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The Relation between τ_n-Tilting Modules and n-term Silting Complexes

Intan Muchtadi-Alamsyah[1], **Yann Palu**[2]

[1]Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung
Bandung, Indonesia

[2]LAMFA
Université de Picardie Jules Verne
Amiens, France

email: ntan@math.itb.ac.id, yann.palu@u-picardie.fr

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Abstract

In this paper, we define support τ_n-tilting modules over a finite dimensional k-algebra A. We establish a bijection between support τ_n-tilting modules and n-term silting complexes in the bounded homotopy category of finitely generated projective A-modules, generalizing the result of Adachi, Iyama and Reiten for support τ-tilting modules and two-term silting complexes.

1 Introduction

In representation theory, one may use tilting modules and associated tilting functors to compare the module categories of two algebras A and B. The reflection functors, introduced by Bernstein, Gelfand and Ponomarev [7] in 1973 motivated tilting theory. Then Auslander, Platzeck and Reiten [6] in 1979 reformulated them and as generalisations, tilting functors were studied

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by Brenner and Butler [8]. As further generalisations, in 1982, Happel and Ringel [13] defined tilted algebras and tilting modules.

Tilting theory is one sort of generalisation of Morita equivalence. Indeed, generalised tilting modules also yield derived equivalences. In 1989 Rickard [16] proved that given two finite-dimensional algebras A and B, the bounded derived categories of A and B are equivalent if and only if there exists a tilting complex T over B such that A is the endomorphism algebra of T. Tilting complexes are generalisations of generalised tilting modules [10]. In [14] Muchtadi-Alamsyah studied the endomorphism ring of n-term tilting complexes as generalisation of 2-term tilting complexes.

Silting complexes are generalisations of tilting complexes [9, 15]. In 2014 τ-tilting theory was introduced by Adachi, Iyama and Reiten [1], thus giving a completion of tilting theory from the point-of-view of mutation, arising from the theory of cluster algebras [11, 12]. Support tilting modules (tilting modules) are generalised to support τ-tilting modules (τ-tilting modules) [2, 5, 17, 18]. Moreover, Adachi, Iyama and Reiten prove the following:

Theorem (Adachi–Iyama–Reiten [1]): *Let A be a finite-dimensional k-algebra. Then there is an explicit bijection between support τ-tilting modules and two-term silting complexes in the bounded homotopy category of finitely generated projective A-modules $K^b(\operatorname{proj} A)$.*

A natural question arises:

Question: *Can one generalise this result to n-term silting complexes?*

In this note, we propose a definition of support τ_n-tilting modules and of n-term silting complexes, and we prove an explicit bijection between support τ_n-tilting A-modules and n-term silting complexes in $K^b(\operatorname{proj} A)$.

2 Definitions and Notations

For a finite dimensional basic algebra A over an algebraically closed field k, the category of finitely generated left A-modules is denoted by $\operatorname{mod} A$ and the category of finitely generated projective left A-modules is denoted by $\operatorname{proj} A$. If $M \in \operatorname{mod} A$, $\operatorname{add} M$ is the category of all direct summands of finite direct sums of copies of M, and the number of non isomorphic indecomposable direct summands of M is denoted by $d(M)$.

If $M \in \operatorname{mod} A$ then the right A-module $\operatorname{Hom}_A(M, A)$ is denoted by M^*, whereas the dual, the right A-module $\operatorname{Hom}_k(M, k)$ is denoted by $D(M)$. If $X_1 \to X_0 \to M \to 0$ is a minimal projective presentation of M, the transpose

$Tr(M)$ of M is the cokernel of $X_0^* \to X_1^*$. The Auslander-Reiten translation is defined as $\tau = DTr$. The syzygy of M, is the kernel of $X_0 \to M$, and is denoted by $\Omega(M)$. We denote by $\tau_n = \tau\Omega^{n-2}$.

