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Solving the singularly perturbation problems of delay differential equations numerically

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Abstract

Singular perturbation problems with delay argument have arisen in practical problems such as in modeling neuronal stochastic behavior, rotation movement of sunflower, respiratory physiology processes, and optimal control theory. This study is important because solving a real problem analytically is tedious, so we propose a numerical method to approximate these problems. In this article, the multistep block method is developed to solve problems concerning second order delay differential equations that involved singular perturbation problems. The process consists of using block technique where three points approximate solutions are computed at the same time. Each of the three points is directly integrated once and twice and approximated by using Lagrange interpolation. Predictor and corrector techniques are applied to obtain the accuracy of the method with the corrector

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having one order higher than the predictor. The delay solutions are obtained by using the finite difference method or using the stored delay solutions. The boundary conditions are transformed to the initial conditions via the shooting technique. The reliability of the method is tested for order, consistency, convergence and zero stability. Additionally, the functionality of the method proposed is shown by solving some numerical problems.

1 Introduction

Delay differential equation (DDE) occurs in problems related to science and engineering, such as biology involving neurons in brain cells. In 1965, Stein [1] proposed a realistic model for neuronal stochastic behavior. Stein's model makes assumptions about random synaptic input, such as that excitatory and inhibitory signals arise at random and are modeled as Poisson processes and that depolarization in the membrane potential increases and decreases by small amounts of voltage. In another case, the input signals can be viewed as a Wiener distribution as presented by Lange and Miura [2] where the distribution is constructed as the second order DDE of the general boundary value problem (BVP); i.e., a small parameter ϵ is multiplied with the second order derivative y'' and this DDE is named as the singular perturbation problem. This physiological problem motivated Lange and Miura to follow an asymptotic approach to solve the specific class of singular perturbation DDE in [2] as defined below:

$$\epsilon y''(x) + a(x)y'(x-\tau) + b(x)y(x) = f(x), \quad x \in (0,1), \quad \epsilon \in (0,1), \quad (1.1)$$

involving the boundary conditions (BCs) of the following:

$$y(x) = \phi(x), \quad x \in [-\tau, 0], \quad y(0) = \phi(0) = \xi, \quad y(1) = \eta, \quad (1.2)$$

where τ is the positive delay term that satisfies $\tau \in (0,1)$ and $\tau = mh$ with m being a positive integer and h being the step size while $\phi(x)$, $f(x)$, $a(x)$, and $b(x)$ are continuous functions. The similar BVP is also significant in variational problems in control theory, in addition to modeling the membrane potential for neurons.

The research then proceeded with Kadalbajoo and Sharma [3] using Taylor series to approximate the delay solution $y'(x-\tau)$ in Eq. (1.1), then solved by using the central and forward difference method to estimate $y''(x)$ and $y'(x)$ respectively. Kadalbajoo and Sharma [4] solved another form of singular perturbation of the DDE problem where the delay in the derivative of DDE does not exist but occurs in its dependent variable $y(x-\tau)$ with ϵ^2. Therefore, as it is not suitable to apply Taylor series to approximate the delay solution, they employed the standard method of finite difference.

The fitted mesh technique was used by Kadalbajoo and Kumar [5] to produce a piecewise uniform mesh and the fitted mesh wass used with the B-spline collocation method. After that, Kumar [6] extended the study in [5] by constructing a geometric mesh instead of a uniform mesh and replacing the B-spline approach with the finite difference method along with the geometric mesh. After approximating the delay solution with the Taylor series, Andargie and Reddy [7] then used a parameter fitted scheme and a three-term recurrence relation was established that can be solved using the Thomas algorithm.

From the literature, this singular perturbed DDE problem was never solved using the multistep method. As a result, this motivated our study in this paper.

