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Numerical Solution of Backward Fuzzy SDEs with Time Delayed Coefficients

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Abstract

In this work, we consider the fuzzy stochastic differential delay equation and study the numerical solution of backward fuzzy stochastic differential delay equations (FSDDEs). Finally, we examine numerical convergence for FSDDEs.

1 Introduction

Malrnowski and Michta [1] established existence and uniqueness of solutions for Stochastic Fuzzy Differential Equations (SFDE) with an application. In addition, they studied the continuous at an initial point and stability properties. The existence and uniqueness of solutions for SFDE were established by Malrnowski [2] who considered the strong uniqueness for strong solutions. Michta [3] presented approaches for SFDE with a semimartingale integrator as well as the existence of fuzzy solution. Narayanamoorthy and Yookesh [4] proposed an approximate solution to fuzzy delay differential equations and compared it with the exact solution for fuzzy delay differential equation. Buckwar [5] discussed the problem of a numerical solution of SDDE of

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Itô's form. Moreover, he studied numerical convergence for stochastic differential equations with time delay coefficients and presented some numerical examples to show numerical convergence by using Euler-Maruyama formula. Delong and Imkeller [6, 7] introduced a class of BSDEs with time delay and discussed the existence and uniqueness of solution BSDEs with time delay. In addition, they established the existence and uniqueness as well as the Malliavin's differentiability of the solution for BSDEs with delayed time. Lukasz Delong [8, 9] presented a significant study for applications of time-delayed backward stochastic differential equations. Moreover, he introduced some examples of Pricing, Hedging and Portfolio management problems which could be established in the framework of BSDDEs. Furthermore, he investigated the solutions of BSDEs with time-delayed generators of a moving average type.

In this work, we introduce the theoretical studies on numerical convergent of backward fuzzy stochastic differential delay equations that has the following

$$\Upsilon(T) = \xi + \int_t^T f(s, \Upsilon(s), \Psi(s), \Upsilon_s, \Psi_s)ds - \int_t^T \Psi(s)dW(s), \tag{1.1}$$

where $W_t, 0 \leq t \leq T$ is a Brownian motion defined on the probability space (Ω, Γ, P), and $T < \infty$ is a finite time horizon. The coefficients f at time s and the terminal condition ξ depend on the past values of a solution $(\Upsilon_s, \Psi_s) = (\Upsilon(s+\alpha), \Psi(s+\alpha))_{-T\leq\alpha\leq 0}$.

2 Preliminaries and Notations

In this section, we present some notations and assumptions used in the sequel. Therefore, we consider standard d-dimensional Brownian motions $W_t, 0 \leq t \leq T$, defined on the complete probability space (Ω, Γ, P), $\{\Gamma_t\}_{0\leq t\leq T}$ denotes the natural filtration with σ-algebra P of Γ_t-progressively measurable subsets of $\Omega \times [0, T]$. Moreover, we consider the Euclidean norm $|\cdot|$ in $\mathbb{R}^q$ and $\mathbb{R}^{q\times d}$, we present the following spaces:

i) Let $L^2_{-T}(\mathbb{R}^{q\times d})$ is the space of measurable function $\Psi : [-T, 0] \to \mathbb{R}^{q\times d}$ such that $\int_{-T}^0 |\Psi(t)|^2 \mathrm{dt} < \infty$.

ii) Let $L^\infty_{-T}(\mathbb{R}^q)$ be the space of measurable function $\Upsilon : [-T, 0] \to \mathbb{R}^q$ such that $\sup_{-T\leq t\leq 0} |\Upsilon(t)|^2$
$< \infty$.

iii) Let $H_T^2(\mathbb{R}^n)$ be the space of Γ-predictable processes $Y : \Omega\times[0,T] \to \mathbb{R}^n$ such that $E\int_0^T |\Upsilon(t)|^2 \mathrm{dt} < \infty$.

iv) Let $S_T^2(\mathbb{R}^q)$ be the space of Γ-adapted, product measurable processes $\Upsilon : \Omega \times [0,T] \to \mathbb{R}^q$ such that $E[\sup_{0\le t\le T} |\Upsilon(t)|^2] < \infty$.