Remark 2.1. *Note that $\tau = \tau_2$ in our convention. Our notation slightly differs from the one that is currently used in higher homological algebra: what we call τ_n here is more commonly denoted τ_{n-1}. We chose to shift the usual notation so that τ_n-tilting modules correspond to n-term complexes.*

Definition 2.2. *Let $M \in \operatorname{mod} A$.*

1. *If $\operatorname{Hom}(M, \tau_n M) = 0$ and $Ext^i(M, M) = 0$ for all $i \in \{1, \cdots, n-1\}$, then we call M a* **strongly τ_n-rigid** *module.*
2. *If M is a strongly τ_n-rigid module and $d(M) = d(A)$, then we call M a* **τ_n-tilting** *module.*
3. *If M is a τ_n-tiling $(A/\langle e\rangle)$-module, where e is an idempotent of A, then we call M a* **support τ_n-tilting** *module.*

Let **sτ_n-tilting**A be the set of isomorphism classes of basic support τ_n-tilting modules.

Definition 2.3. *Let $M \in \operatorname{mod} A$ and $X \in \operatorname{proj} A$.*

1. *If M is a strongly τ_n-rigid module and $\operatorname{Hom}_A(X, M) = 0$, then we call (M, X) a* **strongly τ_n-rigid pair**.
2. *If (M, X) is a strongly τ_n-rigid pair and $d(M) + d(X) = d(A)$, then we call (M, X) a* **support τ_n-tilting pair**.

Proposition 2.4. *Let M be in $\operatorname{mod} A$ with $X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_0 \xrightarrow{d_0} M \to 0$ as its minimal projecive presentation. Let $N \in \operatorname{mod} A$, with $Ext^i(M, N) = 0$ for all $i = 1, ..., n-1$.*

1. *The following sequence*

$$0 \to \operatorname{Hom}_A(N, \tau\Omega^{n-2}M) \to D\operatorname{Hom}_A(X_{n-1}, N)$$
$$\xrightarrow{D\operatorname{Hom}(d_{n-1},N)} D\operatorname{Hom}_A(X_{n-2}, N) \xrightarrow{D\operatorname{Hom}(d_{n-2},N)} \cdots \xrightarrow{D\operatorname{Hom}(d_2,N)}$$
$$D\operatorname{Hom}_A(X_1, N) \xrightarrow{D\operatorname{Hom}(d_1,N)} D\operatorname{Hom}_A(X_0, N) \xrightarrow{D\operatorname{Hom}(d_0,N)} D\operatorname{Hom}_A(M, N) \to 0$$

is exact.

2. $\mathrm{Hom}_A(N, \tau\Omega^{n-2}M) = 0$ *if and only if the sequence*

$$\mathrm{Hom}_A(X_0, N) \overset{\mathrm{Hom}(d_1,N)}{\longrightarrow} \mathrm{Hom}_A(X_1, N) \overset{\mathrm{Hom}(d_2,N)}{\longrightarrow} \cdots \overset{\mathrm{Hom}(d_{n-2},N)}{\longrightarrow}$$

$$\mathrm{Hom}_A(X_{n-2}, N) \overset{\mathrm{Hom}(d_{n-1},N)}{\longrightarrow} \mathrm{Hom}_A(X_{n-1}, N) \to 0$$

is exact.

Proof. 1. The following sequence

$$0 \to \tau\Omega^{n-2}M \to \nu X_{n-1} \to \nu X_{n-2} \cdots \to \nu X_0$$

is exact. By applying $\mathrm{Hom}(N, -)$, the following diagram

$$\begin{array}{llll} 0 \to \mathrm{Hom}_A(N, \tau\Omega^{n-2}M) \to & \mathrm{Hom}_A(N, \nu X_{n-1}) \to \cdots \to & \mathrm{Hom}_A(N, \nu X_0) & \\ & \quad\quad \downarrow\cong & \quad \downarrow\cong & \\ & D\,\mathrm{Hom}_A(X_{n-1}, N) \to \cdots \to & D\,\mathrm{Hom}_A(X_0, N) & \to D\,\mathrm{Hom}_A(M, N) \to 0 \end{array}$$

is a diagram of exact sequences.

