International Journal of Mathematics and
Computer Science, **16**(2021), no. 1, 659–675

Neutrosophic $\mathcal{N}$-ideals in Ternary Semigroups

Warud Nakkhasen

Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Faculty of Science
Mahasarakham University
Mahasarakham 44150, Thailand

email: warud.n@msu.ac.th

(Received September 19, 2020, Accepted November 2, 2020)

Abstract

The objective of this paper is to extend the concept of neutrosophic $\mathcal{N}$-ideals in semigroups to ternary semigroups and investigate some of its properties. Moreover, consider characterizations of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals by using the notion of neutrosophic $\mathcal{N}$-products. Furthermore, we show that the homomorphic preimage and the onto homomorphic image of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals are also neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals in ternary semigroups.

1 Introduction

The notion of ternary algebraic systems was first introduced by Lehmer [9] in 1932 who investigated certain ternary algebraic systems, called triplexes, which turned out to be commutative ternary groups. The notion of ternary semigroups was known to Banach who, by an example, verified that a ternary semigroup does not necessarily reduce to an ordinary semigroup. The ideal theory in ternary semigroups was studied by Siosn [15]. In 2010, Santiago and Bala [14] developed the theory of ternary semigroups.

Key words and phrases: Neutrosophic $\mathcal{N}$-structure, neutrosophic $\mathcal{N}$-ideal, neutrosophic $\mathcal{N}$-product.
AMS (MOS) Subject Classifications: 08A72, 20N10, 20M12.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

Zadeh [18] introduced the degree of membership truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [2] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache [16] introduced the degree of indeterminacy/neutrality (i) and defined the neutrosophic set on three components

$$(\text{t, i, f}) = (\text{truth, indeterminacy, falsehood}).$$

These three functions are completely independent. Later, Smarandache [17] considered a more general platform which extends the concepts of the classic sets and fuzzy sets, intuitionistic fuzzy sets and interval intuitionistic fuzzy sets. In 2009, Jun et al. [5] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures. Khan et al. [7] discussed neutrosophic $\mathcal{N}$-structures and their applications in semigroups. This structure was studied by many mathematicians (e.g., [1, 11, 6, 8]). In 2019, Elavarasan et al. [4] introduced the notion of neutrosophic $\mathcal{N}$-ideals in semigroups and investigated some of their properties. Recently, Rattana and Chinram [12, 13] extended the concept of neutrosophic $\mathcal{N}$-structures in n-ary groupoids and ternary semigroups.

In this paper, we investigate the extension of neutrosophic $\mathcal{N}$-ideals from semigroups to ternary semigroups and study some of their properties. Moreover, we consider characterizations of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals by using the concept of neutrosophic $\mathcal{N}$-products. Furthermore, we show that the homomorphic preimage and the onto homomorphic image of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals are also a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal in ternary semigroups.

2 Preliminaries

A nonempty set X with a ternary operation $[\ ] : X \times X \times X \to X$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, is called a *ternary semigroup* [9] if it satisfies the following associative law holds:

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$.

Let $(S, \cdot)$ be a semigroup. Then, we define the ternary operation $[\ ]$ on S by $[abc] = (ab)c$ for all $a, b, c \in S$. So, $(S, [\ ])$ is a ternary semigroup. This shows that every semigroup is a ternary semigroup. Conversely, Banach showed that a ternary semigroup does not necessarily reduce to a semigroup.

For example, $S = \{-i, 0, i\}$ is a ternary semigroup under the multiplication over complex numbers, while $S = \{-i, 0, i\}$ is not a semigroup under complex number multiplication.

For any nonempty subsets A, B and C of a ternary semigroup X, let

$$[ABC] = \{[abc] \mid a \in A, b \in B, c \in C\}.$$

A nonempty subset A of a ternary semigroup X is called a *ternary subsemigroup* of X if $[AAA] \subseteq A$; a *left ideal* of X if $[XXA] \subseteq A$; a *lateral ideal* of X if $[XAX] \subseteq A$; a *right ideal* of X if $[AXX] \subseteq A$; an *ideal* of X if A is a left, right and lateral ideal of X, see [3].

Let $\{a_i \mid i \in \Lambda\}$ be a family of real numbers. We have

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite;} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite;} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

For any two real numbers a and b, we write $a \vee b$ and $a \wedge b$ instead of $\vee\{a, b\}$ and $\wedge\{a, b\}$, respectively.

We denote the family of all functions from a nonempty set X to $[-1, 0]$ by $\mathcal{F}(X, [-1, 0])$. An element of $\mathcal{F}(X, [-1, 0])$ is called a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X). An ordered pair (X, f) of X and an $\mathcal{N}$-function f on X is called an *$\mathcal{N}$-structure*. A *neutrosophic $\mathcal{N}$-structure* over X [7] is defined to be the structure

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function* on X, respectively.

Note that every neutrosophic $\mathcal{N}$-structure X_N over X satisfies the condition: $-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0$ for all $x \in X$.

Let $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ and $X_M := \dfrac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-structures over X.

(i) X_N is called a *neutrosophic $\mathcal{N}$-substructure* of X_M, denoted by $X_N \subseteq X_M$, if it satisfies:

$$T_N(x) \geq T_M(x), I_N(x) \leq I_M(x), F_N(x) \geq F_M(x)$$

for all $x \in X$. If $X_N \subseteq X_M$ and $X_M \subseteq X_N$, we say that $X_N = X_M$.

