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New Method for Solving Nonlinear equations

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Abstract

In this paper, we show a new method for solving nonlinear and transcendental equations. Our method generalizes the well known Newton-Raphson method.

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1 Halley's Method Rediscovered

One of the most popular methods for solving nonlinear equations is the Newton-Raphson Method. In this work, we improve this method in order to accelerate the convergence. Let f be a smooth function of class C^3 defined on some open interval J and let a be some number in this interval. Then there exists a small $\delta \geq 0$ such that following formula:

$$\begin{aligned} y &= f(x) \approx f(a) - \frac{2f'(a)^2}{f''(a)} + \\ & 2f'(a)^2 \left(\frac{1}{f''(a)} + \frac{8(x-a)f'(a)}{16f'(a)^2 + 3\delta^2 f''(a) - f'(a)\left(8(x-a) + 2\delta^2 f'''(a)\right)} \right) \end{aligned} \tag{1.1}$$

Now, we may solve equation (1.1) for x to obtain

$$\begin{aligned} x &\approx a + \frac{16f'(a)^2 + \left(3f''(a)^2 - 2f'(a)f'''(a)\right)\delta^2}{8f'(a)f''(a)} - \\ & \frac{f'(a)\left(16f'(a)^2 + 3f''(a)^2\delta^2 - 2f'(a)f'''(a)\delta^2\right)}{4f''(a)\left(2f'(a)^2 + f''(a)(y - f(a))\right)}. \end{aligned} \tag{1.2}$$

Now, suppose that $y = f(\rho) = 0$. Then from (1.2) we have the following approximation for the root ρ near a

$$\begin{aligned} \rho &\approx a + \frac{16f'(a)^2 + \left(3f''(a)^2 - 2f'(a)f'''(a)\right)\delta^2}{8f'(a)f''(a)} - \\ & \frac{f'(a)\left(16f'(a)^2 + 3f''(a)^2\delta^2 - 2f'(a)f'''(a)\delta^2\right)}{4f''(a)\left(2f'(a)^2 - f''(a)f(a)\right)}. \end{aligned} \tag{1.3}$$

If we set $\delta = 0$, then we obtain the Halley-Chebyshev Method:

$$\rho \approx a - \frac{2f'(a)f(a)}{2f'(a)^2 - f''(a)f(a)}. \tag{1.4}$$

Now, setting $\delta = f''(a) = 0$, we obtain the Newton-Raphson Method.

2 Main results

We propose the following iteration method for solving the equation $f(x) = 0$. We take some suitable initial value $x_0 = a$ and we define the following sequence for $n = 0, 1, 2, ...$

$$x_{n+1} \approx x_n + \frac{16f'(x_n)^2 + \left(3f''(x_n)^2 - 2f'(x_n)f'''(x_n)\right)\delta^2}{8f'(x_n)f''(x_n)} - \frac{f'(x_n)\left(16f'(x_n)^2 + 3f''(x_n)^2\delta^2 - 2f'(x_n)f'''(x_n)\delta^2\right)}{4f''(x_n)\left(2f'(x_n)^2 - f''(x_n)f(x_n)\right)}. \quad (2.5)$$

In order to avoid the evaluation of the derivatives, we choose some small number δ and we make use of the following finite difference formulas as follows :

$$f'(a) \approx \frac{-f_{-3} + 9f_{-2} - 45f_{-1} + 45f_1 - 9f_2 + f_3}{60\delta}. \quad (2.6)$$

$$f''(a) \approx \frac{2f_{-3} - 27f_{-2} + 270f_{-1} - 490f_0 + 270f_1 - 27f_2 + 2f_3}{180\delta^2}. \quad (2.7)$$

$$f'''(a) \approx \frac{-7f_{-4} + 72f_{-3} - 338f_{-2} + 488f_{-1} - 488f_1 + 338f_2 - 72f_3 + 7f_4}{240\delta^3}. \quad (2.8)$$