Therefore, the spaces $H_T^2(\mathbb{R}^{q\times d})$ and $S_T^2(\mathbb{R}^q)$ are done with the norm $||\Psi||^2_{H_T^2} = E\int_0^T |\Psi(t)|^2 dt$ and $||\Upsilon||^2_{S_T^2} = E[\sup_{0\le t\le T} |\Upsilon(t)|^2]$, respectively. We consider the following backward stochastic differential equations with time delayed coefficients:

$$\begin{cases} d(\Upsilon(t)) = f(t, \Upsilon(t), \Psi(t), \Upsilon_t, \Psi_t)dt - \Psi(t)dW(t), 0 \le t \le T, \\ \Upsilon_T = \xi(\Upsilon_T, \Psi_T), -T \le t \le 0, \end{cases} \tag{2.2}$$

where f is Borel-measurable function at time set depends on the past values of the solution $\Upsilon_s = (\Upsilon(s+\alpha))_{-T\le\alpha\le0}$ and $\Psi_s = (\Psi(s+\alpha))_{-T\le\alpha\le0}$ with the terminal condition ξ. We always consider the sets $\Psi(t) = 0$ and $\Psi(t) = \Psi(0)$ for $t < 0$. In our work, we need the following assumptions:

(H1): For all $-\lambda \le s < t \le 0$, there exists $K_1 \ge 0$ and such that

$$E[|\xi(t) - \xi(s)|^2] \le K_1(t-s).$$

(H2): Suppose that $f : \Omega\times[0,T]\times\mathbb{R}^q\times\mathbb{R}^{q\times d}\times L^\infty_{-T}(\mathbb{R}^q)\times L^2_{-T}(\mathbb{R}^{q\times d}) \to \mathbb{R}^q$ is product measurable. Then there exist positive constants K_2, K_3 and a finite measure θ on $[-\lambda, 0]$ such that

$$|f(t, \Upsilon^1, \Psi^1, \Upsilon^1_t, \Psi^1_t) - f(t, \Upsilon^2, \Psi^2, \Upsilon^2_t, \Psi^2_t)|^2 \le K_2(|\Upsilon^1 - \Upsilon^2|^2 + |\Psi^1 - \Psi^2|^2)$$

$$+K_3\Big(\int_{-T}^0 |\Upsilon^1(t+\alpha) - \Upsilon^2(t+\alpha)|^2\theta(\mathrm{d}\alpha) + \int_{-T}^0 |\Psi^1(t+\alpha) - \Psi^2(t+\alpha)|^2\theta(\mathrm{d}\alpha)\Big).$$

for all $t \in [0,T]$, $(\Upsilon^1, \Psi^1), (\Upsilon^2, \Psi^2) \in \mathbb{R}^q\times\mathbb{R}^{q\times d}$, $(\Upsilon^1_t, \Psi^1_t), (\Upsilon^2_t, \Psi^2_t) \in L^\infty_{-T}(\mathbb{R}^q)\times L^2_{-T}(\mathbb{R}^{q\times d})$.

(H3)

$$E\int_0^T |f(t, 0, 0, 0, 0)|^2 \mathrm{dt} < \infty.$$

(H4)

$$f(t, \cdot, \cdot, \cdot, \cdot) = 0,$$

for $t < 0$.

3 Fuzzy Solution of BSDDEs

Let $\mu(\mathbb{R}^q)$ be the family of all nonempty, compact and convex subsets of $\mathbb{R}^q$. Now, we denote the fuzzy set space of $\mathbb{R}^q$ as $A(\mathbb{R}^q)$; i.e., the set of functions $v : \mathbb{R}^q \to [0,1]$ such that $[v]^\alpha \in \mu(\mathbb{R}^q)$ for every $\alpha \in [0,1]$, where $[v]^\alpha = \{a \in \mathbb{R}^q : v(a) \geq \alpha\}$ for $\alpha \in [0,1]$ and $[v]^0 = \{a \in R^q : v(a) > 0\}$. Let (Ω, Γ, P) be a complete probability space with a filtration $\{\Gamma_t\}_{t\in[0,T]}, T \in (0,\infty)$, satisfying usual conditions. A mapping $\Upsilon : \Omega \to A(\mathbb{R}^q)$ is said to be a fuzzy random variable, if $[\Upsilon]^\alpha : \Omega \to \mu(\mathbb{R}^q)$ is an Γ-measurable multifunction for all $\alpha \in [0,1]$.

