International Journal of Mathematics and
Computer Science, **16**(2021), no. 2, 809–817

Norm inequalities and applications

Wasim Audeh

Department of Mathematics
Faculty of Arts and Sciences
University of Petra
Amman, Jordan

email: waudeh@uop.edu.jo

(Received November 30, 2020, Accepted December 29, 2020)

Abstract

Let A, B and X be operators on a complex separable Hilbert space such that A and B are positive semidefinite. If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, then

$$||X^*AX + XBX^*|| \leq \frac{1}{2}\max\left(\|S\|, \|T\|\right) + \frac{1}{2}\|V\|,$$

where

$$S = \left(g(A)XX^*f(A) + f(A)XX^*g(A)\right),$$
$$T = \left(g(B)X^*Xf(B) + f(B)X^*Xg(B)\right),$$

and

$$V = \left(g(A)X^2f(B) + f(A)X^2g(B)\right).$$

This inequality extends an inequality of Kittaneh, which improved an earlier inequality of Davidson and Power.

1 Introduction

Throughout this paper, the set of all bounded linear operators on a complex separable Hilbert space is denoted by $\mathbb{B}(\mathbb{H})$ and the two sided ideal of compact

Key words and phrases: Norm, Singular value, Positive operator, Inequality.
AMS (MOS) Subject Classifications: 15A18, 15A42, 47A63, 47B02, 47B15.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

operators is denoted by $\mathbb{K}(\mathbb{H})$. The j^{th} largest singular value of A is denoted by $s_j(A)$ which is the eigenvalue of the positive operator $|A|$. The singular values satisfy the following property

$$s_j(A) \leq s_j(B) \text{ if and only if } s_j(A \oplus A) \leq s_j(B \oplus B) \tag{1.1}$$

for $j = 1, 2, ...$ The spectral (usual operator) norm is denoted by $\|.\|$, where $\|A\| = s_1(A)$ and the Schatten p-norms are denoted by $\|.\|_p$, where $\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{1/p}$ for $1 \leq p \leq \infty$. The wider class of unitarily invariant norms are denoted by $|||.|||$ which satisfy the invariance property: $|||UAV||| = |||A|||$ for $A \in \mathbb{M}_n$ and U, V are unitary.

Kittaneh [9] proved that if $A, B \in \mathbb{B}(\mathbb{H})$ are positive semidefinite, then

$$|||(A+B) \oplus 0||| \leq |||(A \oplus B)||| + \left|\left|\left|A^{1/2}B^{1/2} \oplus A^{1/2}B^{1/2}\right|\right|\right|. \tag{1.2}$$

Specifying this inequality to the operator norm $\|.\|$ and the Schatten p-norms, respectively, we have

$$||A+B|| \leq \max\{\|A\|, \|B\|\} + \left\|A^{1/2}B^{1/2}\right\|. \tag{1.3}$$

and

$$\|A+B\|_p \leq \left(\|A\|_p^p + \|B\|_p^p\right)^{1/p} + 2^{1/p}\left\|A^{1/2}B^{1/2}\right\|_p. \tag{1.4}$$

We give a generalization of the inequalities (1.2), (1.3), and (1.4).

Let $A, B, C \in \mathbb{K}(\mathbb{H})$. The authors in [7] proved the arithmetic-geometric mean inequality for singular values

$$2s_j(AB^*) \leq s_j(A^*A + B^*B) \tag{1.5}$$

for $j = 1, 2, ...$ The authors in [11] showed that

$$s_j(A - B) \leq s_j(A \oplus B) \tag{1.6}$$

for $j = 1, 2, ...$, while the author in [10] showed that if $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. Then

$$2s_j(B) \leq s_j \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \tag{1.7}$$

for $j = 1, 2, ...$ In [5], it was shown that if A is self-adjoint, $B \geq 0$ and $\pm A \leq B$, then

$$2s_j(A) \leq s_j((B + A) \oplus (B - A)) \tag{1.8}$$

for $j = 1, 2, ...$ In this paper, we provide an inequality which is equivalent to the inequalities (1.5)-(1.8). This inequality states that if $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

$$s_j((A - C) + (B - B^*)) \leq s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right)$$

for $j = 1, 2, ...$ and several applications are also given. More detailed and impressive generalizations on this topic have been discussed recently in [3] and [4].

