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Greybody factors for various black holes in perfect fluid spheres

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Abstract

Hawking radiation is emitted from black holes. At the black hole event horizon, the Hawking radiation is the exact blackbody radiation. However, while propagating, the Hawking radiation is modified by the curvature of spacetime. Therefore, it is not considered as blackbody radiation when travelling out of the event horizon. Greybody factors are the transmission probabilities with which the Hawking radiation can be transmitted by gravitational potential to spatial infinity. This

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potential is the curvature of spacetime generated by a black hole. In this paper, the greybody factors for various black holes in perfect fluid spheres are calculated. The formula of the lower bound on the transmission probability is used to obtain the greybody factors. The bounds can give us a qualitative understanding of the quantum nature of black holes. Finally, we calculate the Hawking temperature and entropy of blackholes.

1 Introduction

General Relativity is the gravitation created by the internal curvature of spacetime. Consider the motion of an object bound to a spherical surface without any force. Such object moves along the geodesics of the sphere [1]. Therefore, spacetime will be bent around every object with mass. The Einstein field equation is the centerpiece of general relativity.

The Einstein field equation explains the gravity is the result from spacetime curvature due to the energy-momentum. The idea is that the existence of energy curves spacetime [2]. The Einstein field equation can be written in this form;

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature, G is the Newtons gravitational constant, and $T_{\mu\nu}$ is the stress-energy. Einsteins field equations are the second-order differential equations for $g_{\mu\nu}$, which are extremely difficult to solve. Thus, in order to actually solve Einsteins equations, we often use the simplifying assumption that the metric has a significant degree of symmetry [3, 4, 5].

Perfect fluid sphere is one of such assumptions that reduces the complexity of the Einstein field equation. Perfect fluid spheres satisfy the following criteria: no viscosity, no heat conduction, and geometrically isotropic ($p_r = p_t$) [6]. Through these three properties of a perfect fluid sphere, the stress energy tensor takes the form [6];

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{pmatrix}. \tag{1.2}$$

From the Einstein field equations, the Isotropy ($p_r = p_t$) leads to the constraint

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}. \tag{1.3}$$

We can then use this constraint to solve the Einstein field equations.

2 Perfect fluid spheres

We can apply two specific theorems on perfect fluid spheres and then analyze the results [6];
The specific geometry in the Schwarzschild metric must first be defined by

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{B(r)} dr^2 + r^2 d\Omega^2, \tag{2.4}$$

with the assumption of a perfect fluid sphere. We can then apply the two following theorems to the perfect fluid sphere.

Theorem 1 (1st BVW theorem [6]). Suppose $\{\zeta(r), B(r)\}$ represents a perfect fluid sphere, and is defined by

$$\Delta(r) = \Big[\frac{\zeta(r)}{\zeta(r) + r\zeta'(r)}\Big]^2 r^2 \exp\{2 \int \frac{\zeta'(r)}{\zeta(r)} \frac{\zeta(r) - r\zeta'(r)}{\zeta(r) + r\zeta'(r)} dr\}. \tag{2.5}$$

Then $\{\zeta(r), B(r) + \lambda\Delta(r)\}$ is also a perfect fluid sphere.

Theorem 2 (2nd BVW theorem [6]). Suppose $\{\zeta(r), B(r)\}$ represents a perfect fluid sphere, and is defined by

$$Z(r) = \sigma + \varepsilon \int \frac{r dr}{\zeta(r)^2 \sqrt{B(r)}}. \tag{2.6}$$

Then $\{\zeta(r) Z(\zeta, B), B(r)\}$ is also a perfect fluid sphere.

The result, after applying the two theorems on the perfect fluid sphere, is still a perfect fluid sphere. We will then analyze which perfect fluid sphere will be transformed into a blackhole.

