International Journal of Mathematics and
Computer Science, **16**(2021), no. 4, 1653–1664

Neo balancing numbers

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(Received April 5, 2021, Accepted May 26, 2021)

Abstract

If a positive integer $n \geq 2$ is a solution of the equation

$$1+2+3+\cdots+(n-1) = (n-1)+(n-0)+(n+1)+(n+2)+\cdots+(n+r)$$

for some integer r, n is called a neo balancing number and r is called a neo balancer corresponding to neo balancing number n. The purpose of this paper is to establish a generating function of neo balancing numbers and recurrence relations for neo balancing numbers. Moreover, we prove the relations between neo balancing numbers and balancing numbers.

1 Introduction

The definition of balancing numbers was introduced by Behera and Panda [3]. An integer $n \in \mathbb{Z}^+$ is called a balancing number if n is a solution of

$$1+2+3+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$$

Key words and phrases: Neo balancing numbers, balancing numbers, Fibonacci numbers, Diophantine equation.
AMS (MOS) Subject Classifications: 11B37, 11B39.
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ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

for some $r \in \mathbb{Z}^+$. So r is called the balancer corresponding to the balancing number n. Then, they found that n is a balancing number if and only if n^2 is a triangular number. Also n is a balancing number if and only if $8n^2+1$ is a perfect square. In addition, they found the generating function of balancing numbers, the non-linear first order recurrence, the second order linear recurrence, recurrence relations for balancing numbers, nonrecursive form for balancing numbers and an application of balancing numbers to a Diophantine equation. For example, 6, 35, 204 and 1189 are balancing numbers with balancers 2, 14, 84 and 492, respectively. The n^{th} balancing number is denoted by B_n, the n^{th} balancer is denoted by R_n and the n^{th} Lucas-balancing number is denoted by $C_n = \sqrt{8B_n^2+1}$. Set initial values $B_1 = 1$ and $B_2 = 6$ and so on to standardize the notation at par with Fibonacci numbers [6]. Some results established by Behera and Panda [3] can be stated with this new convention as follows:
The second order linear recurrence is

$$B_{n+1} = 6B_n - B_{n-1} \text{ for } n \geq 2.$$

Remark 1.1. *We can set* $B_0 = 6B_1 - B_2 = 6(1) - 6 = 0$.

The non-linear first order recurrences are

$$B_{n+1} = 3B_n + \sqrt{8B_n^2+1}; \qquad n = 2,\ 3, \ldots$$

and

$$B_{n-1} = 3B_n - \sqrt{8B_n^2+1}; \qquad n = 1,\ 2, \ldots.$$

The numbers B_n also satisfy the following relation:

$$B_n = B_{r+1}B_{n-r} - B_rB_{n-r-1}; \qquad r = 1,\ 2, \ldots,\ n-2.$$

The Binet form is

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}; \qquad n = 1,\ 2, \ldots$$

where $\lambda_1 = 3+\sqrt{8}$ and $\lambda_2 = 3-\sqrt{8}$.
Another interesting relation is

$$B_{n+1}B_{n-1} = (B_n+1)(B_n-1) \qquad n = 1,\ 2, \ldots.$$

Some results on balancing number and balancer are

$$B_n = \frac{(2R_n+1) + \sqrt{8R_n^2+8R_n+1}}{2}; \qquad n = 1,\ 2, \ldots$$

and

$$R_n = \frac{-(2B_n+1)+\sqrt{8B_n^2+1}}{2}; \qquad n = 1,\ 2, \ldots.$$

Later on, Panda [6] established some other interesting arithmetic-type, de-Moivre's-type and trigonometric-type properties of balancing numbers. Panda also established an important property concerning the greatest common divisor of two balancing numbers.

2 Neo balancing numbers

Let n be a positive integer solution of

$$1+2+3+\cdots+(n-1) = (n-1)+(n+0)+(n+1)+(n+2)+\cdots+(n+r) \quad (2.1)$$

for some integer r. Then n is called the neo balancing number and r is called the neo balancer corresponding to the neo balancing number n. For example, 2, 7, 36 and 205 are neo balancing numbers with neo balancers -1, 1, 13 and 83, respectively. Since we can rewrite equation (2.1) in the form

$$(n-1)^2 = \frac{(n+r)(n+r+1)}{2} \quad (2.2)$$

and

$$r = \frac{-(2n+1)+\sqrt{8(n-1)^2+1}}{2}, \quad (2.3)$$

we will obtain the precious things. From (2.2), n is a neo balancing number if and only if $(n-1)^2$ is a triangular number. Also, by (2.3), n is a neo balancing number if and only if $8(n-1)^2+1$ is a perfect square. In this paper the n^{th} neo balancing number is denoted by P_n and the n^{th} Lucas-balancing number is denoted by $Q_n = \sqrt{8(P_n-1)^2+1}$. Set initial values $P_1 = 2$ and $P_2 = 7$ and so on.

