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Analytical and Approximate Trigonometric Solution to Duffing-Helmholtz Equation

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Abstract

In this paper, we give the analytical solution and an approximate trigonometric solution to the undamped and unforced Duffing-Helmholtz equation. We also provide high accurate formulas to approximate the Weierstrass elliptic function and its period.

1 Introduction

Ordinary and partial differential equations both play important roles in explaining many phenomena that occur in nature or in medical engineering,

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biotechnology, economic, ocean, plasma physics, among others. The Duffing-Helmholtz equation is considered as one of the most important differential equations because it demonstrates the scenario and mechanism of various nonlinear phenomena that occur in nonlinear dynamic systems. It is one of the most common models for analyzing and modeling many nonlinear phenomena in various fields of science such as the mechanical engineering, electrical engineering, plasma physics, to mention a few. Mathematically, the Duffing-Helmholtz oscillator is a second-order ordinary differential equation with a nonlinear restoring force having the form

$$\ddot{x} + px + qx^2 + rx^3 = 0, \tag{1.1}$$

The analytical solution to (1.1) for any given initial conditions may be written in terms of the Weierstrass elliptic function. This is the function $\wp = \wp(t; g_2, g_3)$, where the numbers g_2 and g_3 are called its elliptic invariants. This function obeys the ode

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3. \tag{1.2}$$

The main period of the function $\wp$ is given by

$$T = 2\int_a^\infty \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}, \tag{1.3}$$

where a is the greatest real root to the cubic

$$4x^3 - g_2x - g_3 = 0. \tag{1.4}$$

Suppose that a is the greatest real root of the cubic (1.4) and let b and $-(a+b)$ be the other two roots. Then

$$\begin{aligned} T &= 2\int_a^\infty \frac{dx}{\sqrt{4x^3 - g_2x - g_3}} \\ &= 2\int_a^\infty \frac{dx}{\sqrt{4(x-a)(x-b)(x+a+b)}} \\ &= \frac{2}{\sqrt{a-b}}\left|K\left(\frac{3a}{b-a}+2\right)\right|. \end{aligned} \tag{1.5}$$

In general, if $Re(a) > \max(Re(b), Re(c))$, then

$$\int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{4iF\left(i\sinh^{-1}\left(\sqrt{\frac{a-b}{u-a}}\right)\Big|\frac{a-c}{a-b}\right) + 2K\left(\frac{c-b}{a-b}\right)}{\sqrt{a-b}} \text{ for } u > Re(a). \tag{1.6}$$

The number $K(m)$ may be approximated by means of the formula

$$K(m) \approx \frac{5\pi}{18} - \frac{32\pi}{9(9m-16)}. \tag{1.7}$$

The error for $-1 \leq m \leq 1/4$ is given by

$$\max_{-1\leq m\leq 1/4} \left| \frac{5\pi}{18} - \frac{32\pi}{9(9m-16)} - K(m) \right| = 0.00844. \tag{1.8}$$

This formula allows us to solve the equation $K(m) = \mu$ for $1.31 \leq \mu \leq 1.68$ as follows:

$$m = K^{-1}(\mu) \approx \frac{16}{9} - \frac{64\pi}{9(18\mu - 5\pi)}. \tag{1.9}$$

2 Unforced and Undamped Duffing-Helmholtz Equation and its Analytical Solution

Let us consider the standard (undamped) Duffing-Helmholtz equation in the absence of friction and excitation forces [1]

$$\ddot{x} + px + qx^2 + rx^3 = 0,\ x = x(t), \tag{2.10}$$

subject to the following initial conditions

$$x(0) = x_0\ \&\ x'(0) = \dot{x}_0. \tag{2.11}$$

The general solution to Eq. (2.10) may be written in terms of the elliptic Weierstrass function $\wp$. We give a solution to the i.v.p. (2.10)-(2.11) in terms of Weierstrass elliptic functions. To solve this problem, we consider the following:

$$x(t) = A + \frac{B}{1 + C\wp\,(t + t_0; g_2, g_3)}, \tag{2.12}$$

where $BC \neq 0$.

