Structural Properties and Submonoids of Generalized Cohypersubstitutions

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Abstract

Generalized cohypersubstitutions of type $\tau = (n_i)_{i \in I}$ are mappings which send the $n_i$-ary cooperations symbols to coterms of type $\tau$. We define the operations $+_CG$ and $\oplus CG$ on the set $Cohyp_G(\tau)$ and consider some submonoids of the monoid $Cohyp_G(\tau)$. Finally, we give some structural properties and the relationship among submonoids.

1 Introduction

The topic cohypersubstitution of type $\tau$ in universal algebra has gained interest among many authors. In 2009, Denecke and Seangsura [2] initially introduced and used the main tool in the study of cohyperidentities. They defined coalgebras, coidentities, cohyperidentities and applied all the concepts to construct the monoid of cohypersubstitutions of type $\tau$. In 2013,
Jermjitpornchai and Seangsura [4] generalized the concepts in [2] by studying generalized cohypersubstitutions. They introduced the coterms involving generalized superpositions, discovered some algebraic-structural properties, and constructed the monoid of generalized cohypersubstitutions of type \( \tau = (n_i)_{i \in I} \). The structural properties and special elements of the monoid of generalized cohypersubstitutions of type \( \tau = (2) \), \( \tau = (3) \) and \( \tau = (n) \) has since been studied by many other authors. In this paper we focus on the generalized cohypersubstitutions of type \( \tau \) and define the operation on \( \text{Cohyp}_G(\tau) \), and give some algebra-structural properties of the set of all generalized cohypersubstitutions. In addition, we focus on some submonoid of the monoid \( \text{Cohyp}_G(\tau) \), and study some structural properties and the relationship among submonoids.

2 Generalized Cohypersubstitutions

In this section we provide the basic concept of the monoid of all generalized cohypersubstitutions which is very useful in this paper.

Let \( A \) be a non-empty set and \( n \in \mathbb{N} \). Define the union of \( n \) disjoint copies of \( A \) by \( A^{\text{lin}} := n \times A \), where \( n = \{1, 2, \ldots, n\} \). This is called the \( n \)-th copower of \( A \). An element \((i, a)\) in this copower corresponds to the element \( a \) in the \( i \)-th copy of \( A \), where \( i \in n \). For some natural number \( n \), a mapping \( f^A : A \rightarrow A^{\text{lin}} \) is a co-operation on \( A \); \( n \) is called the arity of the co-operation \( f^A \). Every \( n \)-ary co-operation \( f^A \) on the set \( A \) can be uniquely expressed as the pair of mappings \((f^A_1, f^A_2)\), where \( f^A_1 : A \rightarrow n \) gives the labeling used by \( f^A \) of mapping elements to copies of \( A \), and \( f^A_2 : A \rightarrow A \) shows what element of \( A \) any element is mapped to, so \( f^A(a) = (f^A_1(a), f^A_2(a)) \). We denote the set of all \( n \)-ary co-operations defined on \( A \) by \( cO_A^{(n)} = \{f^A : A \rightarrow A^{\text{lin}}\} \).

Let \( \tau = (n_i)_{i \in I} \) and \((f_i)_{i \in I}\) be an indexed set of co-operation symbols which \( f_i \) has arity \( n_i \) for each \( i \in I \). Let \( \bigcup \{e^n_j | n \geq 1, n \in \mathbb{N}, 0 \leq j \leq n - 1\} \) be a set of symbols which disjoint from \( \{f_i | i \in I\} \) such that \( e^n_j \) has arity \( n \) for each \( 0 \leq j \leq n - 1 \). The coterms of type \( \tau \) are defined as follows:

(i) For every \( i \in I \) the co-operation symbol \( f_i \) is an \( n_i \)-ary coterm of type \( \tau \).

(ii) For every \( n \geq 1 \) and \( 0 \leq j \leq n - 1 \) the symbol \( e^n_j \) is an \( n \)-ary coterm of type \( \tau \).

(iii) If \( t_1, \ldots, t_n \) are \( n \)-ary coterms of type \( \tau \), then for every \( i \in I \), \( f_i[t_1, \ldots, t_n] \) is an \( n \)-ary coterm of type \( \tau \), and if \( t_0, \ldots, t_{n-1} \) are \( m \)-ary coterm of
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Denoted by $CT'_\tau(n)$ the set of all $n$-ary coterm of type $\tau$, and $CT'\tau := \bigcup_{n \geq 1} CT'_\tau(n)$ the set of all coterm of type $\tau$.

Let $m \in \mathbb{N}^n$. A *generalized superposition* of a coterm $S'^m : CT'^m_m \times CT'\tau \to CT'\tau$ defined inductively by the following steps:

(i) If $t = e^n_i$ and $0 \leq i \leq m-1$, then $S'^m(e^n_i, t_0, \ldots, t_{m-1}) = e^n_i$, where $t_0, \ldots, t_{m-1} \in CT'\tau$.

(ii) If $t = e^n_i$ and $0 < m \leq i \leq n-1$, then $S'^m(e^n_i, t_0, \ldots, t_{m-1}) = e^n_i$, where $t_0, \ldots, t_{m-1} \in CT'\tau$.

(iii) If $t = f_i[s_1, \ldots, s_n]$, then

\[
S'^m(t, t_1, \ldots, t_m) = f_i(S'^m(s_1, t_1, \ldots, t_m), \ldots, S'^m(s_n, t_1, \ldots, t_m)),
\]

where $S'^m(s_1, t_1, \ldots, t_m), \ldots, S'^m(s_n, t_1, \ldots, t_m) \in CT'\tau$.