2.1 Function generating neo balancing numbers

In this section we will show some functions that generate neo balancing numbers. We will show that if x is a balancing number, then

$$p(x) = 3(x-1)+1+\sqrt{8(x-1)^2+1}$$

is also a neo balancing numbers.

Theorem 2.1. *For any neo balancing number x, $p(x)$ and $(p \circ p \circ \cdots \circ p)(x)$ are also neo balancing numbers.*

Proof. Since x is a neo balancing number, we obtain $8(x-1)^2+1$ is a perfect square and we have

$$\begin{aligned} 8(p(x)-1)^2+1 &= 8(9)(x-1)^2 + 2(8)(3)(x-1)\sqrt{8(x-1)^2+1} + 8^2(x-1)^2 + 9 \\ &= (8(x-1))^2 + 2[(8)(x-1)][(3)\sqrt{8(x-1)^2+1}] + [3\sqrt{8(x-1)^2+1}]^2 \\ &= [8(x-1) + 3\sqrt{8(x-1)^2+1}]^2 \end{aligned}$$

is a perfect square too. Then $p(x)$ is a neo balancing number. By applying $p(x)$ repeatedly, it follows that $(p \circ p \circ \cdots \circ p)(x)$ is also a neo balancing number.

2.2 Finding the next neo balancing numbers

We have shown that $p(x)$ generate neo balancing numbers. In this section we will show in addition that $p(x)$ is the function generating next neo balancing numbers.

Theorem 2.2. *If x is a neo balancing number, then the next neo balancing number is $p(x) = 3(x-1)+1+\sqrt{8(x-1)^2+1}$ and consequently, the previous one is $p^{-1}(x) = 3(x-1)+1-\sqrt{8(x-1)^2+1}$.*

Proof. Define a function $p : [0,\infty) \to [2,\infty)$, by $p(x) = 3(x-1)+1+\sqrt{8(x-1)^2+1}$. Since

$$p'(x) = 3 + \frac{8(x-1)}{\sqrt{8(x-1)^2+1}} > 0,$$

p is strictly increasing. It is clear that $x < p(x)$, so p is injective. Thus p^{-1} exists. Since p is strictly increasing, p^{-1} is also strictly increasing. Let $u(x) = p^{-1}(x)$. Thus $u(x) = 3(x-1)+1 \pm \sqrt{8(x-1)^2+1}$. Since p is strictly increasing, $p(x) \neq p^{-1}(x)$. Then we obtain $u = 3(x-1)+1-\sqrt{8(x-1)^2+1}$. Since n is a neo balancing number if and only if $8(n-1)^2+1$ is a perfect square and $8(u-1)^2+1 = [8(u-1)-3\sqrt{8(u-1)^2+1}]^2$, we have $u = p^{-1}(x)$ is a neo balancing number. We let $P_n = p(P_{n-1})$ such that $P_0 = 1$ for $n = 1, 2, \ldots$. Thus, $P_1 = 2$, $P_2 = 7$, $P_3 = 36$ and so on. Now we will prove by the method of induction.

Let H_i be the hypothesis that there is no neo balancing number between P_i and P_{i+1}. Since $P_1 = 2$ and $P_2 = 7$ it is clear that there is no neo balancing number between P_1 and P_2. Assume H_n is true. We will show that H_{n+1} is true, by contradiction. Suppose H_{n+1} is false, so there is a neo balancing number δ such that $P_{n+1} < \delta < P_{n+2}$. Thus, $p^{-1}(P_{n+1}) < p^{-1}(\delta) < p^{-1}(P_{n+2})$. We will get $P_n < p^{-1}(\delta) < P_{n+1}$. It is a contradiction, so H_{n+1} is true. Therefore the neo balancing number next to x is $p(x)$. Since $p(p^{-1}(x)) = x$, it follows that $p^{-1}(x)$ is the largest neo balancing number less than x.