In these formulas, $f_j := f(a+j\delta)$ for any j. The following code performs the algorithm in Wolfram Mathematica.

```
Clear[x, y, z, a, δ, j];
fp[a_] := - (f[a - 3 δ] - 9 f[a - 2 δ] + 45 f[a - δ] - 45 f[a + δ] + 9 f[a + 2 δ] - f[a + 3 δ]) / (60 δ);
fpp[a_] :=
  - (-2 f[a - 3 δ] + 27 f[a - 2 δ] - 270 f[a - δ] - 270 f[a + δ] + 27 f[a + 2 δ] - 2 f[a + 3 δ] + 490 f[a]) / (180 δ^2);
fppp[a_] :=
  - 1/(240 δ^3) (7 f[a - 4 δ] - 72 f[a - 3 δ] + 338 f[a - 2 δ] - 488 f[a - δ] + 488 f[a + δ] - 338 f[a + 2 δ] +
     72 f[a + 3 δ] - 7 f[a + 4 δ]);
```

```
x[0] := N[a];
y[0] := N[a]; z[0] := N[a];
x[n_] :=
  x[n] =
   x[n - 1] + (3 f[x[n - 1]] (-2 fp[x[n - 1]]^2 + f[x[n - 1]] × fpp[x[n - 1]]))/(6 fp[x[n - 1]]^3 - 6 f[x[n - 1]] × fp[x[n - 1]] × fpp[x[n - 1]] + f[x[n - 1]]^2 fppp[x[n - 1]]);
 (*Proposed Method*)
y[n_] := y[n] = y[n - 1] + (2 f[y[n - 1]] × fp[y[n - 1]])/(-2 fp[y[n - 1]]^2 + f[y[n - 1]] × fpp[y[n - 1]]) (*Halley-Chebyshev*)
z[n_] := z[n] = z[n - 1] - f[z[n - 1]]/fp[z[n - 1]]; (*Newton-Raphson*)
```

2.1 Numerical Experiments

Let us consider the problem of finding the zeros of the Zeta Riemann function on the critical line. Hardy proved that there are infinitely many zeros on the critical line. In table 1 we illustrate the evolution of the Halley-Chebyshev Method, the Newton-Raphson Method, and our proposed method for different values of δ. We must choose a suitable initial value for the desired root. For the n-th zero of the Zeta Riemann function on the critical line we have the following approximate formula [1], [2], [6]:

$$\rho_n \approx \tilde{\rho}_n := \frac{1}{2} + \frac{2\pi(n - 11/8)\pi}{W((n - 11/8)/e)} i,\ n \geq 1. \tag{2.9}$$

Here W stands for the Lambert function. See Table 1.

n	ρ_n	$\lvert\rho_n - \tilde{\rho}_n\rvert$	n	ρ_n	$\lvert\rho_n - \tilde{\rho}_n\rvert$
100	$0.5 + 236.524\,i$	0.536962	600	$0.5 + 939.024\,i$	0.272636
150	$0.5 + 318.853\,i$	0.153322	650	$0.5 + 1001.35\,i$	0.254336
200	$0.5 + 396.382\,i$	0.379872	700	$0.5 + 1062.92\,i$	0.233906
250	$0.5 + 470.774\,i$	0.193655	750	$0.5 + 1123.1\,i$	0.478165
300	$0.5 + 541.847\,i$	0.693494	800	$0.5 + 1183.71\,i$	0.13398
350	$0.5 + 611.774\,i$	0.272982	850	$0.5 + 1243.54\,i$	0.00436351
400	$0.5 + 679.742\,i$	0.122888	900	$0.5 + 1302.35\,i$	0.336821
450	$0.5 + 746.499\,i$	0.234496	950	$0.5 + 1361.39\,i$	0.0592523
500	$0.5 + 811.184\,i$	0.263354	1000	$0.5 + 1419.42\,i$	0.0952836
550	$0.5 + 875.985\,i$	0.416264	1050	$0.5 + 1477.44\,i$	0.170379

Table 1. Some zeros of zeta Riemann function.

Now, suppose we are interested in finding a good approximate value for the 123456789-nth zero. Let $\delta = 0.01$. After only three iterations the proposed method (2.5) gives

$$\rho_{123456789} = 0.5 + 51962348.20140442997217178i$$

with accuracy of with $|\zeta(\rho_{123456789})| = 1.0056 \times 10^{-8}$. This says that the method is highly accurate. The Chebyshv-Halley Method gives the value $\rho_{123456789} = 51962348.186967$ with $|\zeta(\rho_{123456789})| = 0.61524$. Using the Newton-Raphson Method, we obtain a less accurate result.

The main problem in finding zeroes of Zeta Riemann function lies on its evaluation of $\zeta(0.5 + it)$ for large t. For $n = 10^{13}$, our method gives $\rho_{10^{13}} = 0.5 + 2445999556030.2470703125i$ with $|\zeta(\rho_{10^{13}})| = 0.0302637$.

3 Conclusions

We proposed a method that is very effective for finding roots of such functions as the Zeta Riemann function. The method employed here does not make use of evaluating the derivatives of the function whose roots we want to find. Instead, we use finite differences formulas. The accuracy will depend on the choice of the parameter value δ. We expect that this method will be useful for solving other transcendental equations.

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