Substituting (2.12) into the ordinary differential equation $\ddot{x} + px + qx^2 + rx^3 = 0$ gives

$$\frac{1}{2(1 + C\wp^3)} \sum_{j=0}^{3} K_j \wp^j = 0, \tag{2.13}$$

with

$$\begin{aligned}
K_3 &= 2C^2\left(A^3Cr + A^2Cq + ACp + 2B\right), \\
K_2 &= 2C\left(3A^3Cr + 3A^2BCr + 3A^2Cq + 2ABCq + 3ACp + BCp - 6B\right), \\
K_1 &= C\left(6A^3r + 12A^2Br + 6A^2q + 6AB^2r + 8ABq + 6Ap + 2B^2q - 3BCg_2 + 4Bp\right), \\
K_0 &= A^3r + 6A^2Br + 2A^2q + 6AB^2r + 4ABq + 2Ap + \\
&\quad 2B^3r + 2B^2q - 4BC^2g_3 + BCg_2 + 2Bp.
\end{aligned}$$

Equating the coefficients K_j to zero yield an algebraic system. Solving this system, we finally get

$$\begin{aligned}
B &= -\frac{6A\left(A^2r + Aq + p\right)}{3A^2r + 2Aq + p}, C = \frac{12}{3A^2r + 2Aq + p}, \\
g_2 &= -\frac{1}{12}\left(3r^2A^4 + 4qrA^3 + 6prA^2 - p^2\right), \\
g_3 &= \frac{1}{216}\left[\left(9pr^2 - 3q^2r\right)A^4 + \left(12pqr - 4q^3\right)A^3 + \left(18p^2r - 6pq^2\right)A^2 + 2p^3\right], \quad (2.14)
\end{aligned}$$

The values of t_0 and A can be determined from the initial conditions $x(0) = x_0$ and $x'(0) = \dot{x}_0$ and

$$\ddot{x}(0) + px(0) + qx^2(0) + rx^3(0) = 0. \quad (2.15)$$

We have

$$t_0 = \pm\wp^{-1}\left(\frac{x_0 - A - B}{C(A - x_0)}; g_2, g_3\right). \quad (2.16)$$

The number A is a solution to the quartic

$$3rA^4 + 4qA^2 + 6pA - (3rx_0^4 + 4qx_0^3 + 6px_0^2 + 6\dot{x}_0^2) = 0. \quad (2.17)$$

Example 1

The solution of the i.v.p.

$$\begin{cases} \ddot{x} + x + 2x^2 + 3x^3 = 0, \\ x(0) = 1 \text{ and } x'(0) = 1, \end{cases} \quad (2.18)$$

according to the relation (2.12) is given by

$$x(t) = 1.07627 - \frac{2.72078}{1 + 0.762858\wp(t - 0.148317; -7.16667, 0.675926)}. \quad (2.19)$$

In Figure 1, the comparison with the approximate analytic solution (2.19) and the approximate numerical solution using RK4 is investigated.