3 Properties of neo balancing numbers and balancing numbers

3.1 Recurrence relations between neo balancing numbers and balancing numbers

We have known that $P_0 = 1$ $P_1 = 2$, $P_2 = 7$, $P_3 = 36$, and so on. If P_n is the n^{th} neo balancing number, then

$$P_{n+1} = 3(P_n - 1) + 1 + \sqrt{8(P_n - 1)^2 + 1} \tag{3.4}$$

and

$$P_{n-1} = 3(P_n - 1) + 1 - \sqrt{8(P_n - 1)^2 + 1}. \tag{3.5}$$

It is clear from (3.4) and (3.5) that the neo balancing numbers obey the following recurrence relation:

$$P_{n+1} = 6P_n - P_{n-1} - 4 \tag{3.6}$$

or

$$\overline{P_{n+1}} = 6\overline{P_n} - \overline{P_{n-1}} \tag{3.7}$$

where $\overline{P_n} = P_n - 1$. Also, the balancing numbers obey the following recurrence relation [3]:

$$B_{n+1} = 6B_n - B_{n-1}. \tag{3.8}$$

Using the recurrence relation (3.6), (3.7) and (3.8) we can obtain some other interesting relations concerning neo balancing numbers and balancing numbers.

Theorem 3.1. *Let B_n be the n^{th} balancing number, P_n be the n^{th} neo balancing number and $1 \le k \le n$ for any positive integers n and k. Then we have the following relations.*

(a) $P_{n+1}P_{n-1} = (P_n + 5)(P_n - 1)$.
(b) $P_{n+1}P_{n-1} + 9 = (P_n + 2)^2$.
(c) $P_n = B_k\overline{P_{n-k}} - B_{k-1}\overline{P_{n-k-1}} + 1$.
(d) $P_n = B_{n-1} + 1$.
(e) $B_n = P_{n+1} - 1$.
(f) $P_n \cdot B_n = P_nP_{n+1} - P_n$.
(g) $P_n = \overline{P_{k+1}} \cdot \overline{P_{n-k}} - \overline{P_k} \cdot \overline{P_{n-k-1}} + 1$.
(h) $P_{2n+1} = \overline{P_{n+1}}^2 - \overline{P_n}^2 + 1$.
(i) $P_{2n} = \overline{P_n}(\overline{P_{n+1}} - \overline{P_{n-1}}) + 1$.

Proof. The proof of (a) and (b) directly follow from (3.4) and (3.5). The proof of (c) is base on mathematical induction on k. Clearly, (c) is true for $n > 1$ and $k = 1$. From (3.7) and assume that (c) is true for $k = r$, i.e., $P_n = B_r\overline{P_{n-r}} - B_{r-1}\overline{P_{n-r-1}} + 1$ Thus,

$$\begin{aligned} B_{r+1}\overline{P_{n-r-1}} - B_r\overline{P_{n-r-2}} + 1 &= (6B_r - B_{r-1})\overline{P_{n-r-1}} - B_r\overline{P_{n-r-2}} + 1 \\ &= 6B_r\overline{P_{n-r-1}} - B_{r-1}\overline{P_{n-r-1}} - B_r\overline{P_{n-r-2}} + 1 \\ &= B_r(6\overline{P_{n-r-1}} - \overline{P_{n-r-2}}) - B_{r-1}\overline{P_{n-r-1}} + 1 \\ &= B_r\overline{P_{n-r}} - B_{r-1}\overline{P_{n-r-1}} + 1 = P_n. \end{aligned}$$

Therefore, (c) is true for $k = r + 1$. This completes the proof of (c). The proofs of (d) and (e) follow by replacing $k = n - 1$ in (c) and n by$n + 1$ in (d), respectively. From (e), it follows that

$$P_n \cdot B_n \;=\; P_n(P_{n+1} - 1) = P_nP_{n+1} - P_n.$$

This completes the proof of (f). From (c) and (e), it follows that

$$\begin{aligned} P_n = B_k\overline{P_{n-k}} - B_{k-1}\overline{P_{n-k-1}} + 1 &= (P_{k+1} - 1)\overline{P_{n-k}} - (P_k - 1)\overline{P_{n-k-1}} + 1 \\ &= \overline{P_{k+1}} \cdot \overline{P_{n-k}} - \overline{P_k} \cdot \overline{P_{n-k-1}} + 1. \end{aligned}$$

This completes the proof of (g). Finally, the proof of (h) follows by replacing n by $2n+1$ and k by n in (g). Similarly, the proof of (i) follows by replacing n by $2n$ and k by n in (g). This completes the proof of Theorem 3.1.