3 Black holes in Perfect fluid spheres

Schwarzschild considered the curvature of spacetime around a star as perfectly spherical and non-revolving, and then defined the distance from the center of the star to the boundaries as the Schwarzschild radius. The curvature of spacetime is so powerful that even light can be confined within it. Stars that explode and collapse to a size smaller than the radius of Schwarzschild will be transformed into black holes. A spherical surface called the Event Horizon will be generated, which is the same size as the Schwarzschild radius. If an object moves too close to the Event Horizon, it will be pulled in by the enormous force of attraction, and will no longer be able to escape, even if that object is traveling at the speed of light [7]. There are four types of black holes, which are classified according to their properties of rotation and charge; namely, Schwarzschild black hole, Kerr black hole, ReissnerNordstrm black hole, and Kerr-Newman black hole [8]. In this paper, we are interested in the Schwarzschild black hole because it is a static black hole which does not rotate and has no electric charge, making it possible to calculate the radius of the perfect fluid sphere in this paper. It also has a static spherically symmetric geometry in relation to the Schwarzschild coordinates.

The Schwarzschild radius also corresponding and coincident with the event horizon [9]. At smaller radius, light will be confined within the black hole. Schwarzschild radius thus becomes [10]

$$r_s = \frac{2GM}{c^2} \approx 2.95\frac{M}{M_{sun}}. \tag{3.7}$$

We start with the specific geometry in the Schwarzschild metric

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2. \tag{3.8}$$

We consider [18]

$$G_{\hat{r}\hat{r}} = \frac{-2rB(r)\zeta(r) + \zeta(r) - \zeta(r)B(r)}{r^2\zeta(r)}. \tag{3.9}$$

Considering the Einstein field equation

$$G_{\hat{r}\hat{r}} = 8\pi G T_{\hat{r}\hat{r}}, \tag{3.10}$$

the pressure inside the perfect fluid sphere is given by

$$p = \frac{G_{\hat{r}\hat{r}}}{8\pi G} = \frac{1}{8\pi G}\frac{-2rB(r)\zeta(r) + \zeta(r) - \zeta(r)B(r)}{r^2\zeta(r)}. \tag{3.11}$$

So, the radius r of a perfect fluid sphere, must satisfy [11]

$$p(r) = 0. \tag{3.12}$$

If $r < r_s$, a perfect fluid sphere becomes a black hole, the Schwarzschild singularity at $r = 2M$ is a point not a sphere, it should not happen in nature and at $r > 2M$ is the result from the gravitational collapse [12].

TABLE 1. A black hole in the form of a perfect fluid sphere (some selective perfect fluid spheres) [6].

Black holes	**Metrics**
Schwarzschild Exterior	$-(1-\frac{2m}{r})dt^2 + (1-\frac{2m}{r})^{-1}dr^2 + r^2 d\Omega^2$
Kuch 68 II	$-(1-\frac{2m}{r})dt^2 + [(1-\frac{2m}{r})(1+C(2r-2m)^2)]^{-1}dr^2 + r^2 d\Omega^2$

A black hole in the form of a perfect fluid sphere as shown in Table 1. The Schwarzschild Exterior is considered as a black hole because we used the coefficient in front of dr^2 to match with various perfect fluid spheres such as S1, K-O III and the group of Tolman to find the condition through which perfect fluid spheres become a black hole, with the Schwarzschild Exterior having a coefficient in front of dr^2 equal to dt^2. Kuch 68 II is a type of Schwarzschild black hole, and we use the coefficient in front of dr^2 of the Kuch 68 II to match with other perfect fluid spheres such as M-W II and Heint IIa ($C = 0$) to find the condition in which perfect fluid spheres become a black hole.

We then use the black holes in relation to the perfect fluid spheres to calculate the potential of the black holes using the Klein-Gordon equation.

3.1 Klein-Gordon equation

The Klein-Gordon equation is one of the relativistic wave equations that describes the spinless particles behavior [13]. The Klein-Gordon equation can be written in the form

$$\frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \psi = 0, \tag{3.13}$$

where $g_{\mu\nu}$ is the metric tensor, $g^{\mu\nu}$ is the inverse of the metric tensor, and g is the determinant of the metric tensor. We can use this equation to transform into the Regge-Wheeler equation that is similar to the Schrodinger equation.