2 Derivation of Method

We would like to extend the derivation of the Jaaffar et al. [9] which is the two points block multistep method to the three points block multistep method. The second order of the ordinary differential equation (ODE) could be described as:

$$y'' = f\big(x, y, y'\big). \tag{2.3}$$

We implement the Block approach; i.e., three points approximate solutions are simultaneously calculated in a sequence of block for every iteration along the interval $[0,1]$. The ODE (2.3) was integrated once and twice from both sides at each point in that block which are first point y_{i+1}, second point y_{i+2}, and third point y_{i+3}, with $h = x_{i+1} - x_i$ as the step size.

First point:

$$y'_{i+1} = y'_i + \int_{x_i}^{x_{i+1}} f\big(x, y, y'\big)dx. \tag{2.4}$$

$$y_{i+1} = y_i + hy'_i + \int_{x_i}^{x_{i+1}} (x_{i+1} - x) f\big(x, y, y'\big)dx. \tag{2.5}$$

Second point:

$$y'_{i+2} = y'_i + \int_{x_i}^{x_{i+2}} f\big(x, y, y'\big)dx. \tag{2.6}$$

$$y_{i+2} = y_i + 2hy'_i + \int_{x_i}^{x_{i+2}} (x_{i+2} - x) f\big(x, y, y'\big) dx. \tag{2.7}$$

Third point:

$$y'_{i+3} = y'_i + \int_{x_i}^{x_{i+3}} f\big(x, y, y'\big) dx. \tag{2.8}$$

$$y_{i+3} = y_i + 3hy'_i + \int_{x_i}^{x_{i+3}} (x_{i+3} - x) f\big(x, y, y'\big) dx. \tag{2.9}$$

For the first, second and third points, the functions $f\big(x, y, y'\big)$ are replaced in Eq.(2.4)-(2.9) by Lagrange interpolation polynomials $L_1(x)$, $L_2(x)$, and $L_3(x)$ respectively. Assume that x in Eq.(2.4)-(2.9) is $x = x_{i+3} + sh$ and $dx = hds$. Thus, Eq.(2.4)-(2.9) became

$$\begin{aligned}
y'_{i+1} &= y'_i + h \int_{-3}^{-2} L_1(s) ds, \\
y_{i+1} &= y_i + hy'_i - h^2 \int_{-3}^{-2} (2 + s) L_1(s) ds, \\
y'_{i+2} &= y'_i + \int_{-3}^{-1} L_2(s) ds, \\
y_{i+2} &= y_i + 2hy'_i - h^2 \int_{-3}^{-1} (1 + s) L_2(s) ds, \\
y'_{i+3} &= y'_i + \int_{-3}^{0} L_3(s) ds, \\
y_{i+3} &= y_i + 3hy'_i - h^2 \int_{-3}^{0} (s) L_3(s) ds.
\end{aligned}$$

Solving the integration above with Maple, we eventually get the three point multistep method as follows:

$$
\begin{aligned}
y'_{i+1} &= y'_i + h\left(\frac{251}{720}f_{i+1} + \frac{646}{720}f_i - \frac{264}{720}f_{i-1} + \frac{106}{720}f_{i-2} - \frac{19}{720}f_{i-3}\right),\\
y_{i+1} &= y_i + hy'_i + h^2\left(\frac{135}{1440}f_{i+1} + \frac{752}{1440}f_i - \frac{246}{1440}f_{i-1} + \frac{96}{1440}f_{i-2} - \frac{17}{1440}f_{i-3}\right),\\
y'_{i+2} &= y'_i + h\left(\frac{29}{90}f_{i+2} + \frac{124}{90}f_{i+1} + \frac{24}{90}f_i + \frac{4}{90}f_{i-1} - \frac{1}{90}f_{i-2}\right),\\
y_{i+2} &= y_i + 2hy'_i + h^2\left(\frac{5}{90}f_{i+2} + \frac{104}{90}f_{i+1} + \frac{78}{90}f_i - \frac{8}{90}f_{i-1} + \frac{1}{90}f_{i-2}\right),\\
y'_{i+3} &= y'_i + h\left(\frac{27}{80}f_{i+3} + \frac{102}{80}f_{i+2} + \frac{72}{80}f_{i+1} + \frac{42}{80}f_i - \frac{3}{80}f_{i-1}\right),\\
y_{i+3} &= y_i + 3hy'_i + h^2\left(\frac{15}{160}f_{i+3} + \frac{144}{160}f_{i+2} + \frac{378}{160}f_{i+1} + \frac{192}{160}f_i - 9f_{i-1}\right).
\end{aligned}
\tag{2.10}
$$

To maximize the method's precision, the predictor-corrector strategy is applied. Up to now, we only derive the corrector formula for the three points multistep method. The derivation of the predictor formula is in the same procedure as the corrector, but with one order less than the corrector.