3.2 Nonrecursive form for neo balancing numbers

In this section, we shall obtain another nonrecursive form for P_n by solving the recurrence relation (3.7) as a difference equation. We rewrite the recurrence relation (3.7) in the form

$$\overline{P_{n+1}} - 6\overline{P_n} + \overline{P_{n-1}} = 0. \tag{3.9}$$

Then we have a second-order linear homogeneous difference equation whose auxiliary equation is

$$\lambda^2 - 6\lambda + 1 = 0. \tag{3.10}$$

The roots $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$ of (3.10) are real and unequal. Thus

$$\overline{P_n} = A\lambda_1^n + B\lambda_2^n. \tag{3.11}$$

Solving for A and B, we obtain

$$A = \frac{1}{\lambda_1 - \lambda_2} \quad \text{and} \quad B = -\frac{1}{\lambda_1 - \lambda_2}.$$

Substituting these values into (3.11), we get

$$\overline{P_n} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\ ;\ n = 0, 1, 2, 3, \ldots$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$. Then we obtain the following theorem for neo balancing numbers.

Theorem 3.2. *If P_n is the n^{th} neo balancing number, then its Binet form is*

$$\overline{P_n} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\ ;\ n = 0, 1, 2, 3, \ldots,$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

Theorem 3.3. *If a and b are natural numbers and $a > b$, then*

$$\overline{P_{a+b}} \cdot \overline{P_{a-b}} = (\overline{P_a} - \overline{P_b})(\overline{P_a} + \overline{P_b}).$$

Proof. The proof directly follows from Theorem 3.2.

Theorem 3.4. *If a and b are natural numbers and $a > b$, then*

$$\frac{\overline{P_{a+b}}}{\overline{P_{a-b}}} = \frac{(\overline{P_a} - \overline{P_b})(\overline{P_a} + \overline{P_b})}{\overline{P_{a-b}}^2}.$$

Proof. The proof directly follows from Theorem 3.3.

4 The connection of analogous properties of neo balancing numbers with some properties

Panda [6] introduced some fascinating properties of balancing numbers. In this section we will introduce some properties of neo balancing numbers as follows.

4.1 Arithmetic properties of neo balancing numbers

We know that, if x and y are real or complex numbers, then $(x+y)(x-y) = x^2 - y^2$. So we obtained an analogous property of neo balancing numbers as Theorem 3.3.

Theorem 4.1. *If n is a natural number, then*
(a) $\overline{P_1} + \overline{P_3} + \overline{P_5} + \cdots + \overline{P_{2n-1}} = \overline{P_n}^2$.
(b) $\overline{P_2} + \overline{P_4} + \overline{P_6} + \cdots + \overline{P_{2n}} = \overline{P_n} \cdot \overline{P_{n+1}}$.
(c) $\overline{P_1} + \overline{P_2} + \overline{P_3} + \cdots + \overline{P_{2n}} = \overline{P_n}(\overline{P_n} + \overline{P_{n+1}})$.

Proof. From relation (h) of Theorem 3.1, we obtain $\overline{P_{2n+1}} = \overline{P_{n+1}}^2 - \overline{P_n}^2$, so (a) follows. From relation (i) of Theorem 3.1, we replace n by $n+1$ as follows $\overline{P_{2n+2}} = \overline{P_{n+2}} \cdot \overline{P_{n+1}} - \overline{P_{n+1}} \cdot \overline{P_n}$, so (b) follows. Finally, the identity (c) directly follows from (a) and (b).

4.2 De-Moivre properties of neo balancing numbers

The complex identity $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$ is known as the de-Moivre's formula [1]. The following theorem looks like de-Moivre's formula.

Theorem 4.2. *If n and r are natural numbers, then* $(Q_n + \sqrt{8}\overline{P_n})^r = Q_{nr} + \sqrt{8}\overline{P_{nr}}$.

Proof. Form Theorem 3.2, we obtain

$$Q_n^2 = 8\overline{P_n}^2 + 1 = 8\left[\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\right] + 1 = \left[\frac{\lambda_1^n - \lambda_2^n}{2}\right]^2.$$

Therefore,

$$Q_n = \left[\frac{\lambda_1^n - \lambda_2^n}{2}\right].$$

Since

$$Q_n + \sqrt{8}\overline{P_n} = \left[\frac{\lambda_1^n - \lambda_2^n}{2}\right] + \sqrt{8}\left[\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\right] = \lambda_1^n,$$

we obtain $(Q_n + \sqrt{8}\overline{P_n})^r = (\lambda_1^n)^r = \lambda_1^{nr} = Q_{nr} + \sqrt{8}\overline{P_{nr}}$.