2 Main Results

A generalization of inequality (1.2) is given in the following theorem, which is based on a folklore lemma (see, e.g., [6]). All functions considered in this paper are nonnegative continuous functions on $[0, \infty)$, where $f(t)g(t) = t$ for all $t \in [0, \infty)$.

Lemma 2.1. *Let $A, B \in \mathbb{B}(\mathbb{H})$ such that AB is self-adjoint. Then*

$$|||AB||| \leq |||Re(BA)|||\,.$$

Theorem 2.2. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that $A,\ B \geq 0$. Then*

$$|||(X^*AX + XBX^*) \oplus 0||| \leq \frac{1}{2}\,|||S \oplus T||| + \frac{1}{2}\,|||V \oplus V^*|||\,, \tag{2.9}$$

where

$$S = (g(A)XX^*f(A) + f(A)XX^*g(A))\,,$$

$$T = (g(B)X^*Xf(B) + f(B)X^*Xg(B))\,,$$

and

$$V = \left(g(A)X^2f(B) + f(A)X^2g(B)\right).$$

Proof. Let $S = \begin{bmatrix} X^*f(A) & Xf(B) \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} g(A)X & 0 \\ g(B)X^* & 0 \end{bmatrix}$. Since $f(A)g(A) = A$ and $f(B)g(B) = B$, we have

$$ST = \begin{bmatrix} X^*AX + XBX^* & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$TS = \begin{bmatrix} g(A)XX^*f(A) & g(A)X^2f(B) \\ g(B)X^{*2}f(A) & g(B)X^*Xf(B) \end{bmatrix}.$$

Since ST is self-adjoint and using Lemma 2.1, we get

$$|||(X^*AX + XBX^*) \oplus 0||| \leq \frac{1}{2} \left|\left|\left| \begin{bmatrix} S & V \\ V^* & T \end{bmatrix} \right|\right|\right|.$$

By the triangle inequality, we conclude that

$$\begin{aligned} |||(X^*AX + XBX^*) \oplus 0||| &\leq \frac{1}{2} \left|\left|\left| \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} \right|\right|\right| + \frac{1}{2} \left|\left|\left| \begin{bmatrix} 0 & V \\ V^* & 0 \end{bmatrix} \right|\right|\right| \\ &= \frac{1}{2} |||S \oplus T||| + \frac{1}{2} |||V \oplus V^*||| . \end{aligned}$$

□

Remark 2.3. *Letting $X = I$ and $f(t) = g(t) = t^{1/2}$ in inequality (2.9) leads to inequality (1.2).*

Corollary 2.4. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that A, $B \geq 0$. Then*

$$||X^*AX + XBX^*|| \leq \frac{1}{2} \max(\|S\|, \|T\|) + \frac{1}{2} \|V\|, \tag{2.10}$$

where

$$S = (g(A)XX^*f(A) + f(A)XX^*g(A)),$$
$$T = (g(B)X^*Xf(B) + f(B)X^*Xg(B)),$$

and

$$V = \left(g(A)X^2f(B) + f(A)X^2g(B)\right).$$

Proof. Inequality (2.10) follows by considering the spectral norm in inequality (2.9). □

Remark 2.5. *Letting $X = I$ and $f(t) = g(t) = t^{1/2}$ in inequality (2.10), we get inequality (1.3).*

Corollary 2.6. *Let $A, B, X \in \mathbb{B}(\mathbb{H})$ such that $A, B \geq 0$. Then*

$$||X^*AX + XBX^*||_p^p \leq \frac{1}{2}\left(\|S\|_p^p + \|T\|_p^p\right)^{1/p} + 2^{1/p}\|V\|_p^p, \tag{2.11}$$

where

$$S = (g(A)XX^*f(A) + f(A)XX^*g(A)),$$
$$T = (g(B)X^*Xf(B) + f(B)X^*Xg(B)),$$

and

$$V = \left(g(A)X^2f(B) + f(A)X^2g(B)\right).$$

Proof. Inequality (2.11) follows by considering the Schatten p-norms in inequality (2.9). □

Remark 2.7. *Letting $X = I$ and $f(t) = g(t) = t^{1/2}$ in inequality (2.11), we get inequality (1.4).*

In the rest of this paper, we will assume that all considered operators are compact operators.