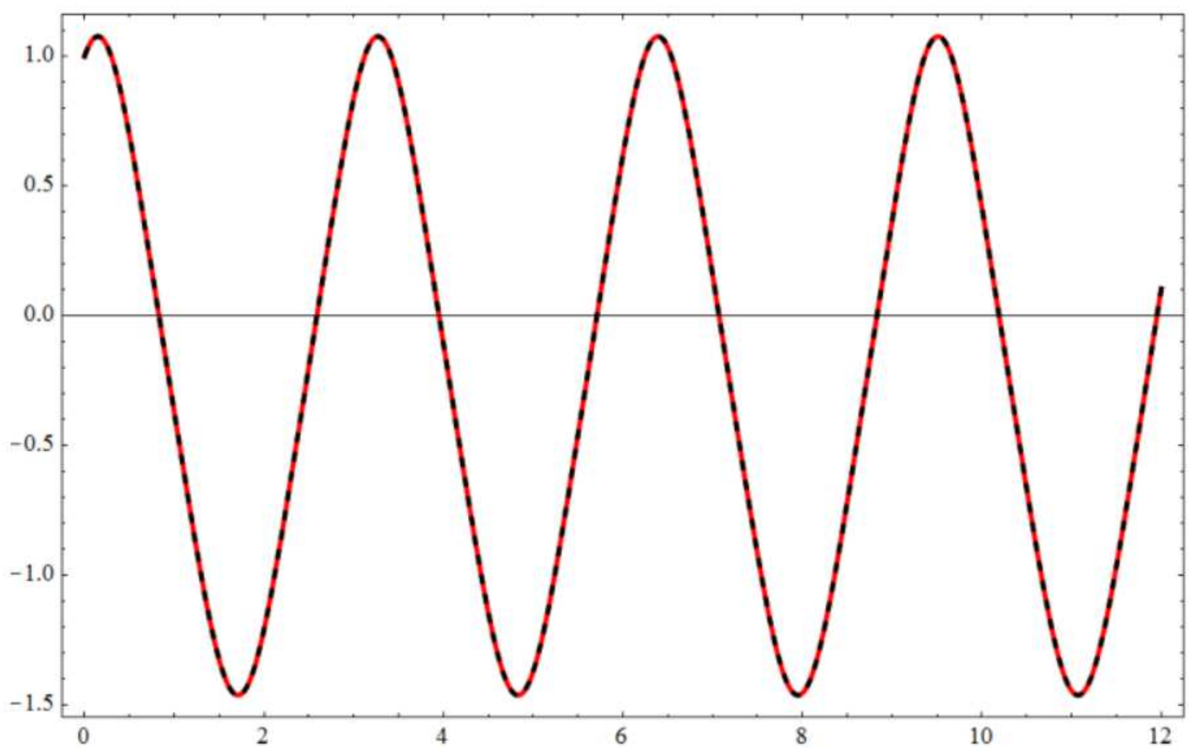

Figure 1: A comparison between solution (2.19) and the approximate numerical solution using RK4.

The periodicity of solution (2.19) is given by

$$T = 2\int_{0.0938538}^{\infty} \frac{1}{\sqrt{4x^3 + 7.16667x - 0.675926}}\,dx = 3.12129.$$

3 Trigonometric Approximation for elliptic Weierstrass function and its inverse

It is known that

$$\mathrm{cn}(\sqrt{\omega}t, m) = \\ 1 - \frac{6}{(4m+1)\left(1+\frac{12}{(4m+1)\omega}\wp\left(t; \frac{1}{12}(16m^2-16m+1)\omega^2, \frac{1}{216}(2m-1)(32m^2-32m-1)\omega^3\right)\right)} \quad (3.20)$$

On the other hand,

$$\wp(t;g_2,g_3) = -\frac{\sqrt{g_2}(4m+1)}{2\sqrt{48m^2-48m+3}} + \sqrt{\frac{3g_2}{16m^2-16m+1}} \cdot \frac{1}{1-\operatorname{cn}\left(\sqrt{2}\sqrt[4]{\frac{3g_2}{16m^2-16m+1}}t, m\right)}, \tag{3.21}$$

where

$$m = \frac{1}{4}(2-\sqrt{\zeta+3}). \tag{3.22}$$

Here, ζ is the least in magnitude root of the cubic

$$4\left(g_2^3-27g_3^2\right)z^3 - 27g_2^3 z + \left(23g_2^3+112g_3^2\right) = 0. \tag{3.23}$$

In the case when $g_2^3-27g_3^2=0$, we have only one real root :

$$z = \frac{733g_3^2}{27g_2^3}. \tag{3.24}$$

Then

$$m = \frac{1}{2} - \frac{1}{54}\sqrt{730} \approx -3.428\,2\times 10^{-4}.$$

Let $g_2^3-27g_3^2 \neq 0$. The discriminant of the cubic (3.23) reads

$$\Delta = 432\left(g_2^3-27g_3^2\right)\left(200g_2^9+9131g_3^2g_2^6+126560g_3^4g_2^3+338688g_3^6\right). \tag{3.25}$$

3.1 First Case. $\Delta > 0$.

The cubic equation (3.23) has three real roots:

$$z_k = \frac{3g_2^{3/2}}{\sqrt{g_2^3-27g_3^2}}\cos\left(\frac{1}{3}\left(2\pi k + \cos^{-1}\left(-\frac{\sqrt{g_2^3-27g_3^2}\left(23g_2^3+112g_3^2\right)}{27g_2^{9/2}}\right)\right)\right),\ k=-1,0,1. \tag{3.26}$$