The above definition can be written as the following forms:

(i) If $t = e^n_i$ and $0 \leq i \leq m-1$, then $e^n_i[t_0, \ldots, t_{m-1}] = e^n_i$, where $t_0, \ldots, t_{m-1} \in CT'\tau$.

(ii) If $t = e^n_i$ and $0 < m \leq i \leq n-1$, then $e^n_i[t_0, \ldots, t_{m-1}] = e^n_i$, where $t_0, \ldots, t_{m-1} \in CT'\tau$.

(iii) If $t = f_i[s_1, \ldots, s_n]$, then

\[
(f_i[s_1, \ldots, s_n])[t_1, \ldots, t_m] = f_i(s_1[t_1, \ldots, t_m], \ldots, s_n[t_1, \ldots, t_m]),
\]

where $s_1[t_1, \ldots, t_m], \ldots, s_n[t_1, \ldots, t_m] \in CT'\tau$.

**Definition 2.1.** [1] For arbitrary coterm $t, t_0, \ldots, t_{m-1} \in CT'\tau$,

\[
t[s_0[t_0, \ldots, t_{m-1}], \ldots, s_{n-1}[t_0, \ldots, t_{m-1}]] = (t[s_0, \ldots, s_{n-1}])[t_0, \ldots, t_{m-1}].
\]

A *generalized co hypersubstitution* of type $\tau$ is a mapping $\sigma : \{f_i \mid i \in I\} \to CT'\tau$. The extension of $\sigma$ is a mapping $\hat{\sigma} : CT'\tau \to CT'\tau$ which is inductively defined by the following steps:

(i) $\hat{\sigma}(e^n_j) := e^n_j$ for every $n \geq 1$ and $0 \leq j \leq n-1$,

(ii) $\hat{\sigma}(f_i) := \sigma(f_i)$ for every $i \in I$,

(iii) $\hat{\sigma}(f_i[t_1, \ldots, t_n]) := \sigma(f_i)[\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]]$ for $t_1, \ldots, t_n \in CT'^{(n)}$.
Denoted by $\text{Cohyp}_G(\tau)$ the set of all generalized cohypersubstitutions of type $\tau$.

**Definition 2.2.** [4] If $t, t_1, \ldots, t_n \in CT$ and $\sigma \in \text{Cohyp}_G(\tau)$, then

$$\hat{\sigma}(t[t_1, \ldots, t_n]) = \hat{\sigma}(t)[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)].$$

**Proposition 2.3.** [1] For arbitrary coterms $t, t_0, \ldots, t_{n-1}$ and $\sigma_1, \sigma_2 \in \text{Cohyp}_G(\tau)$,

$$(\hat{\sigma}_1 \circ \hat{\sigma}_2) = \hat{\sigma}_1 \circ \hat{\sigma}_2.$$

Define a binary operation $\circ_{CG} : \text{Cohyp}_G(\tau) \times \text{Cohyp}_G(\tau) \to \text{Cohyp}_G(\tau)$ on the set of all generalized cohypersubstitutions, $\text{Cohyp}_G(\tau)$, by $\sigma_1 \circ_{CG} \sigma_2 := \hat{\sigma}_1 \circ \hat{\sigma}_2$ for all $\sigma_1, \sigma_2 \in \text{Cohyp}_G(\tau)$ where $\circ$ is the usual composition of mappings. Let $\sigma_{id}$ be the generalized cohypersubstitution such that $\sigma_{id}(f_i) := f_i$ for all $i \in I$. Then $\sigma_{id}$ is an identity element in $\text{Cohyp}_G(\tau)$. So, $\text{Cohyp}_G(\tau) := (\text{Cohyp}_G(\tau), \circ_{CG}, \sigma_{id})$ is a monoid and called the monoid of generalized cohypersubstitutions of type $\tau$. The algebraic structural-properties of the monoid $\text{Cohyp}_G(\tau)$, see [4].

Throughout this paper, we use the following notations:

- $\sigma_t :=$ the generalized cohypersubstitution $\sigma$ of type $\tau$ which maps $f$ to the coterms $t$,
- $e^n_j :=$ the injection symbol for all $0 \leq j \leq n - 1, n \in \mathbb{N}$,
- $E :=$ the set of all injection symbols i.e. $E := \{e^n_j \mid n, j \in \mathbb{N}\}$,
- $E(t) :=$ the set of all injection symbols which occur in the coterms $t$.

## 3 Algebraic-structural Properties of Generalized Cohypersubstitutions

**Definition 3.1.** A nonempty set $R$ together with two binary operations, denoted by $+$ and $\cdot$ respectively, is said to be a left(right) seminear-ring if $(R, +)$ and $(R, \cdot)$ are semigroups and satisfy the left (right) distributive law; i.e., for all $a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c((a + b) \cdot c = a \cdot c + b \cdot c)$.

Now, we define the binary operation $+_{CG}$ on the set $\text{Cohyp}_G(\tau)$ by

$$(\sigma_1 +_{CG} \sigma_2)(f_i) := (\sigma_2(f_i))[\sigma_1(f_i), \ldots, \sigma_1(f_i)] \in CT,$$

for all $\sigma_1, \sigma_2 \in \text{Cohyp}_G(\tau)$. Then we have the following propositions:
Proposition 3.2. For any \( \sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau) \) and \( i \in I \),
\[
\sigma_1 + CG (\sigma_2 + CG \sigma_3) = (\sigma_1 + CG \sigma_2) + CG \sigma_3.
\]