3 Method Analysis

3.1 Method Order

To check the order of method, the corrector formula (2.10) is written in a matrix form of the linear multistep method (LMM) for second order as follows:

$$\alpha Y_i = h\beta Y'_i + h^2\gamma F_i, \tag{3.11}$$

hence,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{i-3} \\ y_{i-2} \\ y_{i-1} \\ y_i \\ y_{i+1} \\ y_{i+2} \\ y_{i+3} \end{bmatrix} = h \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y'_{i-3} \\ y'_{i-2} \\ y'_{i-1} \\ y'_i \\ y'_{i+1} \\ y'_{i+2} \\ y'_{i+3} \end{bmatrix}$$

$$+ h^2 \begin{bmatrix} -\frac{19}{720} & \frac{106}{720} & -\frac{264}{720} & \frac{646}{720} & \frac{251}{720} & 0 & 0 \\ -\frac{17}{1440} & \frac{96}{1440} & -\frac{246}{1440} & \frac{752}{1440} & \frac{135}{1440} & 0 & 0 \\ 0 & -\frac{1}{90} & \frac{4}{90} & \frac{24}{90} & \frac{124}{90} & \frac{29}{90} & 0 \\ 0 & \frac{1}{90} & -\frac{8}{90} & \frac{78}{90} & \frac{104}{90} & \frac{5}{90} & 0 \\ 0 & 0 & -\frac{3}{80} & \frac{42}{80} & \frac{72}{80} & \frac{102}{80} & \frac{27}{80} \\ 0 & 0 & -\frac{9}{160} & \frac{192}{160} & \frac{378}{160} & \frac{144}{160} & \frac{15}{160} \end{bmatrix} \begin{bmatrix} f_{i-3} \\ f_{i-2} \\ f_{i-1} \\ f_i \\ f_{i+1} \\ f_{i+2} \\ f_{i+3} \end{bmatrix}. \tag{3.12}$$

The matrix above is then written in linear difference operator and its derivatives are expanded with Taylor series. The coefficient matrix C_r is obtained as follows:

$$
\begin{aligned}
C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + ... + \alpha_s,\\
C_1 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + s\alpha_s - (\beta_0 + \beta_1 + \beta_2 + ... + \beta_s),\\
C_2 &= \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 + ... + s^2\alpha_s) - (\beta_1 + 2\beta_2 + 3\beta_3 + ... + s\beta_s)\\
&\quad - (\gamma_0 + \gamma_1 + \gamma_2 + ... + \gamma_s),\\
C_3 &= \frac{1}{3!}(\alpha_1 + 2^3\alpha_2 + 3^3\alpha_3 + ... + s^3\alpha_s) - \frac{1}{2!}(\beta_1 + 2^2\beta_2 + 3^2\beta_3 + ... + s^2\beta_s)\\
&\quad - (\gamma_1 + 2\gamma_2 + 3\gamma_3 + ... + s\gamma_s),\\
&\vdots\\
C_r &= \frac{1}{r!}(\alpha_1 + 2^r\alpha_2 + 3^r\alpha_3 + ... + s^r\alpha_s) - \frac{1}{(r-1)!}(\beta_1 + 2^{r-1}\beta_2 + 3^{r-1}\beta_3 + ... + s^{r-1}\beta_s)\\
&\quad - \frac{1}{(r-2)!}(\gamma_1 + 2^{r-2}\gamma_2 + 3^{r-2}\gamma_3 + ... + s^{r-2}\gamma_s),\\
&\quad r = 4, 5, 6, ...
\end{aligned}
$$

where $s = 6$. Finally, C_r is given as below:

$$C_0 = C_1 = \ldots = C_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

with non zeros coefficient matrix, C_7 is the error constant of the method as follows:

$$C_7 = \begin{bmatrix} -\frac{3}{160} & -\frac{41}{5040} & -\frac{1}{90} & \frac{1}{315} & -\frac{3}{160} & -\frac{9}{560} \end{bmatrix}^T.$$