3.2 Second Case. $\Delta < 0$.

The cubic equation (3.23) has only one real root:

$$\zeta = \frac{\left(\sqrt{\Delta}-\left(g_2^3-27g_3^2\right)^2\left(23g_2^3+112g_3^2\right)\right)^{2/3}+9g_2^6-243g_3^2g_2^3}{2\left(g_2^3-27g_3^2\right)\sqrt[3]{\sqrt{\Delta}-\left(g_2^3-27g_3^2\right)^2\left(23g_2^3+112g_3^2\right)}}. \tag{3.27}$$

3.3 Third Case. $\Delta = 0$.

Suppose that $g_2^3 - 27g_3^2 = 0$. We have two real roots for cubic equation (3.23).

$$z_1 = \frac{1}{2}\sqrt[3]{\frac{23g_2^3 + 112g_3^2}{g_2^3 - 27g_3^2}},\; z_2 = -\sqrt[3]{\frac{23g_2^3 + 112g_3^2}{g_2^3 - 27g_3^2}}.$$

From (3.21), it follows that the period of Weierstrass elliptic function reads

$$T = 2\int_a^{\infty} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} = 2\sqrt{2}\sqrt[4]{\frac{\zeta}{3g_2}} K\left(\frac{1}{4}(2 - \sqrt{\zeta + 3})\right), \tag{3.28}$$

where a is the greatest real root to the cubic

$$4x^3 - g_2 x - g_3 = 0.$$

Using the identity (3.21), we have the following approximation:

$$\wp(t; g_2, g_3) \approx -\frac{\sqrt{g_2}(4m+1)}{2\sqrt{48m^2 - 48m + 3}} + \frac{\sqrt{3g_2}}{\sqrt{16m^2 - 16m + 1}\left(1 - \cos_m\left(\sqrt{2}\sqrt[4]{\frac{3g_2}{16m^2 - 16m + 1}}t\right)\right)}, \tag{3.29}$$

where m is given by (3.22) and

$$\cos_m(t) := \frac{\sqrt{1+\lambda}\cos(\sqrt{1+\lambda}t)}{\sqrt{1 + \lambda\cos^2(\sqrt{1+\lambda}t)}}, \tag{3.30}$$

being

$$\lambda = \frac{1}{14}\left(\sqrt{m^2 - 144m + 144} - m - 12\right). \tag{3.31}$$

Example 1. Let $g_2 = 2$ and $g_3 = 1$. For this choice $m = 0.0119056$, we have

$$\wp(t; 2, 1) \approx -0.474691 + \frac{2.71867}{1 - \frac{3.73048\cos(2.32484t)}{\sqrt{14. - 0.0835474\cos^2(2.32484t)}}}. \tag{3.32}$$

The period is $T = 2.70262$ and the error on the interval $-T/2 \le t \le T/2$ in the sup norm is $E = 8.52 \times 10^{-7}$ (See Figure 2). The reciprocals of the two functions are plotted on the interval $-T/2 \le t \le T/2$.