Proof. Let \( \sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau) \). We consider
\[
(\sigma_1 + CG (\sigma_2 + CG \sigma_3))(f_i) = (\sigma_2 + CG \sigma_3)(f_i)[\sigma_1(f_i), \ldots, \sigma_1(f_i)]
\]
\[
= (\sigma_3(f_i))[\sigma_1(f_i), \ldots, \sigma_1(f_i)][\sigma_1(f_i), \ldots, \sigma_1(f_i)]
\]
\[
= (\sigma_3(f_i))\left[\sigma_2(f_i), \ldots, \sigma_2(f_i)\right]\left[\sigma_1(f_i), \ldots, \sigma_1(f_i)\right]
\]
\[
= (\sigma_3(f_i))\left[\sigma_1 + CG \sigma_2(f_i), \ldots, \sigma_1 + CG \sigma_2(f_i)\right]
\]
\[
= (\sigma_1 + CG \sigma_2 + CG \sigma_3)(f_i).
\]

Then \( (Cohyp_G(\tau), +CG) \) is a semigroup.

Proposition 3.3. For any \( \sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau) \) and \( i \in I \),
\[
\sigma_1 \circ CG (\sigma_2 + CG \sigma_3) = (\sigma_1 \circ CG \sigma_2) + CG (\sigma_1 \circ CG \sigma_3).
\]

Proof. Let \( \sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau) \). Consider
\[
(\sigma_1 \circ CG (\sigma_2 + CG \sigma_3))(f_i) = \hat{\sigma}_1((\sigma_2 + CG \sigma_3)(f_i))
\]
\[
= \hat{\sigma}_1((\sigma_3(f_i)\left[\sigma_2(f_i), \ldots, \sigma_2(f_i)\right])
\]
\[
= (\hat{\sigma}_1(\sigma_3(f_i))\left[\hat{\sigma}_1(\sigma_2(f_i)), \ldots, \hat{\sigma}_1(\sigma_2(f_i))\right]
\]
\[
= (\sigma_1 \circ CG \sigma_3)(f_i)\left[\sigma_1 \circ CG \sigma_2(f_i), \ldots, \sigma_1 \circ CG \sigma_2(f_i)\right]
\]
\[
= ((\sigma_1 \circ CG \sigma_2) + CG (\sigma_1 \circ CG \sigma_3))(f_i).
\]

For any generalized cohypersubstitutions \( \sigma_1, \sigma_2 \) and \( \sigma_3 \), the right distributive law; i.e., \( (\sigma_1 + CG \sigma_2) \circ CG \sigma_3 = (\sigma_1 \circ CG \sigma_3) + CG (\sigma_2 \circ CG \sigma_3) \), is not true as the following example shows:
Example 3.4. Let $\tau = (2)$; that is, there is a binary cooperation symbol $f$, and $\sigma_1, \sigma_2, \sigma_3 \in \text{Cohyp}_G(\tau)$ such that $\sigma_1(f) = f[e_0^2, f[e_3^2, e_1^2]], \sigma_2(f) = f[f[e_1^2, e_3^2], e_2^2]$, and $\sigma_3(f) = f[e_1^2, e_3^2]$. So,

$$((\sigma_1 \circ CG \sigma_2) \circ CG \sigma_3)(f) = f[f[e_1^2, f[e_3^2, e_2^2]], e_2^2]$$

and

$$((\sigma_1 \circ CG \sigma_3) + CG (\sigma_2 \circ CG \sigma_3))(f) = f[f[e_3^2, e_2^2], e_2^2].$$

Thus $(\sigma_1 + CG \sigma_2) \circ CG \sigma_3 \neq (\sigma_1 \circ CG \sigma_3) + CG (\sigma_2 \circ CG \sigma_3)$.

Then we obtain the following lemma:

Lemma 3.5. $\text{Cohyp}_{G}^{+CG}(\tau) := (\text{Cohyp}_G(\tau), \circ CG, + CG)$ is a left seminear-ring.

Proof. Since $(\text{Cohyp}_G(\tau), \circ CG)$ is a monoid, the proof directly follows from Propositions 3.2 and 3.3. \qed

Next, we define another binary operation $\oplus_{CG}$ on the set $\text{Cohyp}_G(\tau)$ by

$$(\sigma_1 \oplus_{CG} \sigma_2)(f_i) := (\sigma_1(f_i))[\sigma_2(f_i)]^{n_i}_{\text{terms}} \in CT(\tau),$$

for all $\sigma_1, \sigma_2 \in \text{Cohyp}_G(\tau)$. Then we have the following.

Proposition 3.6. For any $\sigma_1, \sigma_2, \sigma_3 \in \text{Cohyp}_G(\tau)$ and $i \in I$,

$$\sigma_1 \oplus_{CG} (\sigma_2 \oplus_{CG} \sigma_3) = (\sigma_1 \oplus_{CG} \sigma_2) \oplus_{CG} \sigma_3.$$

Proof. Let $\sigma_1, \sigma_2, \sigma_3 \in \text{Cohyp}_G(\tau)$. Consider

$$((\sigma_1 \oplus_{CG} \sigma_2) \oplus_{CG} \sigma_3)(f_i) = ((\sigma_1 \oplus_{CG} \sigma_2)(f_i))[\sigma_3(f_i)]^{n_i}_{\text{terms}}$$

$$= (\sigma_1(f_i))[\sigma_2(f_i)]^{n_i}_{\text{terms}}[\sigma_3(f_i)]^{n_i}_{\text{terms}}$$

$$= (\sigma_1(f_i))[\sigma_2(f_i)][\sigma_3(f_i)]^{n_i}_{\text{terms}}[\sigma_3(f_i)], \ldots,$$

$$= (\sigma_1(f_i))[\sigma_2(f_i)][\sigma_3(f_i)]^{n_i}_{\text{terms}}[\sigma_3(f_i))]

= (\sigma_1(f_i))[\sigma_2(\oplus_{CG} \sigma_3)(f_i)], \ldots, (\sigma_2(\oplus_{CG} \sigma_3)(f_i)]

= (\sigma_1(\oplus_{CG} (\sigma_2(\oplus_{CG} \sigma_3)(f_i))+ (\sigma_1(\oplus_{CG} \sigma_3)(f_i))]

= (\sigma_1 \oplus_{CG} (\sigma_2 \oplus_{CG} \sigma_3))(f_i).$$

\qed
Then \((\text{Cohyp}_G(\tau), \oplus_{CG})\) is a semigroup.