(ii) The *union* of X_N and X_M is defined to be a neutrosophic $\mathcal{N}$-structure

$$X_{N\cup M} := \frac{X}{(T_{N\cup M}, I_{N\cup M}, F_{N\cup M})}$$

where $T_{N\cup M}(x) = T_N(x) \wedge T_M(x), I_{N\cup M}(x) = I_N(x) \vee I_M(x)$ and $F_{N\cup M}(x) = F_N(x) \wedge F_M(x)$ for all $x \in X$.

(iii) The *intersection* of X_N and X_M is defined to be a neutrosophic $\mathcal{N}$-structure

$$X_{N\cap M} := \frac{X}{(T_{N\cap M}, I_{N\cap M}, F_{N\cap M})}$$

where $T_{N\cap M}(x) = T_N(x) \vee T_M(x), I_{N\cap M}(x) = I_N(x) \wedge I_M(x)$ and $F_{N\cap M}(x) = F_N(x) \vee F_M(x)$ for all $x \in X$.

Example 2.1. Let $X = \{x, y, z\}$ be a set and let X_N and X_M be the neutrosophic $\mathcal{N}$-structures over X which are given by

$$X_N = \left\{\frac{x}{(-0.3, -0.5, -0.9)}, \frac{y}{(-0.8, -0.2, -0.1)}, \frac{z}{(-0.7, -0.4, -0.5)}\right\},$$
$$X_M = \left\{\frac{x}{(-0.5, -0.3, -0.7)}, \frac{y}{(-0.1, -0.4, -0.8)}, \frac{z}{(-0.1, -0.5, -0.2)}\right\}.$$

Then, X_N and X_M are neutrosophic $\mathcal{N}$-structures over X. Next, the union and intersection of X_N and X_M are defined as follows:

$$X_{N\cup M} = \left\{\frac{x}{(-0.5, -0.3, -0.9)}, \frac{y}{(-0.8, -0.2, -0.8)}, \frac{z}{(-0.7, -0.4, -0.5)}\right\},$$
$$X_{N\cap M} = \left\{\frac{x}{(-0.3, -0.5, -0.7)}, \frac{y}{(-0.1, -0.4, -0.1)}, \frac{z}{(-0.1, -0.5, -0.2)}\right\}.$$

For a subset A of a nonempty X, consider the neutrosophic $\mathcal{N}$-structure over X

$$\chi_A(X_N) = \frac{X}{(\chi_A(T)_N, \chi_A(I)_N, \chi_A(F)_N)},$$

where

$$\chi_A(T)_N : X \to [-1,0], x \mapsto \begin{cases} -1 & \text{if } x \in A; \\ 0 & \text{otherwise,} \end{cases}$$
$$\chi_A(I)_N : X \to [-1,0], x \mapsto \begin{cases} 0 & \text{if } x \in A; \\ -1 & \text{otherwise,} \end{cases}$$
$$\chi_A(F)_N : X \to [-1,0], x \mapsto \begin{cases} -1 & \text{if } x \in A; \\ 0 & \text{otherwise,} \end{cases}$$

which is called the *characteristic neutrosophic $\mathcal{N}$-structure* of A.

Let X_N be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$T_N^\alpha := \{x \in X \mid T_N(x) \leq \alpha\};$$
$$I_N^\beta := \{x \in X \mid I_N(x) \geq \beta\};$$
$$F_N^\gamma := \{x \in X \mid F_N(x) \leq \gamma\}.$$

The set

$$X_N(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called a (α, β, γ)-*level set* of X_N. Note that $X_N(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$.

3 Main Results

In this section, we apply the concept of neutrosophic $\mathcal{N}$-ideals in semigroups to define the notion of neutrosophic $\mathcal{N}$-ideals in ternary semigroups and study some of its basic properties. Throughout this paper, we assume that X is a ternary semigroup unless specified otherwise.

Definition 3.1. [13] *Let X_N be a neutrosophic $\mathcal{N}$-structure over X. Then, X_N is said to be a neutrosophic $\mathcal{N}$-ternary subsemigroup of X if it satisfies the following conditions:*

(i) $T_N([xyz]) \leq \bigvee\{T_N(x), T_N(y), T_N(z)\}$;

(ii) $I_N([xyz]) \geq \bigwedge\{I_N(x), I_N(y), I_N(z)\}$;

(iii) $F_N([xyz]) \leq \bigvee\{F_N(x), F_N(y), F_N(z)\}$,

for all $x, y, z \in X$.

Definition 3.2. *A neutrosophic $\mathcal{N}$-structure X_N over X is called a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of X if it satisfies the following conditions:*

(i) $T_N([xyz]) \leq T_N(z)$ *(resp.,* $T_N([xyz]) \leq T_N(y)$, $T_N([xyz]) \leq T_N(x)$*);*

(ii) $I_N([xyz]) \geq I_N(z)$ *(resp.,* $I_N([xyz]) \geq I_N(y)$, $I_N([xyz]) \geq I_N(x)$*);*

(iii) $F_N([xyz]) \leq F_N(z)$ *(resp.,* $F_N([xyz]) \leq F_N(y)$, $F_N([xyz]) \leq F_N(x)$*),*

for all $x, y, z \in X$.

If X_N is a neutrosophic $\mathcal{N}$-left, $\mathcal{N}$-lateral and $\mathcal{N}$-right ideal of X, then X_N is called a *neutrosophic $\mathcal{N}$-ideal* of X.