Lambert [10] specifies that the order of the system is n when $C_0 = C_1 = C_2 = \ldots = C_{n+1} = 0$ and $C_{n+2} \neq 0$. Subsequently, the proposed method is order 5 and we would like to call it as the three points block multistep method order 5 (3PBM5).

3.2 Consistency of Method

Assume that Z_i, Z_i', and Z_i'' below represent the matrices of theoretical solutions for ODE (2.3),

$$Z_i = \begin{bmatrix} y(x_{i-3}) \\ y(x_{i-2}) \\ y(x_{i-1}) \\ y(x_i) \\ y(x_{i+1}) \\ y(x_{i+2}) \\ y(x_{i+3}) \end{bmatrix}, Z_i' = \begin{bmatrix} y'(x_{i-3}) \\ y'(x_{i-2}) \\ y'(x_{i-1}) \\ y'(x_i) \\ y'(x_{i+1}) \\ y'(x_{i+2}) \\ y'(x_{i+3}) \end{bmatrix}, \text{ and } Z_i'' = \begin{bmatrix} f\big(x_{i-3}, y(x_{i-3}), y'(x_{i-3})\big) \\ f\big(x_{i-2}, y(x_{i-2}), y'(x_{i-2})\big) \\ f\big(x_{i-1}, y(x_{i-1}), y'(x_{i-1})\big) \\ f\big(x_i, y(x_i), y'(x_i)\big) \\ f\big(x_{i+1}, y(x_{i+1}), y'(x_{i+1})\big) \\ f\big(x_{i+2}, y(x_{i+2}), y'(x_{i+2})\big) \\ f\big(x_{i+3}, y(x_{i+3}), y'(x_{i+3})\big) \end{bmatrix}$$

According to Fatunla [11], the local truncation error (LTE) of the LMM (3.11) is introduced as

$$E_i = \alpha Z_i - h\beta Z_i' - h^2 \gamma Z_i'',$$
$$||E_i|| = C_{n+2} h^{n+2} + O(h^{n+3}).$$

where $||.||$ is the maximum norm. The maximum norm of LTE for the 3PBM5 is

$$||E_i|| = h^7 \begin{bmatrix} -\frac{3}{160} & -\frac{41}{5040} & -\frac{1}{90} & \frac{1}{315} & -\frac{3}{160} & -\frac{9}{560} \end{bmatrix}^T.$$

The proposed method, 3PBM5 is consistent when the step size h tends to zero then, $||E_i||$ is also tends to zero. Thus, the 3PBM5 is consistent.

3.3 Zero Stability of Method

Zero stability is required to verify the method's stability for the step size $h \to 0$. If the first characteristic polynomial $\rho(T)$ introduced as follows:

$$\rho(T) = \det\big[A_0 T - A_1\big] = 0$$

have the roots T_j that satisfy $|T_j| \leq 1$ and the roots have multiplicity not exceeding 2 for $|T_j| = 1$, then the 3PBM5 is considered as zero stable.

$$\rho(T) = \det \begin{bmatrix} T & 0 & 0 & 0 & -1 & 0 \\ 0 & T & 0 & 0 & 0 & -1 \\ 0 & 0 & T & 0 & -1 & 0 \\ 0 & 0 & 0 & T & 0 & -1 \\ 0 & 0 & 0 & 0 & T-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & T-1 \end{bmatrix} = 0,$$

$$T^4 (T-1)^2 = 0, \quad T = 0, 0, 0, 0, 1, 1,$$

where $A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Subsequently, 3PBM5 is zero stable.