**Proposition 3.7.** For any \(\sigma_1, \sigma_2, \sigma_3 \in \text{Cohyp}_G(\tau)\) and \(i \in I\),

\[
\sigma_1 \circ_{CG} (\sigma_2 \oplus_{CG} \sigma_3) = (\sigma_1 \circ_{CG} \sigma_2) \oplus_{CG} (\sigma_1 \circ_{CG} \sigma_3).
\]

**Proof.** Let \(\sigma_1, \sigma_2, \sigma_3 \in \text{Cohyp}_G(\tau)\). Consider

\[
(\sigma_1 \circ_{CG} (\sigma_2 \oplus_{CG} \sigma_3))(f_i) = \hat{\sigma}_1((\sigma_2 \oplus_{CG} \sigma_3)(f_i))
\]

\[
= \hat{\sigma}_1((\sigma_2)(f_i)[\sigma_3(f_i), \ldots, \sigma_3(f_i)])
\]

\[
= (\hat{\sigma}_1(\sigma_2))(f_i)[\hat{\sigma}_1(\sigma_3(f_i)), \ldots, \hat{\sigma}_1(\sigma_3(f_i))],
\]

\[
= ((\sigma_1 \circ_{CG} \sigma_2)(f_i))[\sigma_1(\sigma_3(f_i)), \ldots, \sigma_1(\sigma_3(f_i))]
\]

\[
= ((\sigma_1 \circ_{CG} \sigma_2) \oplus_{CG} (\sigma_1 \circ_{CG} \sigma_3))(f_i).
\]

However, the equation \((\sigma_1 \oplus_{CG} \sigma_2) \circ_{CG} \sigma_3 = (\sigma_1 \circ_{CG} \sigma_3) \oplus_{CG} (\sigma_2 \circ_{CG} \sigma_3)\) is not true for all \(\sigma_1, \sigma_2, \sigma_3 \in \text{Cohyp}_G(\tau)\) as the following example shows:

**Example 3.8.** Let \(\tau = (2); \) that is, there is a binary operation symbol \(f\) and \(\sigma_1, \sigma_2, \sigma_3 \in \text{Cohyp}_G(\tau)\) such that \(\sigma_1(f) = f[e_1^2, e_2^2], \sigma_2(f) = f[e_0^2, f[e_3^2, e_1^2]],\) and \(\sigma_3(f) = f[f[e_1^2, e_0^2], e_2^2].\) So

\[
((\sigma_1 \oplus_{CG} \sigma_2) \circ_{CG} \sigma_3)(f) = f[f[f[e_1^2, f[e_2^2, e_3^2]], f[e_3^2, e_1^2]], e_2^2]
\]

and

\[
((\sigma_1 \circ_{CG} \sigma_3) \oplus_{CG} (\sigma_2 \circ_{CG} \sigma_3))(f) = f[e_2^2, e_2^2].
\]

Thus \((\sigma_1 \oplus_{CG} \sigma_2) \circ_{CG} \sigma_3 \neq (\sigma_1 \circ_{CG} \sigma_3) \oplus_{CG} (\sigma_2 \circ_{CG} \sigma_3)\).

So, we have the following lemma.

**Lemma 3.9.** \(\text{Cohyp}^{\oplus_{CG}}_G(\tau) := (\text{Cohyp}_G(\tau), \circ_{CG}, \oplus_{CG})\) is a left seminear-ring.

**Proof.** Since \((\text{Cohyp}_G(\tau), \circ_{CG})\) is a monoid, the proof directly follows from Propositions 3.6 and 3.7. 

4 Submonoids of Generalized Cohypersubstitutions

Definition 4.1. Let $\tau = (n_i)_{i \in I}$, $n_i \in \mathbb{N} \setminus \{0\}$ be a type with a cooperation symbol $f_i$ of the arity $n_i$, for each $i \in I$.

A generalized cohypersubstitution $\sigma$ of type $\tau$ is called a projection generalized cohypersubstitution if the coterm $\sigma(f_i)$ is the injection symbol for each $i \in I$. Let $P_{CG}^\tau(\tau)$ be the set of all projection generalized cohypersubstitutions of type $\tau$.

A generalized cohypersubstitution $\sigma$ of type $\tau$ is called a dual generalized cohypersubstitution if the coterm $\sigma(f_i) = f[\pi^{n_i}_{(0)}, \ldots, \pi^{n_i}_{(n_i-1)}]$, where $\pi$ is a permutation of the set $J = \{0, 1, \ldots, n_i - 1\}$. Let $D_{CG}^\tau(\tau)$ be the set of all such dual generalized cohypersubstitutions of type $\tau$.

A generalized cohypersubstitution $\sigma$ of type $\tau$ is called leftmost if for every $i \in I$, the first injection symbol occurs in $\hat{\sigma}(f_i[\pi^{n_i}_{0}, \ldots, \pi^{n_i}_{n_i-1}])$ is $\pi^{n_i}_{0}$. Let $\text{Left}^{\tau}_{CG}(\tau)$ be the set of all leftmost generalized cohypersubstitutions of type $\tau$.