Note that every neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of a ternary semigroup is a neutrosophic $\mathcal{N}$-ternary subsemigroup, but the neutrosophic $\mathcal{N}$-ternary subsemigroup need not be a neutrosophic $\mathcal{N}$-left ideal or a neutrosophic $\mathcal{N}$-lateral ideal or a neutrosophic $\mathcal{N}$-right ideal as the following example shows.

Example 3.3. Let $X = \{a, b, c, d\}$ and define the ternary operation [ ] on X as follows:

[ ]	a	b	c	d
aa	a	a	a	d
ab	a	a	a	d
ac	a	a	a	d
ad	d	d	d	d

[ ]	a	b	c	d
ba	b	b	b	d
bb	b	b	b	d
bc	b	b	b	d
bd	d	d	d	d

[ ]	a	b	c	d
ca	a	a	a	d
cb	a	a	a	d
cc	a	a	a	d
cd	d	d	d	d

[ ]	a	b	c	d
da	d	d	d	d
db	d	d	d	d
dc	d	d	d	d
dd	d	d	d	d

Then, $(X, [\,])$ is a ternary semigroup [10]. Define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$
\begin{aligned}
T_N(a) &= -0.6, \quad I_N(a) = -0.1, \quad F_N(a) = -0.9;\\
T_N(b) &= -0.6, \quad I_N(b) = -0.1, \quad F_N(b) = -0.9;\\
T_N(c) &= -0.4, \quad I_N(c) = -0.3, \quad F_N(c) = -0.8;\\
T_N(d) &= -0.2, \quad I_N(d) = -0.7, \quad F_N(d) = -0.6.
\end{aligned}
$$

By routine calculations, $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ is a neutrosophic ternary $\mathcal{N}$-subsemigroup of X, but it is not a neutrosophic $\mathcal{N}$-left ideal, because

$$T_N([bda]) = -0.2 \not\leq -0.6 = T_N(a),$$
$$I_N([bda]) = -0.7 \not\geq -0.1 = I_N(a),$$
$$F_N([bda]) = -0.6 \not\leq -0.9 = F_N(a).$$

Example 3.4. Let $X = \{a, b, c, d\}$ and define the ternary operation [ ] on X as follows:

[ ]	a	b	c	d
aa	a	b	a	d
ab	a	b	a	d
ac	a	b	a	d
ad	d	d	d	d

[ ]	a	b	c	d
ba	a	b	a	d
bb	a	b	a	d
bc	a	b	a	d
bd	d	d	d	d

[ ]	a	b	c	d
ca	a	b	a	d
cb	a	b	a	d
cc	a	b	a	d
cd	d	d	d	d

[ ]	a	b	c	d
da	d	d	d	d
db	d	d	d	d
dc	d	d	d	d
dd	d	d	d	d

Then, $(X, [\ ])$ is a ternary semigroup [10]. Now, define a neutrosophic $\mathcal{N}$-structure X_N over X as follows:

$$T_N(a) = -0.9, \quad I_N(a) = -0.2, \quad F_N(a) = -0.8;$$
$$T_N(b) = -0.5, \quad I_N(b) = -0.4, \quad F_N(b) = -0.6;$$
$$T_N(c) = -0.3, \quad I_N(c) = -0.7 \quad F_N(c) = -0.2;$$
$$T_N(d) = -0.9, \quad I_N(d) = -0.2, \quad F_N(d) = -0.8.$$

By routine computations, $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ is a neutrosophic $\mathcal{N}$-left ideal of X, but it is not a neutrosophic $\mathcal{N}$-lateral ideal, because

$$T_N([bab]) = -0.5 \not\leq -0.9 = T_N(a),$$
$$I_N([bab]) = -0.4 \not\geq -0.2 = I_N(a),$$
$$F_N([bab]) = -0.6 \not\leq -0.8 = F_N(a).$$

In addition, X_N is also not a neutrosophic $\mathcal{N}$-right ideal of X, because

$$T_N([acb]) = -0.5 \not\leq -0.9 = T_N(a),$$
$$I_N([acb]) = -0.4 \not\geq -0.2 = I_N(a),$$
$$F_N([acb]) = -0.6 \not\leq -0.8 = F_N(a).$$

Throughout this section, we will prove the following theorems for neutrosophic $\mathcal{N}$-left ideals. For neutrosophic $\mathcal{N}$-lateral ideals and neutrosophic $\mathcal{N}$-right ideals, one can prove similarly.

Theorem 3.5. *Let X_N be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If X_N is a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of X, then the (α, β, γ)-level set of X_N is a left (resp., lateral, right) ideal of X whenever it is nonempty.*

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$-left ideal of X and $X_N(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Let $x, y \in X$ and $a \in X_N(\alpha, \beta, \gamma)$. Then, $T_N(a) \leq \alpha$, $I_N(a) \geq \beta$ and $F_N(a) \leq \gamma$. It follows that $T_N([xya]) \leq T_N(a) \leq \alpha$, $I_N([xya]) \geq I_N(a) \geq \beta$ and $F_N([xya]) \leq F_N(a) \leq \gamma$. Hence, $[xya] \in X_N(\alpha, \beta, \gamma)$. Therefore, $X_N(\alpha, \beta, \gamma)$ is a left ideal of X. □

Theorem 3.6. *Let X_N be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} are left (resp., lateral, right) ideals of X, then X_N is a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of X.*