2. It is clear from (1).

□

We denote by $K^b(\mathrm{proj}\, A)$ the homotopy category of bounded complexes of finitely generated projective A-modules and for $\mathbf{X} \in K^b(\mathrm{proj}\, A)$, we denote by thick $\mathbf{X}$, the smallest full subcategory of $K^b(\mathrm{proj}\, A)$ that contains $\mathbf{X}$ and is closed under (positive and negative) shifts, cones, isomorphisms, and direct summands.

Definition 2.5. *Assume* $\mathbf{X} \in K^b(\mathrm{proj}\, A)$.

1. *If for any* $i > 0$, $\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(\mathbf{X}, \mathbf{X}[i]) = 0$, *we call* $\mathbf{X}$ *a* **presilting** *complex.*

2. *If* $\mathbf{X}$ *is a presilting complex and also* thick $\mathbf{X} = K^b(\mathrm{proj}\, A)$, *we call* $\mathbf{X}$ *a* **silting** *complex.*

The set of isomorphism classes of basic silting complexes is denoted by silting A.

Proposition 2.6. $d(\mathbf{X}) = d(A)$ *for any* $\mathbf{X} \in$ silting A.

Proof See [3, Theorem 2.27, Corollary 2.28]

Definition 2.7. $\mathbf{X} = (X_i, d_i)$ *in* $K^b(\text{proj}\, A)$ *is an* n-**term complex** *if* $X_i = 0$ *for all* $i \notin \{0, \cdots, -(n-1)\}$ *and* $H^i\mathbf{X} = 0$ *for all* $i \notin \{0, -(n-1)\}$.

Let n-**presilting**A (respectively, n-**silting**A) be the set of isomorphism classes of basic n-term presilting (respectively, silting) complexes.

Definition 2.8. *Let* $\mathbf{X}, \mathbf{Y}$ *be objects of* $K^b(\text{proj}\, A)$. *We denote by* $\mathbf{X} \geq \mathbf{Y}$ *if for any positive integer* $i > 0$,$\text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{X}, \mathbf{Y}[i]) = 0$.

Proposition 2.9. *Let* A *be a finite dimensional* k*-algebra,* $\mathbf{X} \in K^b(\text{proj}\, A)$, *and* $m > 0$. *Then length*$(\mathbf{X}) \leq m$ *if and only if* $A[s] \geq \mathbf{X} \geq A[m+s-1]$ *for some integer* s.

Proof. See [4, Prop 2.9] □

Proposition 2.10. *If* $\mathbf{X}$ *is a silting complex in* $K^b(\text{proj}\, A)$ *and* $\mathbf{Y} \in K^b(\text{proj}\, A)$ *with* $\mathbf{X} \geq \mathbf{Y} = \mathbf{Y_0}$ *then there exist triangles*

$$\mathbf{Y_1} \xrightarrow{g_1} \mathbf{X_0} \xrightarrow{f_0} \mathbf{Y_0} \to \mathbf{Y_1}[1],$$
$$\vdots$$
$$\mathbf{Y_m} \xrightarrow{g_m} \mathbf{X_{m-1}} \xrightarrow{f_{m-1}} \mathbf{Y_{m-1}} \to \mathbf{T_m}[1],$$
$$0 \xrightarrow{g_{l+1}} \mathbf{X_l} \xrightarrow{f_l} \mathbf{Y_l} \to 0$$

for some $m \geq 0$ *and* f_i *is a minimal right* add $\mathbf{X}$*-approximation.*

Proof. See [3, Prop 2.23] □

3 Main Results

Let A be a finite dimensional k-algebra.