A generalized cohypersubstitution $\sigma$ of type $\tau$ is called rightmost if for every $i \in I$, the last injection symbol occur in $\hat{\sigma}(f_i[\pi^{n_i}_{0}, \ldots, \pi^{n_i}_{n_i-1}])$ is $\pi^{n_i}_{n_i-1}$. Let $\text{Right}^{\tau}_{CG}(\tau)$ be the set of all rightmost generalized cohypersubstitutions of type $\tau$.

A generalized cohypersubstitution $\sigma$ of type $\tau$ is called outermost if for every $i \in I$, the first injection symbol occurs in $\hat{\sigma}(f_i[\pi^{n_i}_{0}, \ldots, \pi^{n_i}_{n_i-1}])$ is $\pi^{n_i}_{0}$ and the last injection symbol is $\pi^{n_i}_{n_i-1}$. Let $\text{Out}^{\tau}_{CG}(\tau)$ be the set of all outermost generalized cohypersubstitutions of type $\tau$. Note that $\text{Out}^{\tau}_{CG}(\tau) = \text{Left}^{\tau}_{CG}(\tau) \cap \text{Right}^{\tau}_{CG}(\tau)$.

A generalized cohypersubstitution $\sigma$ of type $\tau$ is called regular if, for every $i \in I$, each of injection symbols $\pi^{n_i}_{0}, \ldots, \pi^{n_i}_{n_i-1}$ occurs in $\hat{\sigma}(f_i[\pi^{n_i}_{0}, \ldots, \pi^{n_i}_{n_i-1}])$. Let $\text{Reg}^{\tau}_{CG}(\tau)$ be the set of all regular generalized cohypersubstitutions of type $\tau$.

A generalized cohypersubstitution $\sigma$ of type $\tau$ is called pre-generalized cohypersubstitution if the coterm $\sigma(f_i)$ is not the injection symbol. Let $\text{Pre}^{\tau}_{CG}(\tau)$ be the set of all pre-generalized cohypersubstitutions of type $\tau$.

Proposition 4.2. Let $\tau$ be a type of generalized cohypersubstitution. The sets $P_{CG}^\tau(\tau) \cup \{\sigma_{id}\}$, $D_{CG}^\tau(\tau)$, $\text{Left}^{\tau}_{CG}(\tau)$, $\text{Right}^{\tau}_{CG}(\tau)$, $\text{Out}^{\tau}_{CG}(\tau)$, $\text{Reg}^{\tau}_{CG}(\tau)$ and $\text{Pre}^{\tau}_{CG}(\tau)$ are submonoids of $\text{Cohyp}_{CG}(\tau)$. 

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Proof. Obviously $\sigma_{id}$ belongs to all of the sets. We only show that all of the sets are closed under the binary operation $\circ_{CG}$.

Let $\sigma_1, \sigma_2 \in P_{CG}^{inj}(\tau) \cup \{0\}$. Consider the possible four cases as follows:

Case 1. If $\sigma_1, \sigma_2$ are not $\sigma_{id}$, then both $\sigma_1(f_i)$ and $\sigma_2(f_i)$ are injection symbols for each $i \in I$. Thus $(\sigma_1 \circ_{CG} \sigma_2)(f_i) = \hat{\sigma}_1(\sigma_2(f_i)) = \hat{\sigma}_1(e^{n_i}_j) = e^{n_i}_j \in \hat{E}$. 

Case 2. If $\sigma_1 = \sigma_2 = \sigma_{id}$, then

\[
(\sigma_1 \circ_{CG} \sigma_2)(f_i) = (\sigma_{id} \circ_{CG} \sigma_{id})(f_i) = \hat{\sigma}_{id}(\sigma_{id}(f_i)) = \hat{\sigma}_{id}(f_i[e^{n_i}_0, \ldots, e^{n_i}_{n_i-1}]) = \sigma_{id}(f_i)[e^{n_i}_0, \ldots, e^{n_i}_{n_i-1}]
\]

Case 3. If $\sigma_1 \in P_{CG}^{inj}(\tau)$ and $\sigma_2 = \sigma_{id}$, then

\[
(\sigma_1 \circ_{CG} \sigma_2)(f_i) = (\sigma_{id} \circ_{CG} \sigma_{id})(f_i) = \hat{\sigma}_{id}(\sigma_{id}(f_i)) = \hat{\sigma}_{id}(f_i)[e^{n_i}_0, \ldots, e^{n_i}_{n_i-1}].
\]

So, if $\sigma_1(f_i) = e^{n_i}_j; 0 \leq j \leq n_i - 1$, then $(\sigma_1 \circ_{CG} \sigma_2)(f_i) = e^{n_i}_j$ and if $\sigma_1(f_i) = e^{n_i}_k; k \geq n_i$, then $(\sigma_1 \circ_{CG} \sigma_2)(f_i) = e^{n_i}_k$.

Case 4. If $\sigma_2 \in P_{CG}^{inj}(\tau)$ and $\sigma_1 = \sigma_{id}$, then $(\sigma_1 \circ_{CG} \sigma_2)(f_i) = (\sigma_{id} \circ_{CG} \sigma_2)(f_i) = \hat{\sigma}_{id}(\sigma_2(f_i)) = \epsilon^{n_i}_j \in \hat{E}$. Hence $\sigma_1 \circ_{CG} \sigma_2 \in P_{CG}^{inj}(\tau) \cup \{\sigma_{id}\}$.

