \cdots (\Delta - 1)m\}$. Hence, $H_{\odot} = \Delta$. □

The labeling of a caterpillar graph by means of an injective function with 29 vertices is shown in Figure 4.

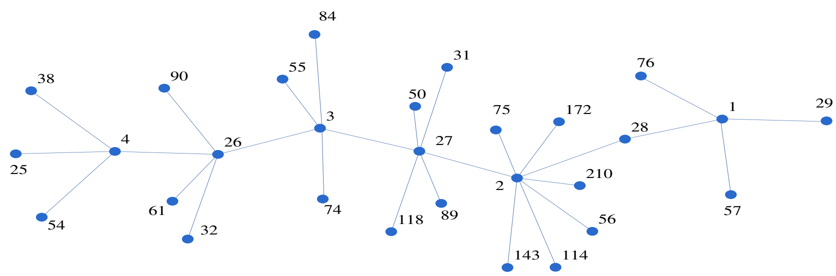

Figure 4: Labeling of caterpillar graph by means of an injective function.

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