Theorem 3.1. *There exists a bijection between*

$$n\text{-}\,\text{silting}\, A \leftrightarrow s\tau_n\text{-}\,\text{tilting}\, A$$

given by $\mathbf{X} \in n$-silting A *maps to* $H^0(\mathbf{X}) \in s\tau$-tilting A *and* $(M, Y) \in s\tau -$ tilting A *maps to* $(X_{n-1} \oplus Y \xrightarrow{(d_{n-1} 0)} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_0) \in n$-silting A *where* $X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_0$ *is a minimal projective presentation of* M.

Proposition 3.2. *Any object in* $n-\text{presilting}\, A$ *is a direct summand of an object in* $n-\text{silting}\, A$.

Proof. Let $\mathbf{X} \in n\text{-presilting}\, A$. By Proposition 2.9, we have $A \geq \mathbf{X} \geq A[n-1]$. Since $\mathbf{X} \geq A[n-1]$, by Proposition 2.10, we have triangles

$$\begin{gathered}\mathbf{V_1} \xrightarrow{g_1} \mathbf{U_0} \xrightarrow{f_0} A[n-1] \to \mathbf{V_1}[1] \\ \mathbf{V_2} \xrightarrow{g_2} \mathbf{U_1} \xrightarrow{f_1} \mathbf{V_1} \to \mathbf{V_2}[1] \\ \vdots \\ \mathbf{V_{n-1}} \xrightarrow{g_{n-1}} \mathbf{U_{n-2}} \xrightarrow{f_{n-2}} \mathbf{V_{n-2}} \to \mathbf{V_{n-1}}[1]\end{gathered}$$

with f_i minimal right add $\mathbf{P}$-approximation. Put $\mathbf{W} = \mathbf{X} \oplus \mathbf{V_{n-1}}$. Since A is a silting object, then $K^b(\text{proj}\, A) = \text{thick}\, \mathbf{W}$.

1. We prove that for any $i > 0$, $\text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{X}, \mathbf{V_{n-1}}[i]) = 0$. As there exists an exact sequence

$$\text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{X}, \mathbf{V_{n-2}}[i-1]) \xrightarrow{0} \text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{X}, \mathbf{V_{n-1}}[i])$$
$$\to \text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{X}, \mathbf{U_{n-2}}[i]) = 0.$$

   We observe that for any $i > 0$, $\text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{X}, \mathbf{V_{n-1}}[i]) = 0$.

2. We will prove that for any $i > 0$, $\text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{V_{n-1}}, \mathbf{W}[i]) = 0$. As the following sequences

$$0 \overset{(1)}{=} \text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{U_{n-2}}, \mathbf{W}[i]) \to \text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{V_{n-1}}, \mathbf{W}[i]) \to$$
$$\text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{V_{n-2}}[-1], \mathbf{W}[i]) \to \text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{U_{n-2}}[-1], \mathbf{W}[i]) \overset{(1)}{=} 0,$$

   are exact, therefore $\text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{V_{n-1}}, \mathbf{W}[i]) \cong \text{Hom}_{K^b(\text{proj}\, A)}(\mathbf{V_{n-2}}[-1], \mathbf{W}[i])$.
   There are exact sequences :

$$0 = \text{Hom}(\mathbf{U_{n-3}}[-1], \mathbf{W}[i]) \to \text{Hom}(\mathbf{V_{n-2}}[-1], \mathbf{W}[i]) \to$$
$$\text{Hom}(\mathbf{V_{n-3}}[-2], \mathbf{W}[i]) \to \text{Hom}(\mathbf{U_{n-3}}[-2], \mathbf{W}[i]) = 0,$$
$$0 = \text{Hom}(\mathbf{U_{n-4}}[-2], \mathbf{W}[i]) \to \text{Hom}(\mathbf{V_{n-3}}[-2], \mathbf{W}[i]) \to$$
$$\text{Hom}(\mathbf{V_{n-4}}[-3], \mathbf{W}[i]) \to \text{Hom}(\mathbf{U_{n-4}}[-3], \mathbf{W}[i]) = 0,$$
$$\vdots$$
$$0 = \text{Hom}(\mathbf{U_0}[2-n], \mathbf{W}[i]) \to \text{Hom}(\mathbf{V_1}[2-n], \mathbf{W}[i]) \to \text{Hom}(A, \mathbf{W}[i]) \to \text{Hom}(\mathbf{U_0}[1-n], \mathbf{W}[i]) = 0.$$