Proof. Assume that $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} are left ideals of X. Suppose that $T_N([abc]) > T_N(c)$ for some $a, b, c \in X$. Then, $T_N([abc]) > t_{\alpha} \geq T_N(c)$ for some $t_{\alpha} \in [-1, 0)$. Hence, $c \in T_N^{t_{\alpha}}$, but $[abc] \notin T_N^{t_{\alpha}}$, which is a contradiction. Thus,

$$T_N([xyz]) \leq T_N(z)$$

for all $x, y, z \in X$. If $I_N([abc]) < I_N(c)$ for some $a, b, c \in X$, then $I_N([abc]) < t_{\beta} \leq I_N(c)$ for some $t_{\beta} \in (-1, 0]$. Thus, $c \in I_N^{t_{\beta}}$, but $[abc] \notin I_N^{t_{\beta}}$. This is a contradiction. So

$$I_N([xyz]) \geq I_N(z)$$

for some $x, y, z \in X$. Now, suppose that $F_N([abc]) > F_N(c)$ for some $a, b, c \in X$. Then, $F_N([abc]) > t_{\gamma} \geq F_N(c)$ for some $t_{\gamma} \in [-1, 0)$. This implies that $c \in F_N^{t_{\gamma}}$, but $[abc] \notin F_N^{t_{\gamma}}$, which is a contradiction. Hence,

$$F_N([xyz]) \leq F_N(z)$$

for all $x, y, z \in X$. Therefore, X_N is a neutrosophic $\mathcal{N}$-left ideal of X. □

Theorem 3.7. *The intersection of two neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals of X is also a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal.*

Proof. Let $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ and $X_M := \dfrac{X}{(T_M, I_M, F_M)}$ be two neutrosophic $\mathcal{N}$-left ideals of X. For every $x, y, z \in X$, we have

$$\begin{aligned} T_{N\cap M}([xyz]) &= T_N([xyz]) \vee T_M([xyz]) \leq T_N(z) \vee T_N(z) = T_{N\cap M}(z), \\ I_{N\cap M}([xyz]) &= I_N([xyz]) \wedge I_M([xyz]) \geq I_N(z) \wedge I_N(z) = I_{N\cap M}(z), \\ F_{N\cap M}([xyz]) &= F_N([xyz]) \vee F_M([xyz]) \leq F_N(z) \vee F_N(z) = F_{N\cap M}(z). \end{aligned}$$

Consequently, $X_{N\cap M}$ is a neutrosophic $\mathcal{N}$-left ideal of X. □

Corollary 3.8. *If $\{X_{N_i} \mid i \in \Lambda\}$ be a family of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals of X, then $X_{\bigcap N_i}$ is also a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of X.*

Let $X_N := \dfrac{X}{(T_N, I_N, F_N)}$, $X_M := \dfrac{X}{(T_M, I_M, F_M)}$ and $X_L := \dfrac{X}{(T_L, I_L, F_L)}$ be neutrosophic $\mathcal{N}$-structures over X. The *neutrosophic $\mathcal{N}$-product* [13] of X_N, X_M and X_L is defined by

$$\begin{aligned} X_N \odot X_M \odot X_L &:= \frac{X}{(T_{N\circ M\circ L}, I_{N\circ M\circ L}, F_{N\circ M\circ L})} \\ &= \left\{ \frac{x}{(T_{N\circ M\circ L}(x), I_{N\circ M\circ L}(x), F_{N\circ M\circ L}(x))} \mid x \in X \right\} \end{aligned}$$

where

$$T_{N\circ M\circ L}(x) = \begin{cases} \bigwedge\limits_{x=[pqr]} \{T_N(p) \vee T_M(q) \vee T_L(r)\} & \text{if } \exists\ p, q, r \in X \text{ such that } x = [pqr] \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{N\circ M\circ L}(x) = \begin{cases} \bigvee\limits_{x=[pqr]} \{I_N(p) \wedge I_M(q) \wedge I_L(r)\} & \text{if } \exists\ p, q, r \in X \text{ such that } x = [pqr] \\ -1 & \text{otherwise,} \end{cases}$$

$$F_{N\circ M\circ L}(x) = \begin{cases} \bigwedge\limits_{x=[pqr]} \{F_N(p) \vee F_M(q) \vee F_L(r)\} & \text{if } \exists\ p, q, r \in X \text{ such that } x = [pqr] \\ 0 & \text{otherwise.} \end{cases}$$

For any $x \in X$, the element $\dfrac{X}{(T_{N\circ M\circ L}, I_{N\circ M\circ L}, F_{N\circ M\circ L})}$ is simply denoted by

$$(X_N \odot X_M \odot X_L)(x) := (T_{N\circ M\circ L}(x), I_{N\circ M\circ L}(x), F_{N\circ M\circ L}(x)).$$

Theorem 3.9. *Let A be a nonempty subset of X. Then the following statements are equivalent:*

(i) *A is a left (resp., lateral, right) ideal of X;*

(ii) *the characteristic neutrosophic $\mathcal{N}$-structure $\chi_A(X_N)$ over X is a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of X.*

Proof. $(i) \Rightarrow (ii)$ Assume that A is a left ideal of X. Let $x, y, z \in X$. If $z \notin A$, then $\chi_A(T)_N([xyz]) \leq 0 = \chi_A(T)_N(z)$; $\chi_A(I)_N([xyz]) \geq -1 = \chi_A(I)_N(z)$ and $\chi_A(F)_N([xyz]) \leq 0 = \chi_A(F)_N(z)$. On the other hand, suppose that $z \in A$. Then, $[xyz] \in A$. It follows that $\chi_A(T)_N([xyz]) = -1 = \chi_A(T)_N(z)$, $\chi_A(I)_N([xyz]) = 0 = \chi_A(T)_N(z)$ and $\chi_A(F)_N([xyz]) = -1 = \chi_A(F)_N(z)$. Therefore, $\chi_A(X_N)$ is a neutrosophic $\mathcal{N}$-left ideal of X.