Definition 3.1. *([2]) Let (Ω, Γ, P) be a complete probability space. We can say $y : \Omega \to \Gamma(\mathbb{R}^q)$ is a fuzzy random variable if, for all $\alpha \in [0,1]$, $[\Upsilon]^\alpha : \Omega \to \mu(\mathbb{R}^q)$ is an Γ-measurable.*

Definition 3.2. *A mapping $\Upsilon : [0,1] \times \Omega \to \Gamma(\mathbb{R}^q)$ is said to be a fuzzy stochastic process, if the mapping $\Upsilon(t,.) = \Upsilon(t) : \Omega \to \Gamma(\mathbb{R}^q)$ is a fuzzy random variable.*

Definition 3.3. *A fuzzy stochastic process Υ is d_∞-continuous if the mappings $\Upsilon(.,W) : [0,1] \to \Gamma(\mathbb{R}^q)$ are d_∞-continuous functions.*

We consider the backward stochastic fuzzy differential equations as:

$$d\Upsilon(t) = f(t, \Upsilon(t), \Psi(t))dt - \Psi(t)dW(t)$$
$$\Psi(T) = \Psi_T,$$

where $f : \Omega \times [0,T] \times \mathbb{R}^q \times \mathbb{R}^{q\times d} \times A(\mathbb{R}^q) \to A(R^q)$ and $\Upsilon_T : \Omega \to A(\mathbb{R}^q)$ is a fuzzy random variable. Now, we can consider the following fuzzy BSDDEs

$$d\Upsilon(t) = f(t, \Upsilon(t), \Upsilon(t), \Upsilon_t, \Psi_t)dt - \Psi(t)dW(t), 0 \leq t \leq T,$$
$$\Upsilon(T) = \xi(\Upsilon_T, \Psi_T), -T \leq t \leq 0,$$

where f is Borel-measurable function at time set depend on the past values of the solution $\Upsilon_s = (\Upsilon(s+\alpha))_{-T\leq\alpha\leq 0}$ and $\Psi_s = (\Psi(s+\alpha))_{-T\leq\alpha\leq 0}$. We always set $\Psi(t) = 0$ and $\Upsilon(t) = \Upsilon(0)$ for $t < 0$.

4 Numerical Scheme for BFSDDEs

In this section, we present a numerical scheme is based upon a discretization of (1). For all integers $n, k \geq 1$ and $t \in [0,T]$, let $-\lambda = t_{-k} < t_{-k+1} <$

$\cdots < 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of interval $[-\lambda, T]$, and denote $\delta = \Delta_{j+1} = t_{j+1} - t_j = \frac{T}{n}, 1 \leq j \leq n, \Delta W_{j+1} = W_{j+1} - W_j$, where $j = 0, 1, \cdots, n-1$ and $\Delta_t = \max_{-\lambda \leq j \leq n-1} \Delta t_j$. Therefore, on the small interval $[t_j, t_{j+1}]$ the equation

$$\Upsilon(t_j) = \Upsilon(t_{j+1}) + \int_{t_j}^{t_{j+1}} f(s, \Upsilon(s), \Psi(s), \Upsilon_s, \Psi_s) ds - \int_{t_j}^{t_{j+1}} \Psi(s) dW(s). \quad (4.3)$$

We can approximate with the discrete equation

$$\Upsilon^n(t_j) = \Upsilon^n(t_{j+1}) + f(t_j, \Upsilon^n_j(t), \Psi^n_j(t), \Upsilon^n_j(t+\alpha), \Psi^n_j(t+\alpha))\delta - \Psi^n_j(t)\Delta W_{j+1},$$

with $\Upsilon(T) = \xi(T)$ on $-T \leq t \leq 0$. Now, we consider a class of BFSDDEs as follows:

$$\Upsilon^n_j(t) = \xi(T) + \int_0^T f(s, \Upsilon^n_j(s), \Psi^n_j(s), \Upsilon^n_j(s+\alpha), \Psi^n_j(s+\alpha)) ds - \int_0^T \Psi^n_j(s) dW(s).$$

Therefore, we can define the Euler-Maruyama approximate solution by

$$\hat{\Upsilon}(t) = \xi(T) + \int_0^T f(s, \hat{\Upsilon}(s), \hat{\Psi}(s), \hat{\Upsilon}(s+\alpha), \hat{\Psi}(s+\alpha)) ds - \int_0^T \hat{\Psi}(s) dW(s).$$

5 Main Results

This section is devoted to discuss numerical convergence of BFSDDE.