Since $A \geq \mathbf{X}$, we have $\mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(A, \mathbf{X}[i]) = 0$ and there are exact sequences

$$0 = \mathrm{Hom}(A, \mathbf{U_{n-2}}[i]) \to \mathrm{Hom}(A, \mathbf{V_{n-2}}[i]) \to \mathrm{Hom}(A, \mathbf{V_{n-1}}[i{+}1]) \to \mathrm{Hom}(A, \mathbf{U_{n-2}}[i{+}1]) = 0,$$

$$0 = \mathrm{Hom}(A, \mathbf{U_{n-3}}[i]) \to \mathrm{Hom}(A, \mathbf{V_{n-3}}[i]) \to \mathrm{Hom}(A, \mathbf{V_{n-2}}[i{+}1]) \to \mathrm{Hom}(A, \mathbf{U_{n-3}}[i{+}1]) = 0,$$

$$\vdots$$

$$0 = \mathrm{Hom}(A, \mathbf{U_0}[i]) \to \mathrm{Hom}(A, A[n{-}1{+}i]) \to \mathrm{Hom}(A, \mathbf{V_1}[i{+}1]) \to \mathrm{Hom}(A, \mathbf{U_0}[i{+}1]) = 0.$$

Finally, we get

$$\begin{array}{rcl} \mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(\mathbf{V_{n-1}}, \mathbf{W}[i]) & \cong & \mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(\mathbf{V_{n-2}}[-1], \mathbf{W}[i]) \\ & \cong & \mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(A, \mathbf{W}[i]) \\ & = & \mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(A, \mathbf{X}[i]) \oplus \mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(A, \mathbf{V_{n-2}}[i]) \\ & \cong & 0 \oplus \mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(A, A[n-1+i]) = 0. \end{array} .$$

By (1) and (2), we see that for any $i > 0$, $\mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(\mathbf{W}, \mathbf{W}[i]) = 0$.

□

Proposition 3.3. *If* $\mathbf{X} \in n-\mathrm{presilting}\,A$ *then* $\mathbf{X} \in n-\mathrm{silting}\,A$ *if and only if* $d(\mathbf{X}) = d(A)$.

Proof. According to Proposition 2.6, $d(\mathbf{X}) = d(A)$. We will show the other direction. Let $\mathbf{X} \in n-\mathrm{presilting}\,A$ with $d(\mathbf{X}) = d(A)$. By Proposition 3.2, $\mathbf{X} \oplus \mathbf{Y}$ is silting for some complex $\mathbf{Y}$. Therefore by Proposition 2.6, $d(\mathbf{X} \oplus \mathbf{Y}) = d(A) = d(\mathbf{X})$. This implies $\mathbf{Y}$ is in $\mathrm{add}\,\mathbf{X}$ and $\mathbf{X}$ is silting.

□

Lemma 3.4. *Let* M, N *in* $\mathrm{mod}\,A$. *Let* $X_{n-1} \xrightarrow{d^X_{n-1}} X_{n-2} \xrightarrow{d^X_{n-2}} \cdots \xrightarrow{d^X_1} X_0 \xrightarrow{d^X_0} M \to 0$ *be a minimal projective presentation of* M *and* $Y_{n-1} \xrightarrow{d^Y_{n-1}} Y_{n-2} \xrightarrow{d^Y_{n-2}} \cdots \xrightarrow{d^Y_1} Y_0 \xrightarrow{d^Y_0} N \to 0$ *be minimal projective presentation of* N. *Denote by* $\mathbf{X} = (X_{n-1} \xrightarrow{d^X_{n-1}} X_{n-2} \xrightarrow{d^X_{n-2}} \cdots \xrightarrow{d^X_1} X_0)$ *and* $\mathbf{Y} = (Y_{n-1} \xrightarrow{d^Y_{n-1}} Y_{n-2} \xrightarrow{d^Y_{n-2}} \cdots \xrightarrow{d^Y_1} Y_0)$ *n-term complexes over* A. *Then the following assertions are equivalent:*