3.2 Regge-Wheeler equation

The Regge-Wheeler equation describes perturbations of the Schwarzschild metric and also plays a significant role in Schwarzschild black hole [14]. The

Regge-Wheeler equation can be written in this form [15]

$$\frac{d^2\Psi}{dr_*^2} + [\omega^2 - V(r)]\Psi(r) = 0, \tag{3.14}$$

where $V(r) = \frac{l(l+1)\zeta(r)^2}{r^2} + r^{-1}\sqrt{\zeta(r)^2 B(r)}\frac{d}{dr}\sqrt{\zeta(r)^2 B(r)}$ is potential of black hole. We can then use this potential to obtain the greybody factors. For the Schwarzschild black holes that have a coefficient in front of dr^2 equal to dt^2, we can obtain the potential in the form [16]

$$V(r) = \frac{l(l+1)f(r)}{r^2} + \frac{f(r)f'(r)}{r^2}. \tag{3.15}$$

Table 2. Potentials of black holes that are obtained from the Regge-Wheeler equation.

Black holes	Potentials
Schwarzschild Exterior	$V(r) = \frac{l(l+1)(1-\frac{2m}{r})}{r^2} + \frac{\frac{2m}{r^2}(1-\frac{2m}{r})}{r}$
Kuch 68 II	$V(r) = \frac{l(l+1)(1-\frac{2m}{r})}{r^2} + \frac{2C(2r-2m)}{r}$

Potentials of black holes as shown in Table 2 are functions that depend on r that are difficult to solve using the exact solution method. In calculating the greybody factors, the methods that are suitable for these potentials are the WKB* and the 2x2 transfer matrix methods; however, for black holes, we cannot specify the value of the greybody factor. Therefore, we calculate the greybody factor using the 2x2 transfer matrix method to obtain the rigorous bound of the transmission and the reflection probabilities.

* WKB (WentzelKramersBrillouin) is the approximation method for solving time-independent Schrdinger equation. The WKB approximation suitable for slowly-varying potential and describe the tunneling.

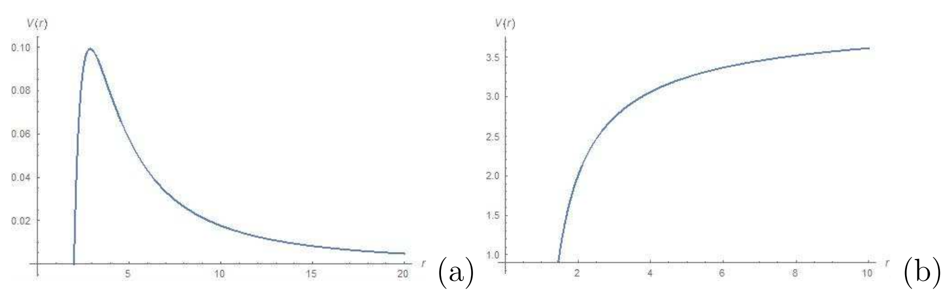

FIGURE 1. Plotting of the potentials of black holes. (a) Potential of the Schwarzschild Exterior black hole for $l = 1, m = 1$. (b) Potential of the Kuch 68 II black hole for $m = 1, l = 1, C = 1$.

4 Transmission and reflection probabilities

Greybody factors are the transmission and reflection probabilities. The transmission and reflection probabilities can be calculated using the 2x2 transfer matrix method to obtain the lower bound on the transmission coefficient and upper bound on the reflection coefficient. The 2x2 transfer matrix method is given by [17]

$$T \geq \operatorname{sech}^2 \frac{1}{2\omega} \int_{-\infty}^{\infty} |V(r)| dr_*, \tag{4.16}$$

where $\frac{dr_*}{dr} = \frac{1}{\sqrt{\zeta(r)^2 B(r)}}$.

For the Schwarzschild black holes that have a coefficient front of dr^2 equal to dt^2, we can obtain dr_* in this form [16]

$$\frac{dr_*}{dr} = \frac{1}{f(r)}. \tag{4.17}$$

Table 3. Transmission probability of black holes.