3.4 Convergence of Method

The Dahlquist convergence theorem states that if LMM is zero stable and consistent, then it is a convergent method. Since the proposed method has attained consistency and zero stability, thus 3PBM5 converges to the accurate solution.

4 Implementation of Method

The 3PBM5 is not a self starting method, thus starting methods are required. The starting methods implemented in this study are the predictor formula of direct Euler's method order 1, then the corrector formula of direct modified Euler's method order 2, and finally the corrector formula of direct Adams Moulton method order 3.

The initial value problem (IVP) is established from the boundary value problem (BVP) by applying partial differentiation and consequently producing two IVPs to be solved simultaneously. The first IVP is formulated as follows:

$$y''(x,t) = f\big(x, y(x,t), y'(x,t)\big),$$

concerning the initial conditions (ICs):

$$y(a,t) = \alpha,\ y'(a,t) = t_1,$$

where t_1 is the first guessing for the missing initial value. The second IVP is

$$q''(x,t) = \frac{\delta f}{\delta y} q(x,t) + \frac{\delta f}{\delta y'} q'(x,t),$$

where $q(x,t) = \dfrac{\delta y}{\delta t}(x,t)$ with ICs:

$$q(a,t) = 0, q'(a,t) = 1.$$

Then, the Newton's method is used to guess the next missing initial value and this process is known as the shooting technique.

For the delay part, the delay solution $y'(x-\tau)$ in Eq.(1.1) is approximated with finite difference method if $(x-\tau) \in [-\tau, 0]$. The delay solution $y'(x-\tau)$ is taken from the previously calculated $y'(x-\tau)$ that has been stored if $(x-\tau) \in [0, 1]$.

5 Numerical Results

Two numerical problems are tested to observe the performance of the 3PBM5. Problem 1 has the exact solution while Problem 2 has no exact solution. Both problems are taken from Kadalbajoo and Sharma [3].

Problem 1:

$$\epsilon y''(x) + y'(x-\tau) - y(x) = 0, \quad x \in (0, 1),$$

BCs: $y(0) = 1, \quad x \in [-\tau, 0] \quad y(1) = 1.$

Theoretical solution:

$$y(x) = \frac{(1-e^{m_2})e^{m_1 x} + (e^{m_1} - 1)e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

$$m_1 = \frac{-1 - \sqrt{1 + 4(\epsilon - \tau)}}{2(\epsilon - \tau)}, \quad m_2 = \frac{-1 + \sqrt{1 + 4(\epsilon - \tau)}}{2(\epsilon - \tau)}.$$

Problem 2:

$$\epsilon y''(x) + e^{-0.5x} y'(x-\tau) - y(x) = 0, \quad x \in (0, 1),$$

BCs: $y(0) = 1, \quad x \in [-\tau, 0] \quad y(1) = 1.$

The following notations are used for Tables 1–4:

h	:	Step size of method.
MAXE	:	Maximum absolute errors.
AVE	:	Average absolute errors.
FCN	:	Total number of function calls in the last guessing iteration.
TS	:	Total number of steps in the last guessing iteration.
t_{last}	:	The guessing value in last guessing iteration.
Error x_{last}	:	Absolute error for x_N at the last guessing iteration.
3PBM5	:	Three Points Block Multistep Method of order five in this paper.
Hoo5	:	Two Points Block Multistep Method of order five in Hoo and Majid [12].
CR	:	Previous methods (Taylor series, exponentially integratng factor, trapezoidal rule and Thomas algorithm) used in Challa and Reddy [8].
KS	:	Previous methods (Taylor series, finite difference schemes, and discrete invariant imbedding algorithm) in Kadalbajoo and Sharma [3].

Table 1: Numerical simulation for solving Problem 1 at $h = 0.01$ when $\epsilon = 0.1$ at $\tau = 0.01$.

Methods	Hoo5	3PBM5	CR	KS
MAXE	2.9249E-03	4.0169E-03	1.1721E-02	1.1824E-02
AVE	2.2661E-04	3.8072E-04	-	-
FCN	301	211	-	-
TS	52	36	-	-
Error x_{last}	4.4409E-16	1.3323E-15	-	-
t_{last}	-6.337424	-6.244217	-	-

Table 2: Numerical simulation for solving Problem 1 at $h = 0.001$ when $\epsilon = 0.1$ at $\tau = 0.01$.