Remark 4.3. *The Fibonacci numbers satisfy a similar property*

$$\left[\frac{L_n + \sqrt{5}F_n}{2}\right]^r = \frac{L_{rn} + \sqrt{5}F_{rn}}{2}.$$

Corollary 4.4. *If n and r are natural numbers, then*

$$(Q_n - \sqrt{8}\overline{P_n})^r = Q_{nr} - \sqrt{8}\overline{P_{nr}}.$$

Proof. Since

$$Q_n - \sqrt{8}\overline{P_n} = \left[\frac{\lambda_1^n - \lambda_2^n}{2}\right] - \sqrt{8}\left[\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\right] = \lambda_2^n,$$

the result follows easily.

4.3 Trigonometric properties of neo balancing numbers

The trigonometric identity

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

is quite well known. Then The following theorem looks like Trigonometric formula.

Theorem 4.5. *If m and n are natural numbers, then*

$$\overline{P_{m+n}} = Q_m\overline{P_n} + Q_n\overline{P_m}.$$

Proof. Since

$$(Q_m + \sqrt{8}\overline{P_m})(Q_n + \sqrt{8}\overline{P_n}) = \lambda_1^m\lambda_1^n = \lambda_1^{m+n} = Q_{m+n} + \sqrt{8}\overline{P_{m+n}}$$

and

$$(Q_m + \sqrt{8}\overline{P_m})(Q_n + \sqrt{8}\overline{P_n}) = Q_mQ_n + 8\overline{P_mP_n} + \sqrt{8}(Q_n\overline{P_m} + Q_m\overline{P_n}),$$

we compare them and equate the irrational part from both sides to obtain

$$\overline{P_{m+n}} = Q_m\overline{P_n} + Q_n\overline{P_m}.$$

Remark 4.6. *The Fibonacci numbers satisfy a similar property*

$$F_{m+n} = \frac{F_mL_n + L_mF_n}{2}.$$

Corollary 4.7. *If m and n are natural numbers, then*

$$Q_{m+n} = Q_mQ_n + 8\overline{P_m}\ \overline{P_n}.$$

Proof. This can be proven using the arguments in the proof of Theorem 4.5.

Corollary 4.8. *If m is natural number, then*

$$\overline{P_{2m}} = 2\overline{P_m}Q_m.$$

Proof. From Theorem 4.5, we replace n by m.

Remark 4.9. *The Fibonacci numbers satisfy a similar property*

$$F_{2n} = F_n L_n.$$

4.4 Properties concerning the greatest common divisor of two balancing numbers

Theorem 4.10. *If m and n are natural numbers, then $\overline{P_m}$ divides $\overline{P_n}$ if and only if m divides n.*

To prove the above theorem we need the following lemmas.

Lemma 4.11. *If m and n are natural numbers, then* $\gcd(\overline{P_n}, Q_n) = 1$.

Proof. Since we have defined $Q_n = \sqrt{8\overline{P_n}^2 + 1}$, we obtain $Q_n^2 = 8\overline{P_n}^2 + 1$. Then $\gcd(\overline{P_n}^2, Q_n^2) = 1$ and thus $\gcd(\overline{P_n}, Q_n) = 1$.

Lemma 4.12. *If n and k are natural numbers, then $\overline{P_k}$ divides $\overline{P_{nk}}$.*

Proof. The proof is based on mathematical induction. Clearly, this lemma is true for $n = 1$. Assuming that this lemma is true for $n = r$, we obtain $\overline{P_k}$ divides $\overline{P_{rk}}$. By Theorem 4.5 , we have

$$\begin{aligned}\overline{P_{(r+1)k}} &= \overline{P_{rk+k}} \\ &= \overline{P_{rk}}Q_k + Q_{rk}\overline{P_k}.\end{aligned}$$

Thus, we obtain that $\overline{P_k}$ divides $\overline{P_{(r+1)k}}$.

Lemma 4.13. *If n and k are natural numbers, then* $\gcd(\overline{P_k}, Q_{nk}) = 1$.

Proof. Since we have shown that $\gcd(\overline{P_n}, Q_n) = 1$ and $\overline{P_k}$ divides $\overline{P_{nk}}$, we obtain that $\gcd(\overline{P_k}, Q_{nk}) = 1$.