Lemma 5.1. *Assume (H1) and (H2). Then*

$$E[\sup_{0 \leq t \leq T} |\Upsilon(t)|^2 + \int_0^T |\Psi(t)|^2 dt] \leq CN,$$

where $N = E[|\xi|^2 + \int_0^T |f(s, 0, 0, 0, 0)|^2 ds]$.

Proof. From Itô's formula $|\Upsilon(t)|^2$, we obtain

$$|\Upsilon(t)|^2 + \int_0^T |\Upsilon(s)|^2 ds + \int_0^T |\Psi(s)|^2 ds \leq |\xi|^2 + 2\int_0^T \langle \Upsilon(s), f(s, \Upsilon(s), \Upsilon(s), \Upsilon_s, \Upsilon_s)\rangle ds$$

$$-2\int_0^T \langle \Upsilon(s), \Upsilon(s) dW(s)\rangle,$$

where $t \in [0,T]$. From Young's inequality and (H2), we have

$$2\int_0^T \langle \Upsilon(s), f(s,\Upsilon(s),\Psi(s),\Upsilon_s,\Psi_s)\rangle ds \le b\int_0^T |\Upsilon(s)|^2 ds + \frac{1}{b}\int_0^T |f(s,\Upsilon(s),\Psi(s),\Upsilon_s,\Psi_s)|^2 ds$$

$$\le b\int_0^T |\Upsilon(s)|^2 ds + \frac{3}{b}\int_0^T |f(s,0,0,0,0)|^2$$

$$+\frac{6k_2^2}{b}\int_0^T (|\Upsilon(s)|^2 + |\Psi(s)|^2) ds$$

$$+\frac{3k_4}{b}\int_0^T \int_{-T}^0 (|\Upsilon(s+\alpha)|^2 + |\Psi(s+\alpha)|^2)\theta(d\alpha) ds.$$

By changing the order of integration, we get

$$\int_0^T \int_{-T}^0 |\Upsilon(s+\alpha)|^2 \theta(d\alpha) ds = \int_{-T}^0 \int_0^T |\Upsilon(s+\alpha)|^2 ds\theta(d\alpha)$$

$$= \int_{-T}^0 \int_0^T |\Upsilon(t)|^2 dt\theta(d\alpha) \le \zeta \int_0^T |\Upsilon(t)|^2 dt,$$

and

$$\int_0^T \int_{-T}^0 |\Psi(s+\alpha)|^2 \theta(d\alpha) ds = \int_{-T}^0 \int_0^T |\Psi(s+\alpha)|^2 ds\theta(d\alpha)$$

$$= \int_{-T}^0 \int_0^T |\Psi(t)|^2 dt\theta(d\alpha) \le \zeta \int_0^T |\Psi(t)|^2 dt,$$

where $\zeta = \int_{-T}^0 \theta(d\alpha)$. Therefore, we have

$$|\Upsilon(t)|^2 + \int_0^T |\Upsilon(s)|^2 ds + \int_0^T |\Psi(s)|^2 ds \le |\xi|^2 + b\int_0^T |\Upsilon(s)|^2 ds + \frac{6k_2^2}{b}\int_0^T (|\Upsilon(s)|^2$$

$$+|\Psi(s)|^2) ds + \frac{3}{b}\int_0^T |f(s,0,0,0,0)|^2 ds + \frac{3k_3\zeta}{b}\int_0^T (|\Upsilon(s)|^2 + |\Psi(s)|^2) ds.$$

By taking the expectation and $t = 0$, we have

$$E|\Upsilon(0)|^2 + C_1 E\int_0^T |\Upsilon(s)|^2 ds + C_2 E\int_0^T |\Psi(s)|^2 ds \le E|\xi|^2 + \frac{3}{b}E\int_0^T |f(s,0,0,0,0)|^2 ds,$$

where $C_1 = 1 - b - \frac{6k_2^2}{b} - \frac{3k_3\zeta}{b}$, $C_2 = \epsilon - \frac{6k_2^2}{b} - \frac{3k_3\zeta}{b}$, $\epsilon > 0$. For sufficiently small k_2 and k_3, choosing $\epsilon > 0$ and $b > 0$ such that $C_1 > 0$ and $C_2 > 0$, then there exists a constant $C > 0$ depending on ϵ, b, k_2, k_3 and ζ such that

$$E\int_0^T |\Upsilon(s)|^2 ds + E\int_0^T |\Upsilon(s)|^2 ds \leq C\{E|\xi|^2 + E\int_0^T |f(s,0,0,0,0)|^2 ds\}.$$