1. $\mathrm{Hom}_A(N, \tau_n M) = 0$ *and for all* $i = 1, \cdots, n-1$, $Ext^i(M, N) = 0$.
2. $\mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(\mathbf{X}, \mathbf{Y}[i]) = 0$ *for all* $i = 1, \cdots, n-1$.

This implies that M *is strongly* τ_n*-rigid if and only if* $\mathbf{X} \in n-\mathrm{presilting}\,A$.

Proof. $(1 \Rightarrow 2)$ Let $i \in \{1, \cdots n-1\}$. Any morphism $f \in \mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(\mathbf{X}, \mathbf{Y}[i])$ is defined by some $f_j \in \mathrm{Hom}_A(X_j, Y_{j-i})$, for all $j \in \{1, \cdots, n-1\}$.

$$\begin{array}{ccccccccc} & & 0 & \to & X_{n-1} & \xrightarrow{d^X_{n-1}} \cdots \to & X_i & \xrightarrow{d^X_i} & X_{i-1} & \xrightarrow{d^X_{i-1}} \cdots \to X_0 \to 0 \\ & & \downarrow & & \downarrow f_{n-1} & & \downarrow f_i & & \downarrow & \\ 0 \to Y_{n-1} & \xrightarrow{d^Y_{n-1}} \cdots \to & Y_{n-i} & \xrightarrow{d^Y_{n-i}} & Y_{n-1-i} & \xrightarrow{d^Y_{n-1-i}} \cdots \to & Y_0 & \to & 0 & \end{array}$$

By Proposition 2.4, since $\mathrm{Hom}_A(N, \tau_n M) = 0$,

$$\mathrm{Hom}_A(X_0, N) \xrightarrow{(d^X_1, N)} \mathrm{Hom}_A(X_1, N) \xrightarrow{(d^X_2, N)} \cdots \xrightarrow{(d^X_{n-2}, N)}$$

$$\mathrm{Hom}_A(X_{n-2}, N) \xrightarrow{(d^X_{n-1}, N)} \mathrm{Hom}_A(X_{n-1}, N) \to 0 \text{ is exact.}$$

As $Ext^{i+1}(M, N) = 0$, then $(d^X_i, N) : \mathrm{Hom}_A(X_{i-1}, N) \to \mathrm{Hom}_A(X_i, N)$ is surjective. Therefore there exists $f_{i-1} : X_{i-1} \to N$ such that $d^Y_0 f_i = f_{i-1} d^X_i$. Moreover, since X_{i-1} is projective, there exists $h_{i-1} : X_{i-1} \to Y_0$ such that $d^Y_0 h_{i-1} = f_{i-1}$. Since $d^Y_0(f_i - h_{i-1} d^X_i) = 0$, we have $h_i : X_i \to Y_1$ with $f_i = d^Y_1 h_i + h_{i-1} d^X_i$.

$$\begin{array}{cccccccccc} & 0 & \to & X_{n-1} & \xrightarrow{d^X_{n-1}} & \cdots \to & X_i \xrightarrow{d^X_i} & X_{i-1} \xrightarrow{d^X_{i-1}} & X_{i-2} \to \cdots \to X_0 \to M \to 0 \\ & \downarrow & & \downarrow f_{n-1} & & & \downarrow f_i & \downarrow f_{i-1} & \downarrow \\ 0 \to Y_{n-1} \to \cdots \to & Y_{n-i} & \xrightarrow{d^Y_{n-i}} & Y_{n-1-i} & \xrightarrow{d^Y_{n-1-i}} & \cdots \to & Y_0 \xrightarrow{d^Y_0} & N \to & 0 \end{array}$$

Now since $f_i d^X_{i-1} = d^Y_1 f_{i-1}$, then $d^Y_1(f_{i-1} - h_i d^X_{i-1}) = 0$, and we have $h_{i+1} : X_{i+1} \to Y_2$ with $f_{i+1} = d^Y_2 h_{i+1} + h_i d^X_{i+1}$. We continue recursively and get that $f_j = d^Y_{j-i+1} h_j + h_{j-1} d^X_j$ for all j between $i+2$ and $n-1$. Hence for all $i > 0$, $\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(\mathbf{X}, \mathbf{Y}[i]) = 0$.