$(ii) \Rightarrow (i)$ Assume that $\chi_A(X_N)$ is a neutrosophic $\mathcal{N}$-left ideal of X. Let $x, y \in X$ and $a \in A$. Then, $\chi_A(T)_N([xya]) \leq \chi_A(T)_N(a) = -1$, $\chi_A(I)_N([xya]) \geq \chi_A(I)_N(a) = 0$, $\chi_A(F)_N([xya]) \leq \chi_A(F)_N(a) = -1$. Hence, $\chi_A(T)_N([xya]) = -1$, $\chi_A(I)_N([xya]) = 0$ and $\chi_A(F)_N([xya]) = -1$. This implies that $[xya] \in A$. Consequently, A is a left ideal of X. □

Theorem 3.10. *Let $\chi_A(X_N), \chi_B(X_N)$ and $\chi_C(X_N)$ be characteristic neutrosophic $\mathcal{N}$-structures over X for any subsets A, B and C of X. Then the following statements hold:*

(i) $\chi_A(X_N) \cup \chi_B(X_N) = \chi_{A\cup B}(X_N)$;

(i) $\chi_A(X_N) \cap \chi_B(X_N) = \chi_{A\cap B}(X_N)$;

(iii) $\chi_A(X_N) \odot \chi_B(X_N) \odot \chi_C(X_N) = \chi_{[ABC]}(X_N)$.

Proof. (i) Let $x \in X$. If $x \in A \cup B$, then $x \in A$ or $x \in B$. Thus,

$$(\chi_A(T)_N \cup \chi_B(T)_N)(x) = \chi_A(T)_N(x) \wedge \chi_B(T)_N(x) = -1 = \chi_{A\cup B}(T)_N(x),$$
$$(\chi_A(I)_N \cup \chi_B(I)_N)(x) = \chi_A(I)_N(x) \vee \chi_B(I)_N(x) = 0 = \chi_{A\cup B}(I)_N(x),$$
$$(\chi_A(F)_N \cup \chi_B(F)_N)(x) = \chi_A(F)_N(x) \wedge \chi_B(F)_N(x) = -1 = \chi_{A\cup B}(F)_N(x).$$

So, $\chi_A(X_N) \cup \chi_B(X_N) = \chi_{A\cup B}(X_N)$. If $x \notin A \cup B$, then

$$(\chi_A(T)_N \cup \chi_B(T)_N)(x) = \chi_A(T)_N(x) \wedge \chi_B(T)_N(x) = 0 = \chi_{A\cup B}(T)_N(x),$$
$$(\chi_A(I)_N \cup \chi_B(I)_N)(x) = \chi_A(I)_N(x) \vee \chi_B(I)_N(x) = -1 = \chi_{A\cup B}(I)_N(x),$$
$$(\chi_A(F)_N \cup \chi_B(F)_N)(x) = \chi_A(F)_N(x) \wedge \chi_B(F)_N(x) = 0 = \chi_{A\cup B}(F)_N(x).$$

Hence, $\chi_A(X_N) \cup \chi_B(X_N) = \chi_{A\cup B}(X_N)$.

(ii) The proof is similar to (i).

(iii) Let $x \in X$. If $x \notin [ABC]$, then

$$(\chi_A(T)_N \odot \chi_B(T)_N \odot \chi_C(T)_N)(x) = 0 = \chi_{[ABC]}(T)_N(x),$$
$$(\chi_A(I)_N \odot \chi_B(I)_N \odot \chi_C(I)_N)(x) = -1 = \chi_{[ABC]}(I)_N(x),$$
$$(\chi_A(F)_N \odot \chi_B(F)_N \odot \chi_C(F)_N)(x) = 0 = \chi_{[ABC]}(F)_N(x).$$

Thus, $\chi_A(X_N) \odot \chi_B(X_N) \odot \chi_C(X_N) = \chi_{[ABC]}(X_N)$. If $x \in [ABC]$, then $x = [abc]$ for some $a \in A, b \in B$ and $c \in C$. It follows that

$$\begin{aligned}
(\chi_A(T)_N \odot \chi_B(T)_N \odot \chi_C(T)_N)(x) &= \bigwedge_{x=[pqr]} \{\chi_A(T)_N(p) \vee \chi_B(T)_N(q) \vee \chi_C(T)_N(r)\} \\
&\leq \chi_A(T)_N(a) \vee \chi_B(T)_N(b) \vee \chi_C(T)_N(c) \\
&= -1 = \chi_{[ABC]}(T)_N(x), \\
(\chi_A(I)_N \odot \chi_B(I)_N \odot \chi_C(I)_N)(x) &= \bigvee_{x=[pqr]} \{\chi_A(I)_N(p) \wedge \chi_B(I)_N(q) \wedge \chi_C(I)_N(r)\} \\
&\geq \chi_A(I)_N(a) \wedge \chi_B(I)_N(b) \wedge \chi_C(I)_N(c) \\
&= 0 = \chi_{[ABC]}(I)_N(x), \\
(\chi_A(F)_N \odot \chi_B(F)_N \odot \chi_C(F)_N)(x) &= \bigwedge_{x=[pqr]} \{\chi_A(F)_N(p) \vee \chi_B(F)_N(q) \vee \chi_C(F)_N(r)\} \\
&\leq \chi_A(F)_N(a) \vee \chi_B(F)_N(b) \vee \chi_C(F)_N(c) \\
&= -1 = \chi_{[ABC]}(F)_N(x).
\end{aligned}$$