Let $\sigma_1, \sigma_2 \in D_{CG}^{inj}(\tau)$. Then $\sigma_1(f_i) = f_i[e^{n_i}_0, \ldots, e^{n_i}_{n_i-1}]$ and $\sigma_2(f_i) = f_i[e^{n_i}_{\pi(0)}, \ldots, e^{n_i}_{\pi(n_i-1)}]$, where $\pi, \pi'$ are permutations of the set $J = \{0, 1, \ldots, n_i-1\}$. Consider

\[
(\sigma_1 \circ_{CG} \sigma_2)(f_i) = \hat{\sigma}_1(\sigma_2(f_i)) = \hat{\sigma}_1(f_i[e^{n_i}_{\pi(0)}, \ldots, e^{n_i}_{\pi(n_i-1)}]) = \hat{\sigma}_1(f_i)[e^{n_i}_{\pi(0)}, \ldots, e^{n_i}_{\pi(n_i-1)}] = (f_i[e^{n_i}_{\pi(0)}, \ldots, e^{n_i}_{\pi(n_i-1)}])[e^{n_i}_{\pi(0)}, \ldots, e^{n_i}_{\pi(n_i-1)}] = f_i[e^{n_i}_{\pi'(0)}, \ldots, e^{n_i}_{\pi'(n_i-1)}] = f_i[e^{n_i}_{\pi'\circ\pi(0)}, \ldots, e^{n_i}_{\pi'\circ\pi(n_i-1)}].
\]

Hence, $\sigma_1 \circ_{CG} \sigma_2 \in D_{CG}^{inj}(\tau)$.

Let $\sigma \in Out_{CG}^{inj}(\tau)$ and $t \in CT_{\tau}$. We will prove by induction on the complexity of the coterm $t$ that the first and last injection symbols occurring
in \( \hat{\sigma}(t) \) agree with the first and last injection symbols, respectively, occurring in \( t \). If \( t = e^n_j \) is an injection symbol, then \( \hat{\sigma}(t) = \hat{\sigma}(e^n_j) = e^n_j \). If \( t = f_i[t_0, \ldots, t_{n_i-1}] \) is a composed coterm where the first and last injection symbol occurring in \( \hat{\sigma}(t_i) \) agree with the first and last injection symbol occurring in \( t_i; 0 \leq l \leq n_i - 1 \), respectively. Suppose that the first injection symbol in \( \hat{\sigma}(t_0) \) is \( e^0_i \) and the last injection symbol in \( \hat{\sigma}(t_{n_i-1}) \) is \( e^{n_i}_{n_i-1} \). Then the first and last injection symbols in \( t \) is \( e^0_i \) and \( e^{n_i}_{n_i-1} \), respectively. Since \( \sigma \in \text{Out}_{\text{CG}}^{\text{inj}}(\tau) \), the first and last injection symbol in \( \hat{\sigma}(t) = (\sigma(f_i))[\hat{\sigma}(t_0), \ldots, \hat{\sigma}(t_{n_i-1})] \) is \( e^0_i \) and \( e^{n_i}_{n_i-1} \), respectively.

Now, we can show that \( \text{Left}_{\text{CG}}^{\text{inj}}(\tau), \text{Right}_{\text{CG}}^{\text{inj}}(\tau) \), and \( \text{Out}_{\text{CG}}^{\text{inj}}(\tau) \) are closed under the operation \( \circ_{\text{CG}} \). Let \( \sigma_1, \sigma_2 \) be generalized cohypersubstitutions, both either leftmost, rightmost or outermost. Then

\[
(\sigma_1 \circ_{\text{CG}} \sigma_2)(f_i[e^0_i, \ldots, e^{n_i}_{n_i-1}]) = (\hat{\sigma}_1 \circ \hat{\sigma}_2)(f_i[e^0_i, \ldots, e^{n_i}_{n_i-1}]) = \hat{\sigma}_1(\hat{\sigma}_2(f_i[e^0_i, \ldots, e^{n_i}_{n_i-1}])),
\]

and it follows from the previous reasoning that this product has the corresponding property.

Let \( \sigma \in \text{Reg}_{\text{CG}}^{\text{inj}}(\tau) \) and \( t \in CT_\tau \). We will prove by induction on the complexity of the coterm \( t \) that the injection symbols occurring in \( t \) and \( \hat{\sigma}(t) \) are the same. If \( t = e^n_j \) is an injection symbol, then \( \hat{\sigma}(t) = \hat{\sigma}(e^n_j) = e^n_j \). If \( t = f_i[t_0, \ldots, t_{n_i-1}] \), where the injection symbol occurring in \( \hat{\sigma}(t_i) \) and \( t_i; 0 \leq l \leq n_i - 1 \) are the same. Since \( \hat{\sigma}(t) = (\sigma(f_i))[\hat{\sigma}(t_0), \ldots, \hat{\sigma}(t_{n_i-1})] \) and \( \sigma \in \text{Reg}_{\text{CG}}^{\text{inj}}(\tau) \), the injection symbols occurring in \( t \) and \( \hat{\sigma}(t) \) are the same. So, if \( \sigma_1, \sigma_2 \in \text{Reg}_{\text{CG}}^{\text{inj}}(\tau) \), then \( (\sigma_1 \circ_{\text{CG}} \sigma_2)(f_i[e^0_i, \ldots, e^{n_i}_{n_i-1}]) = \hat{\sigma}_1(\hat{\sigma}_2(f_i[e^0_i, \ldots, e^{n_i}_{n_i-1}]))) \). It follows from the previous reasoning that this product has the corresponding property.

Finally, let \( \sigma_1, \sigma_2 \in \text{Pre}_{\text{CG}}^{\text{inj}}(\tau) \). It is clear that \( \sigma_1 \circ_{\text{CG}} \sigma_2 \) is again a pre-generalized cohypersubstitution.