$(2 \Rightarrow 1)$ Take any $f \in \mathrm{Hom}_A(X_{n-1}, N)$. Then since X_{n-1} is projective, $d^Y_0 g = f$ for some $g : X_{n-1} \to Y_0$.

$$\begin{array}{cccc} & X_{n-1} & \xrightarrow{d^X_{n-1}} & X_{n-2} \\ & \downarrow g & \searrow f & \\ Y_1 \to & Y_0 & \xrightarrow{d^Y_0} & N \to 0 \end{array}$$

Hence g induces a morphism $\mathbf{X} \to \mathbf{Y}[n-1]$ in $K^b(\mathrm{proj}\, A)$.
As $\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(\mathbf{X}, \mathbf{Y}[n-1]) = 0$, there exists $h_{n-2} : X_{n-2} \to Y_0$ and

$h_{n-1} : X_{n-1} \to Y_1$ such that $g = d_1^Y h_{n-1} + h_{n-2} d_{n-1}^X$. Hence we have $f = d_0^Y (d_1^Y h_{n-1} + h_{n-2} d_{n-1}^X) = d_0^Y h_{n-2} d_{n-1}^X$. Therefore (d_{n-1}^X, N) is surjective and by Proposition 2.4, we conclude that $\mathrm{Hom}_A(N, \tau_n M) = 0$.

Now let $i \in \{1, \cdots n-1\}$. It is clear that $Ext^i(M,N) = Ext^1(\Omega^{i-1}M, N)$ and this is zero if $\mathrm{Hom}_A(X_{i-1}, N) \to \mathrm{Hom}_A(\Omega^i M, N)$ is surjective. Take any $f_i \in \mathrm{Hom}_A(\Omega^i M, N)$. This homomorphism induces $f_i' : X_i \to N$. Since X_i is projective, there exists $g_i : X_i \to Q_0$ such that $d_0^Y g_i = f_i'$.

$$\begin{array}{ccccc} & X_i & \xrightarrow{d_i^X} & X_{i-1} & \\ & \downarrow g_{i-1} & \searrow f_i' & & \\ Y_1 \to & Y_0 & \xrightarrow{d_0^Y} & N \to 0 & \end{array}$$

We see that g_i induces a morphism $\mathbf{X} \to \mathbf{Y}[i]$ in $K^b(\mathrm{proj}\, A)$.
As $\mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(\mathbf{X}, \mathbf{Y}[i]) = 0$, there exists $h_{i-1} : X_{i-1} \to Y_0$ and $h_i : X_i \to Y_1$ such that $g_i = d_1^Y h_i + h_{i-1} d_i^X$. Hence we have $f_i' = d_0^Y (d_1^Y h_i + h_{i-1} d_i^X) = d_0^Y h_{i-1} d_i^X$ and $f_i = d_0^Y h_{i-1} s_i$ where s_i is the inclusion $\Omega^i M \to X_{i-1}$. Therefore (s_i, N) surjective and $Ext^i(M,N) = 0$.

□

Lemma 3.5. *Let $M \in \mathrm{mod}\, A$. Let $\mathbf{X} := (X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0)$ be an n-term complex where $X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} M \to 0$ is a minimal projective presentation of M. Let $Y \in \mathrm{proj}\, A$. Then*

$$\mathrm{Hom}_A(Y, M) = 0 \text{ if and only if } \mathrm{Hom}_{K^b(\mathrm{proj}\, A)}(Y, \mathbf{X}) = 0.$$

By the following result we will see that n-term silting complexes define support τ_n-tilting modules.