Therefore, $\chi_A(X_N) \odot \chi_B(X_N) \odot \chi_C(X_N) = \chi_{[ABC]}(X_N)$. □

Theorem 3.11. *Let X_L be a neutrosophic $\mathcal{N}$-structure over X. Then X_L is a neutrosophic $\mathcal{N}$-left ideal of X if and only if $X_N \odot X_M \odot X_L \subseteq X_L$ for every neutrosophic $\mathcal{N}$-structures X_N and X_M over X.*

Proof. Assume that X_L is a neutrosophic $\mathcal{N}$-left ideal of X. Let X_N and X_M be neutrosophic $\mathcal{N}$-structures over X. Let $x \in X$. Obviously, $X_N \odot X_M \odot X_L \subseteq X_L$ for all $a, b, c \in X$ such that $x \neq [abc]$. Suppose that there exist $a, b, c \in X$ such that $x = [abc]$. We obtain

$$T_L(x) = T_L([abc]) \leq T_L(c) \leq T_N(a) \vee T_M(b) \vee T_L(c),$$
$$I_L(x) = I_L([abc]) \geq I_L(c) \geq I_N(a) \wedge I_M(b) \wedge I_L(c),$$
$$F_L(x) = F_L([abc]) \leq F_L(c) \leq F_N(a) \vee F_M(b) \vee F_L(c).$$

This implies that

$$\begin{aligned}
T_L(x) &\leq \bigwedge_{x=[abc]} \{T_N(a) \vee T_M(b) \vee T_L(c)\} = T_{N\circ M\circ L}(x),\\
I_L(x) &\geq \bigvee_{x=[abc]} \{I_N(a) \wedge I_M(b) \wedge I_L(c)\} = I_{N\circ M\circ L}(x),\\
F_L(x) &\leq \bigwedge_{x=[abc]} \{F_N(a) \vee F_M(b) \vee F_L(c)\} = F_{N\circ M\circ L}(x).
\end{aligned}$$

Therefore, $X_N \odot X_M \odot X_L \subseteq X_L$. Conversely, assume that X_L is a neutrosophic $\mathcal{N}$-structure over X such that $X_N \odot X_M \odot X_L \subseteq X_L$ for every neutrosophic $\mathcal{N}$-structures X_N and X_M over X. Let $x, y, z \in X$ and $a = [xyz]$. Then,

$$\begin{aligned}
T_L([xyz]) &= T_L(a) \leq (\chi_X(T)_N \circ \chi_X(T)_M \circ T_L)(a)\\
&= \bigwedge_{a=[pqr]} \{\chi_X(T)_N(p) \vee \chi_X(T)_M(q) \vee T_L(r)\}\\
&\leq \chi_X(T)_N(x) \vee \chi_X(T)_M \vee T_L(z) = T_L(z),\\
I_L([xyz]) &= I_L(a) \geq (\chi_X(I)_N \circ \chi_X(I)_M \circ I_L)(a)\\
&= \bigvee_{a=[pqr]} \{\chi_X(I)_N(p) \wedge \chi_X(I)_M(q) \wedge I_L(r)\}\\
&\geq \chi_X(I)_N(x) \wedge \chi_X(I)_M \wedge I_L(z) = I_L(z),\\
F_L([xyz]) &= F_L(a) \leq (\chi_X(F)_N \circ \chi_X(F)_M \circ F_L)(a)\\
&= \bigwedge_{a=[pqr]} \{\chi_X(F)_N(p) \vee \chi_X(F)_M(q) \vee F_L(r)\}\\
&\leq \chi_X(F)_N(x) \vee \chi_X(F)_M \vee F_L(z) = F_L(z).
\end{aligned}$$

Consequently, X_L is a neutrosophic $\mathcal{N}$-left ideal of X. □

The proofs of the following theorems are similar to that of Theorem 3.11.

Theorem 3.12. *Let X_M be a neutrosophic $\mathcal{N}$-structure over X. Then X_M is a neutrosophic $\mathcal{N}$-lateral ideal of X if and only if $X_L \odot X_M \odot X_R \subseteq X_M$ for every neutrosophic $\mathcal{N}$-structures X_L and X_R over X.*

Theorem 3.13. *Let X_R be a neutrosophic $\mathcal{N}$-structure over X. Then X_R is a neutrosophic $\mathcal{N}$-right ideal of X if and only if $X_R \odot X_N \odot X_M \subseteq X_R$ for every neutrosophic $\mathcal{N}$-structures X_N and X_M over X.*

Theorem 3.14. *Let X_A, X_N and X_M be neutrosophic $\mathcal{N}$-structures over X. If X_A is a neutrosophic $\mathcal{N}$-left ideal of X, then $X_A \odot X_N \odot X_M$ is also a neutrosophic $\mathcal{N}$-left ideal of X.*