Therefore, for b choosing above, we get

$$\sup_{0\leq t\leq T} |\Upsilon(t)|^2 \leq |\xi|^2 + \frac{3}{b}\int_0^T |f(s,0,0,0,0)|^2 ds + 2\sup_{0\leq t\leq T} |\int_0^T \langle \Upsilon(s), \Psi(s)\rangle dW(s)\rangle|.$$

By Young's inequality and Burkholder-Davis-Gundy inequality, together with above inequality and assumption (H2), there exists a constant $a > 0$ such that

$$2E[\sup_{0\leq t\leq T} |\int_0^T \langle \Upsilon(s), \Psi(s) dW(s)\rangle|] \leq a[a_1 E(\sup_{0\leq t\leq T}\int_0^T |\Upsilon(s)|^2 ds) + \frac{1}{a_1}E(\int_0^T |\Upsilon(s)|^2 ds)],$$

where $a_1 > 0$. Now, choosing $a_1 = \frac{1}{3a}$, for sufficiently small $k_2 > 0$ and $k_3 > 0$, there exists a constant $C > 0$ depending on $\epsilon, b, k_2, k_3, \zeta$, and a such that

$$E[\sup_{0\leq t\leq T} |\Upsilon(t)|^2 + \int_0^T |\Psi(s)|^2 ds] \leq CE[|\xi|^2 + \int_0^T |f(s,0,0,0,0)|^2 ds.$$

The proof is complete.

□

Theorem 5.2. *Under hypotheses (H2)-(H4), the approximate solution (4) will converge to the exact solution (1) in the sense that for all* $t \in [0,T]$, *such that*

$$\lim_{m\to\infty} E|\Upsilon(t) - \Upsilon^m(t)|^2 = 0 and \lim_{m\to\infty} E\int_o^T |\Psi(t) - \Psi^m(t)|^2 dt = 0.$$

Proof. Suppose that $\{\Upsilon_i(t), \Psi_i(t)\}$ and $\{\Upsilon_i^m(t), \Psi_i^m(t)\}$ are the solution of equations (1) and (4), respectively. Therefore,

$$d(\Upsilon_i(t) - \Upsilon_i^m(t)) = [f(t, \Upsilon_i(t), \Psi_i(t), \Upsilon_i(t+\alpha), \Psi_i(t+\alpha))$$

$$-f(t, \Upsilon_i^m(t), \Psi_i^m(t), \Upsilon_i^m(t+\alpha), \Psi_i^m(t+\alpha))]dt - [\Psi_i(t) - \Psi_i^m(t)]dW(t).$$

By applying it$\hat{o}$'s formula to $|\Upsilon_i - \Upsilon_i^m|^2$ and taking the expectation, we have

$$E|\Upsilon_i(t)-\Upsilon_i^m(t)|^2 = 2E\int_0^T \langle \Upsilon_i(s)-\Upsilon_i^m(s), f(s,\Upsilon_i(s),\Psi_i(s),\Upsilon_i(s+\alpha),\Psi_i(s+\alpha))$$
$$-f(s,\Upsilon_i^m(s),\Psi_i^m(s),\Upsilon_i^m(s+\alpha),\Psi_i^m(s+\alpha))\rangle ds$$
$$-2\int_0^T \langle \Upsilon_i(s)-\Psi_i^m(s), \Psi_i(s)-\Psi_i^m(s)dW(s)\rangle.$$

From assumption (H2) and applying Young's inequality, we obtain

$$E|\Upsilon_i(t)-\Upsilon_i^m(t)|^2 \leq 4kE\int_0^T |\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds + \frac{1}{4k}\int_0^T k_2|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds$$
$$+\frac{1}{4k}E[\int_0^T k_3\int_{-T}^0 |\Upsilon_i(s+\alpha)-\Upsilon_i^m(s+\alpha)|^2\theta(d\alpha)ds]$$
$$+\frac{1}{4k}E[\int_0^T k_3\int_{-T}^0 |\Psi_i(s+\alpha)-\Psi_i^m(s+\alpha)|^2\theta(d\alpha)ds]$$
$$-\frac{1}{4k}E\int_0^T |\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds - 4kE\int_0^T |\Psi_i(s)-\Psi_i^m(s)|^2 ds.$$