Proposition 3.6. *Let $\mathbf{X} := (X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0)$ be an n-term complex for A. Let $M = \mathrm{Cok}\, d_1$.*

1. *If $\mathbf{X}$ is a silting complex for A where d_{n-1} is a right minimal homomorphism, then M is a τ_n-tilting module.*
2. *If $\mathbf{X}$ is a silting complex for A then M is a support τ_n-tilting module.*

Proof. 1. Denote by $d_{n-1} = (d_{n-1}' \quad 0) : X_{n-1}' \oplus X_{n-1}'' \to X_{n-2}$, where d_{n-1}' is right minimal. Then $X_{n-1}' \xrightarrow{d_{n-1}'} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{p_1}$

$X_0 \xrightarrow{d_0} M \to 0$ is a minimal projective presentation of M. We will prove that (M, X''_{n-1}) is a support τ_n-tilting pair. As $\mathbf{X}$ is a silting complex, then $\mathrm{Hom}_{K^b(\mathrm{proj}\,A)}(X''_{n-1}, \mathbf{X}) = 0$, and by Lema 3.5 we obtain $\mathrm{Hom}_A(X''_{n-1}, M) = 0$. Therefore (M, X''_{n-1}) is a strongly τ_n-rigid pair. Now we obtain

$$d(M) = d(X'_{n-1} \xrightarrow{d'_{n-1}} X_{n-2} \to \cdots \to X_1 \to X_0),$$

because d'_{n-1} is a minimal projective presentation of M. Thus

$$d(M)+d(X''_{n-1}) = d(X'_{n-1} \xrightarrow{d'_{n-1}} X_{n-2} \to \cdots \to X_1 \to X_0)+d(P''_{n-1}) = d(P),$$

and by Proposition 2.6 this is equal to $d(A)$. We conclude the assertion.

2. For $X''_{n-1} = 0$ in (1), we get the assertion.

□

We will see that support τ_n-tilting modules define n-term silting complexes for A.

Proposition 3.7. *Let* $X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0 \to M \to 0$ *be a minimal projective presentation for* M.

1. *If* M *is* τ_n*-tilting, then* $(X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0)$ *is a* n*-term silting complex.*

2. *If* (M, Y) *is a support* τ_n*-tilting pair then* $(X_{n-1} \oplus Y \xrightarrow{(d_{n-1}, 0)} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0)$ *is a* n*-term silting complex.*

Proof. 2. By Lemma 3.4, $\mathbf{X} := (X_{n-1} \oplus Y \xrightarrow{(d_{n-1}, 0)} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0)$ is a presilting complex. As $X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0 \to M \to 0$ is a minimal projective presentation, we have $d(M) + d(Y) = d(A)$. Thus we have

$$\begin{aligned} d(\mathbf{X}) &= d(X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} X_1 \xrightarrow{d_1} X_0) + d(Y) \\ &= d(M) + d(Y) = d(A). \end{aligned}$$

Therefore $\mathbf{X}$ is silting by Proposition 2.6.

1. For $Q = 0$ in 2 we obtain the assertion.

□

The proof of Theorem 3.1 is obtained by using Proposition 3.6 and Proposition 3.7.

4 Further Research

When the algebra A is cluster-tilted (or more generally 2-Calabi–Yau tilted), there is a functor from a cluster category (or 2-Calabi–Yau triangulated category) $\mathcal{C}$ to $\operatorname{mod} A$. The support τ-tilting modules over A are precisely the images of the cluster-tilting objects of $\mathcal{C}$ under this functor. It seems thus natural to consider n-Calabi–Yau tilted algebras and try to establish a bijection between n-cluster tilting objects and support τ_n tilting modules. Once the bijection is established, one may study mutations and geometric realisation of τ_n-tilting modules and n-term silting complexes.

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