Proof. Assume that X_A is a neutrosophic $\mathcal{N}$-left ideal of X. Let $x, y, z \in X$. If there exist $a, b, c \in X$ such that $z = [abc]$, then $[xyz] = [xy[abc]] = [[xya]bc]$. Then,

$$\begin{aligned}
T_{A\circ N\circ M}(z) &= \bigwedge_{z=[abc]} \{T_A(a) \vee T_N(b) \vee T_M(c)\} \geq \bigwedge_{[xyz]=[[xya]bc]} \{T_A([xya]) \vee T_N(b) \vee T_M(c)\} \\
&= \bigwedge_{[xyz]=[tbc]} \{T_A(t) \vee T_N(b) \vee T_M(c)\} = T_{A\circ N\circ M}([xyz]), \\
I_{A\circ N\circ M}(z) &= \bigvee_{z=[abc]} \{I_A(a) \wedge I_N(b) \wedge I_M(c)\} \leq \bigvee_{[xyz]=[[xya]bc]} \{I_A([xya]) \wedge I_N(b) \wedge I_M(c)\} \\
&= \bigvee_{[xyz]=[tbc]} \{I_A(t) \wedge I_N(b) \wedge I_M(c)\} = I_{A\circ N\circ M}([xyz]), \\
F_{A\circ N\circ M}(z) &= \bigwedge_{z=[abc]} \{F_A(a) \vee F_N(b) \vee F_M(c)\} \geq \bigwedge_{[xyz]=[[xya]bc]} \{F_A([xya]) \vee F_N(b) \vee F_M(c)\} \\
&= \bigwedge_{[xyz]=[tbc]} \{F_A(t) \vee F_N(b) \vee F_M(c)\} = F_{A\circ N\circ M}([xyz]),
\end{aligned}$$

Therefore, $X_A \odot X_N \odot X_M$ is a neutrosophic $\mathcal{N}$-left ideal of X. □

Similarly, we have the following theorem:

Theorem 3.15. *Let X_A, X_N and X_M be neutrosophic $\mathcal{N}$-structures over X. If X_A is a neutrosophic $\mathcal{N}$-right ideal of X, then $X_N \odot X_M \odot X_A$ is also a neutrosophic $\mathcal{N}$-right ideal of X.*

Let $f : X \to Y$ be a function of sets. If $Y_M := \dfrac{Y}{(T_M, I_M, F_M)}$ is a neutrosophic $\mathcal{N}$-structure over Y, the *preimage* [13] of Y_M under f is defined to be a neutrosophic $\mathcal{N}$-structure

$$f^{-1}(Y_M) := \frac{X}{(f^{-1}(T_M), f^{-1}(I_M), f^{-1}(F_M))}$$

over X where $f^{-1}(T_M)(x) = T_M(f(x))$, $f^{-1}(I_M)(x) = I_M(f(x))$ and $f^{-1}(F_M)(x) = F_M(f(x))$ for all $x \in X$.

Theorem 3.16. *Let $f : X \to Y$ be a homomorphism of ternary semigroups. If $Y_M := \dfrac{Y}{(T_M, I_M, F_M)}$ is a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of Y, then the preimage of Y_M under f is a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of X.*

Proof. Assume that $f^{-1}(Y_M) := \dfrac{X}{(f^{-1}(T_M), f^{-1}(I_M), f^{-1}(F_M))}$ is the preimage of Y_M under f. Let $x, y, z \in X$. Then,

$$f^{-1}(T_M)([xyz]) = T_M(f([xyz])) = T_M([f(x)f(y)f(z)]) \leq T_M(f(z)) = f^{-1}(T_M)(z),$$
$$f^{-1}(I_M)([xyz]) = I_M(f([xyz])) = I_M([f(x)f(y)f(z)]) \geq I_M(f(z)) = f^{-1}(I_M)(z),$$
$$f^{-1}(F_M)([xyz]) = F_M(f([xyz])) = F_M([f(x)f(y)f(z)]) \leq F_M(f(z)) = f^{-1}(F_M)(z).$$

Hence, $f^{-1}(Y_M)$ is a neutrosophic $\mathcal{N}$-left ideal of X. □

Let $f : X \to Y$ be an onto function of sets. If $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ is a neutrosophic $\mathcal{N}$-structure over X, then the *image* [13] of X_N under f is defined to be a neutrosophic $\mathcal{N}$-structure

$$f(X_N) := \frac{Y}{(f(T_N), f(I_N), f(F_N))}$$

over Y where

$$f(T_N)(y) = \bigwedge_{x \in f^{-1}(y)} T_N(x), f(I_N)(y) = \bigvee_{x \in f^{-1}(y)} I_N(x),$$
$$f(F_N)(y) = \bigwedge_{x \in f^{-1}(y)} F_N(x).$$

Theorem 3.17. *For an onto homomorphism $f : X \to Y$ of ternary semigroups, let $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over X such that for any nonempty subset A of X there exists $x_0 \in A$ such that $T_N(x_0) = \bigwedge_{z \in A} T_N(z), I_N(x_0) = \bigvee_{z \in A} I_N(z)$ and $F_N(x_0) = \bigwedge_{z \in A} F_N(z)$. If X_N is a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of X, then the image of X_N under f is a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of Y.*

Proof. Assume that $f(X_N) := \dfrac{X}{(f(T_N), f(I_N), f(F_N))}$ is the image of X_N under f. Let $a, b, c \in Y$. Then, $f^{-1}(a) \neq \emptyset, f^{-1}(b) \neq \emptyset$ and $f^{-1}(c) \neq \emptyset$ in X.