By changing the integration order, we have

$$E|\Upsilon_i(t)-\Upsilon_i^m(t)|^2 \leq 4kE\int_0^T |\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds + \frac{k_2}{4k}E\int_0^T |\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds$$
$$+\frac{k_3}{4k}E[\int_{-T}^0\int_0^T |\Upsilon_i(s+\alpha)-\Upsilon_i^m(s+\alpha)|^2 ds\theta(d\alpha)]$$
$$+\frac{k_3}{4k}E[\int_{-T}^0\int_0^T |\Psi_i(s+\alpha)-\Psi_i^m(s+\alpha)|^2 ds\theta(d\alpha)]$$
$$-\frac{1}{4k}E\int_0^T |\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds - 4kE\int_0^T |\Psi_i(s)-\Psi_i^m(s)|^2 ds.$$

And then,

$$E|\Upsilon_i(t)-\Upsilon_i^m(t)|^2 \leq 4kE\int_0^T |\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds + \frac{k_2}{4k}\int_0^T |\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds$$
$$+\frac{k_3}{4k}E[\int_{-T}^0\int_\alpha^{T+\alpha} |\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds\theta(d\alpha)]$$

$$+\frac{k_3}{4k}E[\int_{-T}^{0}\int_{\alpha}^{T+\alpha}|\Psi_i(s)-\Psi_i^m(s)|^2 ds\theta(d\alpha)]$$
$$-\frac{1}{4k}E\int_0^T|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds-4kE\int_0^T|\Psi_i(s)-\Psi_i^m(s)|^2 ds.$$

Suppose that

$$A=\int_{-T}^{0}\theta(d\alpha),\ \int_{\alpha}^{T+\alpha}|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds\le\int_0^T|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds$$

and

$$\int_{\alpha}^{T+\alpha}|\Psi_i(s)-\Psi_i^m(s)|^2 ds\le\int_0^T|\Psi_i(s)-\Psi_i^m(s)|^2 ds.$$

Therefore,

$$E|\Upsilon_i(t)-\Upsilon_i^m(t)|^2\le 4kE\int_0^T|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds+\frac{k_2}{4k}E\int_0^T|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds$$
$$+\frac{k_3A}{4k}E\int_o^T|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds+\frac{k_3A}{4k}E\int_o^T|\Psi_i(s)-\Psi_i^m(s)|^2 ds$$
$$-\frac{1}{4k}E\int_0^T|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds-4kE\int_0^T|\Psi_i(s)-\Psi_i^m(s)|^2 ds.$$

And then, we obtain

$$E|\Upsilon_i(t)-\Upsilon_i^m(t)|^2\le(4k+\frac{k_2}{4k}+\frac{k_3A}{4k}-\frac{1}{4k})E\int_o^T|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds$$
$$+(\frac{k_3A}{4k}-4k)E\int_0^T|\Psi_i(s)-\Psi_i^m(s)|^2 ds.$$

Choosing $k_4=4k+\frac{k_2}{4k}+\frac{k_3A}{4k}-\frac{1}{4k}$ and $k_5=\frac{k_3A}{4k}$, where $k_4,k_5>0$, we have

$$E|\Upsilon_i(t)-\Upsilon_i^m(t)|^2\le k_4E\int_0^T|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds+(k_5-4k)E\int_0^T|\Psi_i(s)-\Psi_i^m(s)|^2 ds.$$

And then,

$$E|\Upsilon_i(t)-\Upsilon_i^m(t)|^2\le k_4E\int_0^T|\Upsilon_i(s)-\Upsilon_i^m(s)|^2 ds.$$

For all $i=0,\cdots,n$ and $t\in[0,T]$, using Gronwall's inequality, we deduce that

$$\lim_{n\to\infty}E[\max_{i=0,\cdots,n}|\Upsilon_i(t)-\Upsilon_i^m(t)|^2]=0,$$

and consequently $\lim_{n\to\infty}E\int_0^T|\Psi(t)-\Psi^m(t)|^2dt=0$.

□

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