Theorem 2.8. *Let* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. *Then*

$$s_j((A-C)+(B-B^*)) \leq s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) \tag{2.12}$$

for $j = 1, 2, ...$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$, $X = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$. Then U is unitary and

$$\begin{aligned} UXU^* &= \frac{1}{2}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\begin{bmatrix} I & I \\ -I & I \end{bmatrix} \\ &= \frac{1}{2}K \geq 0. \end{aligned}$$

where

$$K = \begin{bmatrix} (A+C)-(B+B^*) & (A-C)+(B-B^*) \\ (A-C)-(B-B^*) & (A+C)+(B+B^*) \end{bmatrix}.$$

Applying inequality (1.7) to the operator matrix UXU^* leads to

$$\begin{aligned} s_j\left((A-C)+(B-B^*)\right) &\leq s_j\left(\frac{1}{2}K\right) \\ &= s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) \text{ (by equation (??)).} \end{aligned}$$

Inequality (2.12) has thus been substantiated. $\square$

Corollary 2.9. *Let A, C be positive operators. Then*

$$s_j(A-C) \leq s_j(A \oplus C) \tag{2.13}$$

for $j = 1, 2, ...$

Proof. Letting $B = 0$ in inequality (2.12) leads to inequality (2.13). $\square$

Remark 2.10. *In the sense of Corollary 2.9, Inequality (2.12) is a generalization of inequality (1.6)*

Theorem 2.11. *The following inequalities are equivalent:*

(i) *If* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, *then*

$$2s_j(B) \leq s_j\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

for $j = 1, 2, ...$

(ii) *If* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, *then*

$$s_j((A-C)+(B-B^*)) \leq s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right)$$

for $j = 1, 2, ...$

Proof. $(i) \Longrightarrow (ii)$ This implication follows from the proof of Theorem (2.8).

$(ii) \Longrightarrow (i)$ Let $A = \begin{bmatrix} L & M \\ M^* & N \end{bmatrix}$, $B = 0$, and $C = \begin{bmatrix} L & -M \\ -M^* & N \end{bmatrix}$. Then applying inequality (2.12) for $j = 1, 2, ...$, we get

$$s_j\left((A-C)+(0-0)\right) \leq s_j\left(A \oplus C\right).$$

This leads to

$$2s_j \begin{bmatrix} 0 & M \\ M^* & 0 \end{bmatrix} \leq s_j \left(\begin{bmatrix} L & M \\ M^* & N \end{bmatrix} \oplus \begin{bmatrix} L & -M \\ -M^* & N \end{bmatrix} \right),$$

which is equivalent to

$$2s_j(M \oplus M^*) \leq s_j \left(\begin{bmatrix} L & M \\ M^* & N \end{bmatrix} \oplus \begin{bmatrix} L & -M \\ -M^* & N \end{bmatrix} \right).$$

Applying inequality (1.1), we have

$$2s_j(M) \leq s_j \left(\begin{bmatrix} L & M \\ M^* & N \end{bmatrix} \right),$$

which is precisely inequality (1.7). Theorem 2.11 has thus been substantiated. □

Remark 2.12. *By proving that inequality (2.12) is equivalent to one of the equivalent inequalities which was proven in [5], we will have proven that it is equivalent to all equivalent inequalities.*

The following inequality is equivalent to inequality (1.5).

Theorem 2.13. *Let $A_1, A_2, A_3, A_4, X \in \mathbb{K}(\mathbb{H})$ such that $X \geq 0$ and let*

$$Y = S + T + V + W,$$

where

$$S = X^{1/2} |A_1|^2 X^{1/2},\ T = X^{1/2} |A_2|^2 X^{1/2},$$
$$V = X^{1/2} |A_3|^2 X^{1/2},\ \textit{and}\ W = X^{1/2} |A_4|^2 X^{1/2}.$$