Thus, there exist $x_a \in f^{-1}(a), x_b \in f^{-1}(b)$ and $x_c \in f^{-1}(c)$ such that

$$T_N(x_a) = \bigwedge_{z\in f^{-1}(a)} T_N(z), I_N(x_a) = \bigvee_{z\in f^{-1}(a)} I_N(z), F_N(x_a) = \bigwedge_{z\in f^{-1}(a)} F_N(z),$$
$$T_N(x_b) = \bigwedge_{z\in f^{-1}(b)} T_N(z), I_N(x_b) = \bigvee_{z\in f^{-1}(b)} I_N(z), F_N(x_b) = \bigwedge_{z\in f^{-1}(b)} F_N(z),$$
$$T_N(x_c) = \bigwedge_{z\in f^{-1}(c)} T_N(z), I_N(x_c) = \bigvee_{z\in f^{-1}(c)} I_N(z), F_N(x_c) = \bigwedge_{z\in f^{-1}(c)} F_N(z).$$

It turns out that

$$f(T_N)([abc]) = \bigwedge_{x\in f^{-1}([abc])} T_N(x) \leq T_N([x_a x_b x_c]) \leq T_N(x_c) = \bigwedge_{z\in f^{-1}(c)} T_N(z) = f(T_N)(c),$$
$$f(I_N)([abc]) = \bigvee_{x\in f^{-1}([abc])} I_N(x) \geq I_N([x_a x_b x_c]) \geq I_N(x_c) = \bigvee_{z\in f^{-1}(c)} I_N(z) = f(I_N)(c),$$
$$f(F_N)([abc]) = \bigwedge_{x\in f^{-1}([abc])} F_N(x) \leq F_N([x_a x_b x_c]) \leq F_N(x_c) = \bigwedge_{z\in f^{-1}(c)} F_N(z) = f(F_N)(c).$$

Therefore, $f(X_N)$ is a neutrosophic $\mathcal{N}$-left ideal of Y. □

4 Conclusion

In this paper, we have introduced the concept of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals in ternary semigroups and investigated several their properties. We have also discussed characterizations of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals by using the notion of neutrosophic $\mathcal{N}$-products. Finally, we have shown that the homomorphic preimage and the onto homomorphic image of a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal are also a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal in ternary semigroups. In out future study, we will define the concept of neutrosophic $\mathcal{N}$-bi-ideals in ternary semigroups and investigate their properties.

Acknowledgement.

This research project was financially supported by Mahasarakham University 2021.

References

[1] S. Abdullah, A. F. Ali, Applications of $\mathcal{N}$-structures in implicative filters of BE-algebras, Journal of Intelligent & Fuzzy Systems, **29**, no. 2, (2015), 517–524.

[2] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, **20**, no. 1, (1986), 87–96.

[3] V. N. Dixit, S. A. Dewan, Note on quasi and bi-ideals in ternary semigroups, International Journal of Mathematics and Mathematical Sciences, **18**, (1995), 501-508.

[4] B. Elavarasan, F. Smarandache, Y. B. Jun, Neutrosophic $\mathcal{N}$-ideals in semigroups, Neutrosophic Sets and Systems, **28**, (2019), 273–280.

[5] Y. B. Jun, K. Lee, S. Z. Song, $\mathcal{N}$-ideals of BCK/BCI-algebras, Journal of the Chungcheong Mathematical Society, **22**, (2009), 417–437.

[6] Y. B. Jun, M. A. Öztürk, E. H. Roh, $\mathcal{N}$-structures applied to closed ideals in BCH-algebras, International Journal of Mathematics and Mathematical Sciences, **2010**, Article ID 943565, 9 pages.

[7] M. Khan, S. Anis, F. Smarandache, Y. B. Jun, Neutrosophic $\mathcal{N}$-structures and their applications in semigroups, Annals of Fuzzy Mathematics and Informatics, **14**, no. 6, (2017), 583–598.

[8] K. J. Lee, Y. B. Jun, The essence of subtraction algebras based on $\mathcal{N}$-structures, Communications of the Korean Mathematical Society., **27**, no. 1, (2012), 15–22.

[9] D. H. Lehmer, A ternary analogue of abelian groups, American Journal of Mathematics, **54**, (1932), 329–338.

[10] P. Pornsurat, B. Pibaljommee, Regularities in ordered ternary semigroups, Quasigroups and Related Systems, **27**, (2019), 107–118.

[11] P. Rangsuk, P. Huana, A. Iampan, Neutrosophic $\mathcal{N}$-structures over UP-algebras, Neutrosophic Sets and Systems, **28**, (2009), 87–127.

[12] A. Rattana, R. Chinram, Applications of neutrosophic $\mathcal{N}$-structures in n-ary groupoids, European Journal of Pure and Applied Mathematics, **13**, no. 2, (2020), 200–215.

[13] A. Rattana, R. Chinram, Applications of neutrosophic $\mathcal{N}$-structures in ternary semigroups, Journal of Mathematical and Computational Science, **10**, no. 2, (2020), 1437–1455.

[14] M. L. Santiago, S. S. Bala, Ternary semigroups, Semigroup Forum, **81**, (2010), 380–388.

[15] F. M. Sioson, Ideal theory in ternary semigroups, Mathematica Japonica, **10**, (1965), 63–84.

[16] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Research Press, USA, 1999.

[17] F. Smarandache, Neutrosophic set-a generalization of the intuitionistic fuzzy set, International Journal of Pure and Applied Mathematics, **24**, no. 3, (2005), 287–297.

[18] L. A. Zadeh, Fuzzy sets, Information and Control, **8**, no. 3, (1965), 338–353.