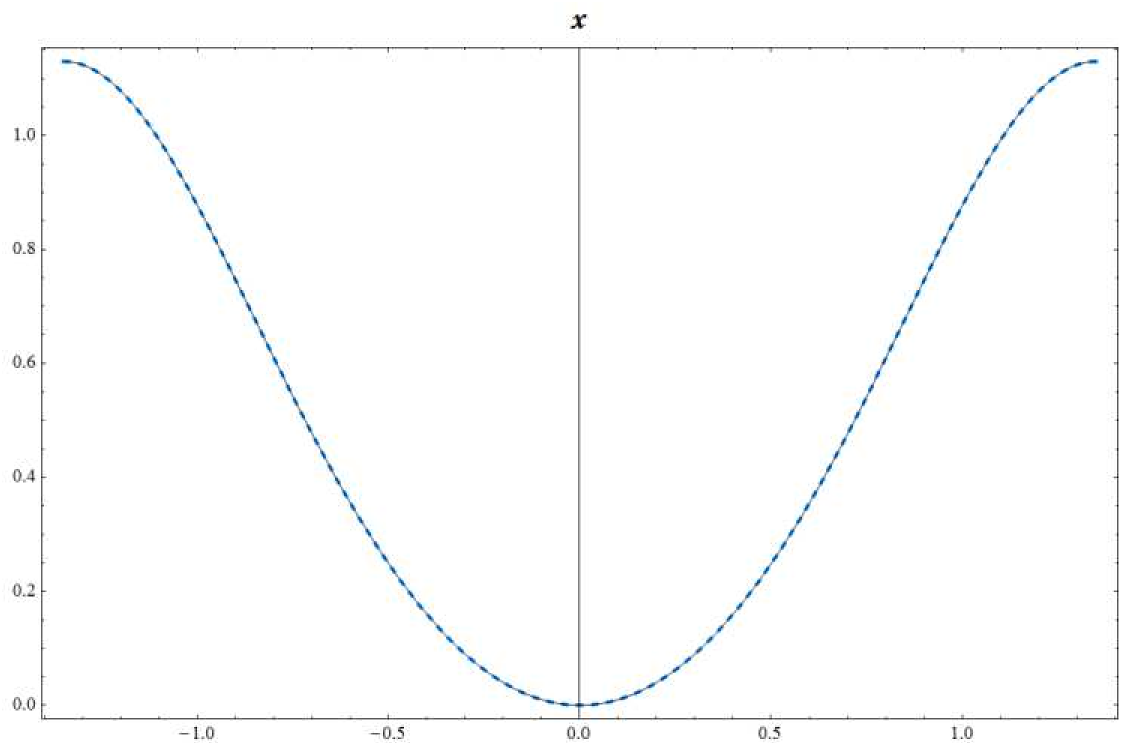

Figure 2.

We now give an approximate expression for the inverse elliptic Weierstrass function. Let

$$a = -\frac{\sqrt{g_2}(4m+1)}{2\sqrt{48m^2-48m+3}} \quad \text{and } b = \frac{\sqrt{3g_2}}{\sqrt{16m^2-16m+1}}.$$

We know that

$$\wp \approx a + \frac{b}{1-\cos_m\left(\sqrt{2}\sqrt[4]{\frac{3g_2}{16m^2-16m+1}}t\right)}.$$

Now, we solve this equation for t to obtain

$$t \approx \pm\frac{\sqrt[4]{16m^2-16m+1}}{\sqrt{2}\sqrt[4]{3}\sqrt[4]{g_2}\sqrt{\lambda+1}}\cos^{-1}\left(\pm\frac{a+b-\wp}{\sqrt{a^2-2ab\lambda-b^2\lambda-2\left(a-b\lambda\right)\wp+\wp^2}}\right), \tag{3.33}$$

and so

$$\wp^{-1}(t;g_2,g_3)\approx\pm\frac{\sqrt[4]{16m^2-16m+1}}{\sqrt{2}\sqrt[4]{3}\sqrt[4]{g_2}\sqrt{\lambda+1}}\cos^{-1}\left(\pm\frac{a+b-t}{\sqrt{a^2-2ab\lambda-b^2\lambda-2\,(a-b\lambda)\,t+\wp^2}}\right). \tag{3.34}$$

Example 2. Suppose we are given

$$\wp(t;-2,-5)=1/5. \tag{3.35}$$

The exact solution is

$$t=\wp^{-1}(1/5;-2,-5)=-1.210122529012178. \tag{3.36}$$

Using (3.34) gives four values:

$$t_1=-1.20908, t_2=1.20908, t_3=-0.763507, t_4=0.763507. \tag{3.37}$$

We choose $t=-1.20908$.
From all previous facts, we establish the following:

General Principle. If the solution to some problem is written in terms of the Weierstrass elliptic function, then such solution can be approximated by means of trigonometric functions.

4 Conclusions

We obtained the exact solution to the undamped Duffing-Helmholtz oscillator equation. We provided a highly accurate trigonometric approximation for the Weierstrass elliptic function and its invers. We also obtained an approximate expression for the elliptic function $K(m)$ and its inverse. We expect that the obtained results will be useful for solving nonlinear ode's in an elementary way.

References

[1] Alvaro H. Salas, Analytic solution to the generalized complex Duffing equation and its application in soliton theory, Applicable Analysis, 2019, DOI: 10.1080/00036811.2019.1698729.