Methods	Hoo5	3PBM5	CR	KS
MAXE	3.1728E-03	3.1184E-03	1.2250E-03	1.2290E-03
AVE	2.2533E-04	2.2370E-04	-	-
FCN	3001	2011	-	-
TS	502	336	-	-
Error x_{last}	2.8866E-15	7.7716E-16	-	-
t_{last}	-6.198468	-6.20529	-	-

Table 3: Approximate solutions for solving Problem 2 at $h = 0.005$ when $\epsilon = 0.1$ for $\tau = 0.00$ and $\tau = 0.01$.

x	$\tau = 0.00$		$\tau = 0.01$	
	3PBM5	Hoo5	3PBM5	Hoo5
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.606387	0.644792	0.594115	0.632400
0.2	0.494656	0.508963	0.485178	0.498822
0.3	0.482622	0.488168	0.476528	0.481609
0.4	0.510366	0.512596	0.506370	0.508341
0.5	0.557894	0.558821	0.555067	0.555859
0.6	0.618936	0.619331	0.616799	0.617126
0.7	0.692271	0.692441	0.690629	0.690765
0.8	0.778778	0.778850	0.777597	0.777653
0.9	0.880430	0.880457	0.879775	0.879794
1.0	0.999994	1.000000	0.999998	1.000000
FCN	409	601	409	601
TS	69	102	69	102
t_{last}	-6.687564	-6.564526	-6.743346	-6.915535

Table 4: Approximate solutions for solving Problem 2 at $h = 0.005$ when $\epsilon = 0.1$ for $\tau = 0.03$ and $\tau = 0.05$.

x	$\tau = 0.03$		$\tau = 0.05$	
	3PBM5	Hoo5	3PBM5	Hoo5
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.567213	0.606826	0.536276	0.577840
0.2	0.465123	0.477708	0.443221	0.454246
0.3	0.464240	0.468474	0.451945	0.455121
0.4	0.498551	0.500051	0.491105	0.492090
0.5	0.549578	0.550135	0.544386	0.544712
0.6	0.612624	0.612838	0.608614	0.608728
0.7	0.687389	0.687473	0.684219	0.684260
0.8	0.775246	0.775278	0.772912	0.772927
0.9	0.878458	0.878468	0.877138	0.877142
1.0	1.000000	1.000000	1.000000	1.000000
FCN	409	601	409	601
TS	69	102	69	102
t_{last}	-6.681274	-6.574174	-6.681318	-6.585132

6 Discussions

The accuracy of 3PBM5 for Problem 1 based on MAXE is better than CR and KS but is compatible with Hoo5 for $h = 0.01$ in Table 1. The accuracy of 3PBM5 is in close agreement with previous methods and multistep method Hoo5 for $h = 0.001$ in Table 2. Tables 3–4 also portray that the approximate solutions of 3PBM5 are compatible with Hoo5 for Problem 2. From Tables 1–4, 3PBM5 has less total function calls (FCN) than Hoo5 since the derivation of 3PBM5 is in a diagonally implicit approach, whereas Hoo5 is fully implicit, so more function calls are required. Moreover, the lesser FCN also because of 3PBM5 computed three points approximate solutions simultaneously which is more than the two points approximate solutions computed by Hoo5. Furthermore, the three points block approach also gives an advantage for 3PBM5 where our proposed method has lesser total iteration steps (TS) than Hoo5.

7 Conclusions

We have compared the proposed method, 3PBM5 with the two points multistep method, Hoo5 based on the method's precision, total function calls and total iteration steps to check the reliability of the block multistep methods. We also compared the accuracy of 3PBM5 with the previous methods that were not multistep methods to observe the capability of multistep method itself. Based on the results, 3PBM5 as a multistep method solved the singular perturbation problems involving DDEs with boundary conditions.

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