Black holes	**Transmission probabilities**
Schwarzschild Exterior	$T \geq sech^2 \left[\frac{1}{2\omega} \frac{m+2l(l+1)m}{4m^2} \right]$
Kuch 68 II	$T \geq sech^2 \frac{1}{2\omega} \frac{l(l+1)a(m+4Cm^3-r+8Cm^2r)}{(r+4Cm^2r)^2} + 2\sqrt{C}\sinh^{-1}[2\sqrt{C}(m-r)] + t\ln[1+ab+4Cm(m-r)] - t\ln[r]$

where $a = \sqrt{1+4C(m-r)^2}$, $b = \sqrt{1+4Cm^2}$, and $t = \frac{4Cm(l(-2+4Cm^2)+l^2(-2+4Cm^2)+b^4)}{b^5}$.

Table 4. Reflection probability of black holes.

Black holes	**Reflection probabilities**
Schwarzschild Exterior	$R \leq \operatorname{sech}^2 \left[\frac{1}{2\omega} \frac{m+2l(l+1)m}{4m^2} \right]$
Kuch 68 II	$R \leq \operatorname{sech}^2 \frac{1}{2\omega} \frac{l(l+1)a(m+4Cm^3-r+8Cm^2r)}{(r+4Cm^2r)^2} + 2\sqrt{C}\sinh^{-1}[2\sqrt{C}(m-r)] + t\ln[1+ab+4Cm(m-r)] - t\ln[r]$

where $a = \sqrt{1+4C(m-r)^2}$, $b = \sqrt{1+4Cm^2}$, and $t = \frac{4Cm(l(-2+4Cm^2)+l^2(-2+4Cm^2)+b^4)}{b^5}$.

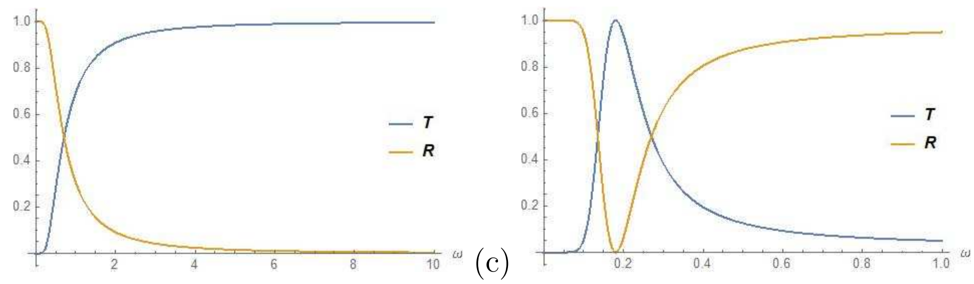

FIGURE 2. Plotting of the relation between the transmission and the reflection probabilities.

The relation between the transmission and the reflection probabilities are plotted as shown in Figure 2, (c) represents the transmission and the reflection probabilities of the Schwarzschild Exterior black hole for $m = 1, l = 1$; from the plots, this potential has a reflection resonance at $\omega = 0$ until ω increases, at which point transmission probability occurs and transmission resonance can be measured. (d) represents the transmission and the reflection probabilities of the Kuch 68 II black hole for $m = 1, l = 1, C = 1$; from the plots, if ω increases, at which point reflection probability occur and increase, while the transmission probability decreases.

5 Temperature of Hawking radiation and entropy

The temperature of the Hawking radiation can be calculated in terms of the surface gravity of the black hole [16];

$$T = \frac{K}{2\pi}, \tag{5.18}$$

where $K = (\sqrt{\zeta^2(r)B(r)})'$.

For Schwarzschild black holes with a coefficient in front of dr^2 equal to dt^2, we can obtain K in this form [16];

$$K = f'(r). \tag{5.19}$$

Entropy is also one of the fundamental properties in thermodynamics, which can be written in the form

$$S = \int \frac{1}{T} dM. \tag{5.20}$$

Table 5. Temperature of Hawking radiation and entropy of black holes.