Therefore, the sets \( D_{\text{CG}}^{\text{inj}}(\tau) \cup \{\sigma_\text{id}\}, D_{\text{CG}}^{\text{inj}}(\tau), \text{Left}_{\text{CG}}^{\text{inj}}(\tau), \text{Right}_{\text{CG}}^{\text{inj}}(\tau), \text{Out}_{\text{CG}}^{\text{inj}}(\tau), \text{Reg}_{\text{CG}}^{\text{inj}}(\tau), \text{Pre}_{\text{CG}}^{\text{inj}}(\tau) \) are submonoids of \( \text{Cohyp}_{\text{CG}}(\tau) \).

**Proposition 4.3.** For any type \( \tau \), the following properties hold:

(i) \( D_{\text{CG}}^{\text{inj}}(\tau) \subset \text{Pre}_{\text{CG}}^{\text{inj}}(\tau) \),

(ii) \( \text{Reg}_{\text{CG}}^{\text{inj}}(\tau) \subset \text{Pre}_{\text{CG}}^{\text{inj}}(\tau) \),

(iii) \( \text{Out}_{\text{CG}}^{\text{inj}}(\tau) \subset \text{Pre}_{\text{CG}}^{\text{inj}}(\tau) \).

**Proof.** The proof is straightforward. \( \square \)
Proposition 4.4. For any type \( \tau \), the sets \( \text{Pre}_{CG}^{\text{inj}}(\tau) \cup \{ \sigma_{id} \}, \text{Left}_{CG}^{\text{inj}}(\tau), \text{Right}_{CG}^{\text{inj}}(\tau), \text{Out}_{CG}^{\text{inj}}(\tau), \text{Reg}_{CG}^{\text{inj}}(\tau) \), and \( \text{Pre}_{CG}^{\text{inj}}(\tau) \) are subsemigroups of \((\text{Cohyp}_{CG}(\tau), +_{CG})\).

**Proof.** We will prove that the sets \( \text{Pre}_{CG}^{\text{inj}}(\tau) \cup \{ \sigma_{id} \}, \text{Left}_{CG}^{\text{inj}}(\tau), \text{Right}_{CG}^{\text{inj}}(\tau), \text{Out}_{CG}^{\text{inj}}(\tau), \text{Reg}_{CG}^{\text{inj}}(\tau) \), and \( \text{Pre}_{CG}^{\text{inj}}(\tau) \) are closed under the operation \( +_{CG} \).

Let \( \sigma_1, \sigma_2 \in \text{Pre}_{CG}^{\text{inj}}(\tau) \). Then \( \sigma_1(f_i) \) and \( \sigma_2(f_i) \) are injection symbols for each \( i \in I \). Since \( (\sigma_1 +_{CG} \sigma_2)f_i := (\sigma_2(f_i))\left[\sigma_1(f_i), \ldots, \sigma_1(f_i)\right] \) and both of the coterm \( \sigma_1(f_i) \) and \( \sigma_2(f_i) \) are injection symbols, this implies that the coterm \( (\sigma_1 +_{CG} \sigma_2)(f_i) \) is an injection symbol. So \( (\sigma_1 +_{CG} \sigma_2)f_i \in \text{Pre}_{CG}^{\text{inj}}(\tau) \).

Let \( \sigma_1, \sigma_2 \in \text{Left}_{CG}^{\text{inj}}(\tau) \). Then \( \text{leftmost}_{inj}(\sigma_1(f_i)) = \text{leftmost}_{inj}(\sigma_2(f_i)) = e^{n_i}_{0} \). By the definition of \( +_{CG} \), we have \( (\sigma_1 +_{CG} \sigma_2)f_i := (\sigma_2(f_i))\left[\sigma_1(f_i), \ldots, \sigma_1(f_i)\right] \). Since \( \text{leftmost}_{inj}(\sigma_1(f_i)) = \text{leftmost}_{inj}(\sigma_2(f_i)) = e^{n_i}_{0} \), if we substitute the coterm \( \sigma_2(f_i) \) by a coterm \( \sigma_1(f_i) \), then we also have \( \text{leftmost}_{inj}((\sigma_2(f_i))\left[\sigma_1(f_i), \ldots, \sigma_1(f_i)\right]) = e^{n_i}_{0} \). So, \( \sigma_1 +_{CG} \sigma_2 \in \text{Left}_{CG}^{\text{inj}}(\tau) \).

Similarly, we can show that \( \sigma_1 +_{CG} \sigma_2 \in \text{Right}_{CG}^{\text{inj}}(\tau) \) and \( \sigma_1 +_{CG} \sigma_2 \in \text{Out}_{CG}^{\text{inj}}(\tau) \).

Let \( \sigma_1, \sigma_2 \in \text{Reg}_{CG}^{\text{inj}}(\tau) \). Then every injection symbols \( e^{n_i}_{0}, \ldots, e^{n_i}_{1} \) occurs in \( \sigma_1(f_i) \) and \( \sigma_2(f_i) \). Since \( (\sigma_1 +_{CG} \sigma_2)(f_i) := (\sigma_2(f_i))\left[\sigma_1(f_i), \ldots, \sigma_1(f_i)\right] \) and each of injection symbols \( e^{n_i}_{0}, \ldots, e^{n_i}_{1} \) occurs in \( \sigma_1(f_i) \) and \( \sigma_2(f_i) \), then every injection symbols \( e^{n_i}_{0}, \ldots, e^{n_i}_{1} \) are in \( (\sigma_1 +_{CG} \sigma_2)(f_i) \). Thus \( \sigma_1 +_{CG} \sigma_2 \in \text{Reg}_{CG}^{\text{inj}}(\tau) \).