Then

$$2s_j \begin{bmatrix} A_1XA_2^* & A_1XA_4^* \\ A_3XA_2^* & A_3XA_4^* \end{bmatrix} \leq s_j(Y) \tag{2.14}$$

for $j = 1, 2, ...$

Proof. Let $S = \begin{bmatrix} A_1X^{1/2} & 0 \\ A_3X^{1/2} & 0 \end{bmatrix}$ and $T = \begin{bmatrix} A_2X^{1/2} & 0 \\ A_4X^{1/2} & 0 \end{bmatrix}$. Then

$$ST^* = \begin{bmatrix} A_1XA_2^* & A_1XA_4^* \\ A_3XA_2^* & A_3XA_4^* \end{bmatrix}$$

and

$$S^*S + T^*T = Y.$$

Applying inequality (1.5) leads to

$$2s_j \begin{bmatrix} A_1XA_2^* & A_1XA_4^* \\ A_3XA_2^* & A_3XA_4^* \end{bmatrix} \leq s_j(Y)$$

for $j = 1, 2, \ldots$ which is precisely inequality (2.14). □

Remark 2.14. *Letting $X = I$ and $A_3 = A_4 = 0$ in Theorem 2.13, inequality (1.5) obtains.*

In the next theorem, we prove that inequality (2.14) is equivalent to inequality (1.5).

Theorem 2.15. *The following statements are equivalent:*

(i) *Let $A, B \in \mathbb{K}(\mathbb{H})$. Then*

$$2s_j(AB^*) \leq s_j(A^*A + B^*B)$$

for $j = 1, 2, \ldots$

(ii) Let $A_1, A_2, A_3, A_4, X \in \mathbb{K}(\mathbb{H})$ such that $X \geq 0$ and let

$$Y = S + T + V + W,$$

where

$$S = X^{1/2} |A_1|^2 X^{1/2},\ T = X^{1/2} |A_2|^2 X^{1/2},$$
$$V = X^{1/2} |A_3|^2 X^{1/2},\ \text{and}\ W = X^{1/2} |A_4|^2 X^{1/2}.$$

Then

$$2s_j \begin{bmatrix} A_1XA_2^* & A_1XA_4^* \\ A_3XA_2^* & A_3XA_4^* \end{bmatrix} \leq s_j(Y)$$

for $j = 1, 2, \ldots$

Proof. $(i) \Longrightarrow (ii)$ This direction follows from the proof of Theorem (2.13).

$(ii) \Longrightarrow (i)$ Letting $A_2 = A_4 = B$, $A_1 = A_3 = A$, and $X = I$ in inequality (2.14) leads to

$$2s_j \begin{bmatrix} AB^* & AB^* \\ AB^* & AB^* \end{bmatrix} \leq s_j(|A|^2 + |B|^2 + |A|^2 + |B|^2).$$

This implies that

$$4s_j(AB^*) \leq 2s_j(A^*A + B^*B),$$

which leads to

$$2s_j(AB^*) \leq s_j(A^*A + B^*B)$$

for $j = 1, 2, \ldots$ □

References

[1] W. Audeh, Some generalizations for singular value inequalities of compact operators, Adv. Oper. Theory, **6,** Article no, 14, (2021), 1–10.

[2] W. Audeh, Singular value inequalities and applications, Positivity, (2020).

[3] W. Audeh, Generalizations for singular value and arithmetic-geometric mean inequalities of operators, J. Math. Anal. Appl., **489,** (2020), 1–8.

[4] W. Audeh, Generalizations for singular value inequalities of operators, Adv. Oper. Theory, **5,** (2020), 371–381.

[5] W. Audeh, F. Kittaneh, Singular value inequalities for compact operators, Linear Algebra Appl., **473,** (2012), 2516–2522.

[6] R. Bhatia, Matrix Analysis, GTM169, Springer-Verlag, New York, 1997.

[7] R. Bhatia, F. Kittaneh, On the singular values of a product of operators, SIAM J. Matrix Anal. Appl., **11,** (1990), 272–277.

[8] R. Bhatia, F. Kittaneh, The matrix arithmetic-geometric mean inequality revisited, Linear Algebra Appl., **428,** (2008), 2177–2191.

[9] F. Kittaneh, Norm inequalities for certain operator sums, J. Funct. Anal., **143,** (1997), 337–348.

[10] Y. Tao, More results on singular value inequalities of matrices, Linear Algebra Appl., **416,** (2006), 724–729.

[11] X. Zhan, Singular values of differences of positive semidefinite matrices, SIAM J. Matrix Anal. Appl., **22,** no. 3, (2000), 819–823.