Black holes	**Temperature**	**Entopy**
Schwarzschild Exterior	$T = \frac{1}{4M\pi}$	$S = 2M^2\pi$
Kuch 68 II	$T = \frac{2CM}{d\pi}$	$S = \frac{\pi(d+\ln[M]-\ln[1+d])}{2C}$

where $d = \sqrt{1 + 4CM^2}$.

In Table 5 shows the temperature of Hawking radiation and entropy of Schwarzschild Exterior and Kuch 68 II black holes.

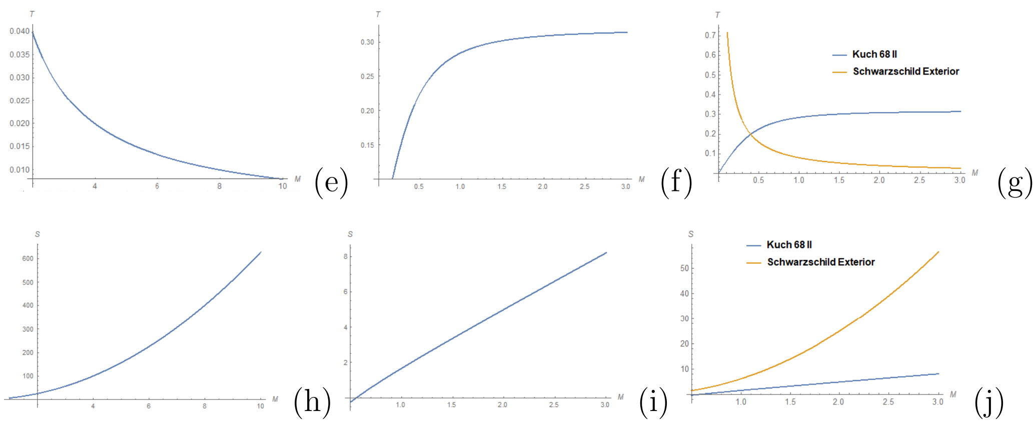

FIGURE 3. Plotting of temperature and entropy of various black holes.

Temperature and entropy of various black holes are plotted as shown in Figure 3, (e) and (h) represent the temperature and entropy of the Schwarzschild Exterior black hole, respectively. From the plots, if M increases, the temperature decreases and the entropy increases. Plots (f) and (i) represent the temperature and entropy of the Kuch 68 II black hole, respectively. From the plots, if M increases, both the temperature and the entropy increase. Plots (g) and (j) represent a comparison of the temperature and entropy of the Kuch 68 II black hole, respectively. From the plots, if M increases, the temperature of the Kuch 68 II becomes higher than that of Schwarzschild Exterior, while the entropy of the Schwarzschild Exterior becomes higher than that of Kuch 68 II.

6 Conclusion

In this paper, we have classified black holes in accordance to perfect fluid spheres, and have also derived the Schwarzschild Exterior black hole and the

Kuch 68 II black hole. We have used these black holes to find the structure of the potentials of the black holes, allowing us to calculate the greybody factors from these potentials. Greybody factors are the transmission and reflection probabilities. We have obtained the rigorous bounds on the transmission and reflection coefficients using the 2x2 transfer matrix method, and the results show that the reflection probability of the Schwarzschild Exterior black hole decreases to zero if the wave's energy increases and if the transmission increase to one. The reflection probability of the Kuch 68 II black hole increases if the wave's energy increases and a transmission occurs, with the transmission resonance occurring at a short interval of wave's energy. Finally, we calculated the temperature of the Hawking radiation and the entropy of black holes, with the results showing that for the Kuch 68 II black hole, the temperature and the entropy are high if the mass of the black hole increases, while Schwarzschild Exterior experiences a temperature decrease and an entropy increase if the mass of the black hole increases. A comparison of the temperatures of these black holes show that the temperature of the Kuch 68 II black hole is higher than that of Schwarzschild Exterior at the equal mass, while the entropy of Schwarzschild Exterior is higher than that of the Kuch 68 II black hole, at the equal mass. From the plotting in figures 1-3, even though the Schwarzschild Exterior black hole and the Kuch 68 II black hole are types of Schwarzschild black hole, these black holes, however, obtained different results.

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