Lemma 4.14. *If n and k are natural numbers and $\overline{P_k}$ divides $\overline{P_n}$, then k divides n.*

Proof. Obviously, $n \geq k$ and this lemma is true for $n = k$. By Euclid's division lemma, there exist integers q and r such that $q \geq 1$, $0 \leq r < k$ and $n = qk + r$. By Theorem 4.5, we have $\overline{P_n} = \overline{P_{qk+r}} = \overline{P_{qk}}Q_r + Q_{qk}\overline{P_r}$. Since $\overline{P_k}$ divides $\overline{P_{qk}}$ and $\gcd(\overline{P_k}, Q_{nk}) = 1$ by previous lemmas, we obtain that $\overline{P_k}$ divides $\overline{P_r}$. Since $r < k$, we obtain that $\overline{P_r} = 0$. Thus, we have $r = 0$ and hence $n = qk$. Therefore k divides n.

Theorem 4.10 directly follows from Lemmas 4.12 and 4.14.

Remark 4.15. *The Fibonacci numbers satisfy a similar property F_m divides F_n if and only if m divides n.*

The following theorem gives a stronger result.

Theorem 4.16. *If m and n are natural numbers, then*

$$\gcd(\overline{P_m}, \overline{P_n}) = \overline{P_{\gcd(m,n)}}.$$

Proof. If $m = n$, then the proof is trivial. Assume without loss of generality that $m < n$. By Euclid's division lemma, there exist integers q_1 and r_1 such that $q_1 \geq 1$, $0 \leq r_1 < m$ and $n = q_1m + r_1$. By Theorem 4.5, we have $\gcd(\overline{P_m}, \overline{P_n}) = \gcd(\overline{P_m}, \overline{P_{q_1m+r_1}}) = \gcd(\overline{P_m}, \overline{P_{q_1m}}Q_{r_1} + Q_{q_1m}\overline{P_{r_1}})$. Since $\overline{P_m}$ divides $\overline{P_{q_1m}}$ and $\gcd(\overline{P_m}, Q_{q_1m}) = 1$, we have $\gcd(\overline{P_m}, \overline{P_n}) = \gcd(\overline{P_m}, \overline{P_{r_1}})$ and $\gcd(m, n) = \gcd(m, q_1m + r_1) = \gcd(m, r_1)$. If $r_1 > 0$, then there exist integers q_2 and r_2 such that $q_2 \geq 1$, $0 \leq r_2 < r_1$ and $m = q_2r_1 + r_2$ such that $\gcd(\overline{P_m}, \overline{P_n}) = \gcd(\overline{P_m}, \overline{P_{r_1}}) = \gcd(\overline{P_{q_2r_1+r_2}}, \overline{P_{r_1}}) = \gcd(\overline{P_{q_2r_1}}Q_{r_2} + Q_{q_2r_1}\overline{P_{r_2}}, \overline{P_{r_1}})$. Since $\overline{P_{r_1}}$ divides $\overline{P_{q_2r_1}}$ and $\gcd(\overline{P_{r_1}}, Q_{q_2r_1}) = 1$, we have $\gcd(\overline{P_m}, \overline{P_n}) = \gcd(\overline{P_{r_2}}, \overline{P_{r_1}})$ and $\gcd(m, n) = \gcd(q_2r_1 + r_2, r_1) = \gcd(r_2, r_1)$. The process may be continued till exists $r_i \neq 0$. Since $r_1 > r_2 > \cdots$, it follows that $r_i \leq m - i$, so that after at most m steps some r_i will be equal to zero. If $r_{k-1} > 0$ and $r_k = 0$, then $\gcd(\overline{P_m}, \overline{P_n}) = \gcd(\overline{P_{r_{k-2}}}, \overline{P_{r_{k-1}}}) = \gcd(\overline{P_{q_kr_{k-1}}}, \overline{P_{r_{k-1}}}) = \overline{P_{r_{k-1}}}$ and $\gcd(m, n) = \gcd(r_{k-2}, r_{k-1}) = \gcd(q_kr_{k-1}, r_{k-1}) = r_{k-1}$. Thus, $\gcd(\overline{P_m}, \overline{P_n}) = \overline{P_{r_{k-1}}} = \overline{P_{\gcd(m,n)}}$.

Remark 4.17. *The Fibonacci numbers satisfy a similar property*

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}.$$

Acknowledgment. This work is supported by Faculty of Science, Burapha University, Thailand.

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