Therefore, \( \text{Pre}_{CG}^{\text{inj}}(\tau) \cup \{ \sigma_{id} \}, \text{Left}_{CG}^{\text{inj}}(\tau), \text{Right}_{CG}^{\text{inj}}(\tau), \text{Out}_{CG}^{\text{inj}}(\tau), \text{Reg}_{CG}^{\text{inj}}(\tau) \), and \( \text{Pre}_{CG}^{\text{inj}}(\tau) \) are subsemigroups of \((\text{Cohyp}_{CG}(\tau), +_{CG})\). \( \square \)

**Proposition 4.5.** Let \( \tau \) be a type of generalized cohypersubstitution. The set \( D_{CG}^{\text{inj}}(\tau) \) is not a subsemigroup of \((\text{Cohyp}_{CG}(\tau), +_{CG})\).

**Proof.** Let \( \sigma_1, \sigma_2 \in D_{CG}^{\text{inj}}(\tau) \). Then \( \sigma_1(f_i) = f_i[e^{n_i}_{0}, \ldots, e^{n_i}_i] \) and \( \sigma_2(f_i) = f_i[e^{n_i}_{0}, \ldots, e^{n_i}_i] \) where \( \pi, \pi' \) are permutations of the set \( J = \{0, \ldots, n_i- \).
1}. Consider

\[
\begin{align*}
(\sigma_1 +_{CG} \sigma_2)(f_i) &= (\sigma_2(f_i))[\sigma_1(f_i), \ldots, \sigma_1(f_i)] \\
&= (f_i[e_{\pi'(0)}^{n_i}, \ldots, e_{\pi'(n_i-1)}^{n_i}]) [f_i[e_{\pi(0)}^{n_i}, \ldots, e_{\pi(n_i-1)}^{n_i}], \ldots, f_i[e_{\pi(0)}^{n_i}, \ldots, e_{\pi(n_i-1)}^{n_i}]], \\
&= f_i[f_i[e_{\pi(0)}^{n_i}, \ldots, e_{\pi(n_i-1)}^{n_i}], \ldots, f_i[e_{\pi(0)}^{n_i}, \ldots, e_{\pi(n_i-1)}^{n_i}]].
\end{align*}
\]

Hence, \(\sigma_1 \circ_{CG} \sigma_2 \notin D_{CG}^{inj}(\tau)\). Therefore, \(D_{CG}^{inj}(\tau)\) is not a subsemigroup of \((\text{Cohyp}_{CG}(\tau), +_{CG})\).

**Proposition 4.6.** For any type \(\tau\), the set \(\text{Pre}_{CG}^{inj}(\tau)\) is a maximal subsemigroup of \((\text{Cohyp}_{CG}(\tau), +_{CG})\).

**Proof.** The proof directly follows from Propositions 4.3, 4.4, and 4.5.

**Proposition 4.7.** For any type \(\tau\), the sets \(\text{P}_{CG}^{inj}(\tau) \cup \{\sigma_{id}\}, \text{Left}_{CG}^{inj}(\tau), \text{Right}_{CG}^{inj}(\tau), \text{Out}_{CG}^{inj}(\tau), \text{Reg}_{CG}^{inj}(\tau), \) and \(\text{Pre}_{CG}^{inj}(\tau)\) are subsemigroups of \((\text{Cohyp}_{CG}(\tau), +_{CG})\).

**Proof.** The proof is similar to that of Proposition 4.4 by considering \((\sigma_1 \oplus_{CG} \sigma_2)(f_i) := (\sigma_1(f_i))[\sigma_2(f_i), \ldots, \sigma_2(f_i)]\).

**Proposition 4.8.** Let \(\tau\) be a type of generalized cohypersubstitution. The set \(D_{CG}^{inj}(\tau)\) is not a subsemigroup of \((\text{Cohyp}_{CG}(\tau), +_{CG})\).

**Proof.** The proof is similar to that of Proposition 4.5.

**Proposition 4.9.** For any type \(\tau\), the set \(\text{Pre}_{CG}^{inj}(\tau)\) is a maximal subsemigroup of \((\text{Cohyp}_{CG}(\tau), +_{CG})\).

**Proof.** The proof directly follows from Propositions 4.3, 4.7, and 4.8.

Now, we have the following theorem:

**Theorem 4.10.** Let \(\tau\) be a type of generalized cohypersubstitution. The sets \(\text{P}_{CG}^{inj}(\tau) \cup \{\sigma_{id}\}, \text{Left}_{CG}^{inj}(\tau), \text{Right}_{CG}^{inj}(\tau), \text{Out}_{CG}^{inj}(\tau), \text{Reg}_{CG}^{inj}(\tau), \) and \(\text{Pre}_{CG}^{inj}(\tau)\) are sub-left seminear-ring of \(\text{Cohyp}_{CG}^{+}(\tau)\).

**Proof.** The proof directly follows from Propositions 4.2 and 4.4.
Theorem 4.11. Let $\tau$ be a type of generalized cohypersubstitution. The sets $P_{CG}^{inj}(\tau) \cup \{\sigma_d\}$, $Left_{CG}^{inj}(\tau)$, $Right_{CG}^{inj}(\tau)$, $Out_{CG}^{inj}(\tau)$, $Reg_{CG}^{inj}(\tau)$, and $Pre_{CG}^{inj}(\tau)$ are sub-left seminear-ring of $Cohyp_{CG}^{G}(\tau)$.

Proof. The proof directly follows from Propositions 4.2 and 4.7.

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