International Journal of Mathematics and
Computer Science, **16**(2021), no. 4, 1117–1129

Structural Properties and Submonoids of Generalized Cohypersubstitutions

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(Received December 15, 2020, Accepted February 13, 2021)

Abstract

Generalized cohypersubstitutions of type $\tau = (n_i)_{i \in I}$ are mappings which send the n_i-ary cooperations symbols to coterms of type τ. We define the operations $+_{CG}$ and $\oplus_{CG}$ on the set $Cohyp_G(\tau)$ and consider some submonoids of the monoid $\underline{Cohyp_G(\tau)}$. Finally, we give some structural properties and the relationship among submonoids.

1 Introduction

The topic cohypersubstitution of type τ in universal algebra has gained interest among many authors. In 2009, Denecke and Seangsura [2] initially introduced and used the main tool in the study of cohyperidentities. They defined coalgebras, coidentities , cohyperidentities and applied all the concepts to construct the monoid of cohypersubstitutions of type τ. In 2013,

Key words and phrases: generalized cohypersubstitutions, left seminear-ring, submonoid, sub-left semonear-ring.
AMS (MOS) Subject Classifications: 08A05, 08A30, 08A40, 16Y99.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

Jermjitpornchai and Seangsura [4] generalized the concepts in [2] by studying generalized cohypersunstitutions. They introduced the coterms involving generalized superpositions, discovered some algebraic-structural properties, and constructed the monoid of generalized cohypersubstitutions of type $\tau = (n_i)_{i\in I}$. The structural properties and special elements of the monoid of generalized cohypersubstitutions of type $\tau = (2), \tau = (3)$ and $\tau = (n)$ has since been studied by many other authors. In this paper we focus on the generalized cohypersubstitutions of type τ and define the operation on $Cohyp_G(\tau)$, and give some algebraic-structural properties of the set of all generalized cohypersubstitutions. In addition, we focus on some submonoid of the monoid $\underline{Cohyp_G(\tau)}$, and study some structural properties and the relationship among submonoids.

2 Generalized Cohypersubstitutions

In this section we provide the basic concept of the monoid of all generalized cohypersubstitutions which is very useful in this paper.

Let A be a non-empty set and $n \in \mathbb{N}$. Define the union of n disjoint copies of A by $A^{\sqcup n} := \underline{n} \times A$, where $\underline{n} = \{1, 2, \ldots, n\}$. This is called *the n-th copower of A*. An element (i, a) in this copower corresponds to the element a in the i-th copy of A, where $i \in \underline{n}$. For some natural number n, a mapping $f^A : A \to A^{\sqcup n}$ is a co-operation on A; n is called *the arity of the co-operation* f^A. Every n-ary co-operation f^A on the set A can be uniquely expressed as the pair of mappings (f_1^A, f_2^A), where $f_1^A : A \to \underline{n}$ gives the labeling used by f^A of mapping elements to copies of A, and $f_2^A : A \to A$ shows what element of A any element is mapped to, so $f^A(a) = (f_1^A(a), f_2^A(a))$. We denote the set of all n-ary co-operations defined on A by $cO_A^{(n)} = \{f^A : A \to A^{\sqcup n}\}$.

Let $\tau = (n_i)_{i\in I}$ and $(f_i)_{i\in I}$ be an indexed set of co-operation symbols which f_i has arity n_i for each $i \in I$. Let $\bigcup\{e_j^n \mid n \geq 1, n \in \mathbb{N}, 0 \leq j \leq n-1\}$ be a set of symbols which disjoint from $\{f_i \mid i \in I\}$ such that e_j^n has arity n for each $0 \leq j \leq n-1$. The *coterms* of type τ are defined as follows:

(i) For every $i \in I$ the co-operation symbol f_i is an n_i-ary coterm of type τ.

(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$ the symbol e_j^n is an n-ary coterm of type τ.

(iii) If $t_1, \ldots, t_{n_i}$ are n-ary coterms of type τ, then for every $i \in I$, $f_i[t_1, \ldots, t_{n_i}]$ is an n-ary coterm of type τ, and if $t_0, \ldots, t_{n-1}$ are m-ary coterm of type τ, then $e_j^n[t_0, \ldots, t_{n-1}]$ is an m-ary coterm of type τ for every $0 \le j \le n-1$.

Denoted by $CT_\tau^{(n)}$ the set of all n-ary coterms of type τ, and $CT_\tau := \bigcup_{n\ge 1} CT_\tau^{(n)}$ the set of all coterms of type τ.

Let $m \in \mathbb{N}^*$. A *generalized superposition* of a coterms $S^m : CT_\tau^m \times CT_\tau \to CT_\tau$ defined inductively by the following steps:

(i) If $t = e_i^n$ and $0 \le i \le m-1$, then $S^m(e_i^n, t_0, \ldots, t_{m-1}) = t_i$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \le i \le n-1$, then $S^m(e_i^n, t_0, \ldots, t_{m-1}) = e_i^n$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \ldots, s_{n_i}]$, then $S^m(t, t_1, \ldots, t_m) = f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$, where $S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m) \in CT_\tau$.

The above definition can be written as the following forms:

(i) If $t = e_i^n$ and $0 \le i \le m-1$, then $e_i^n[t_0, \ldots, t_{m-1}] = t_i$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \le i \le n-1$, then $e_i^n[t_0, \ldots, t_{m-1}] = e_i^n$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \ldots, s_{n_i}]$, then $(f_i[s_1, \ldots, s_{n_i}])[t_1, \ldots, t_m] = f_i(s_1[t_1, \ldots, t_m], \ldots, s_{n_i}[t_1, \ldots, t_m])$, where $s_1[t_1, \ldots, t_m], \ldots, s_{n_i}[t_1, \ldots, t_m] \in CT_\tau$.

Definition 2.1. [1] *For arbitrary coterms* $t, t_0, \ldots, t_{m-1} \in CT_\tau$,

$$t[s_0[t_0, \ldots, t_{m-1}], \ldots, s_{n_i-1}[t_0, \ldots, t_{m-1}]] = (t[s_0, \ldots, s_{n_i-1}])[t_0, \ldots, t_{m-1}].$$

A *generalized cohypersubstitution* of type τ is a mapping $\sigma : \{f_i \,|\, i \in I\} \to CT_\tau$. The extension of σ is a mapping $\hat{\sigma} : CT_\tau \to CT_\tau$ which is inductively defined by the following steps:

(i) $\hat{\sigma}(e_j^n) := e_j^n$ for every $n \ge 1$ and $0 \le j \le n-1$,

(ii) $\hat{\sigma}(f_i) := \sigma(f_i)$ for every $i \in I$,

(iii) $\hat{\sigma}(f_i[t_1, \ldots, t_{n_i}]) := \sigma(f_i)[\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]]$ for $t_1, \ldots t_{n_i} \in CT_\tau^{(n)}$.

Denoted by $Cohyp_G(\tau)$ the set of all generalized cohypersubstitutions of type τ.

Definition 2.2. [4] *If $t, t_1, \ldots, t_n \in CT_\tau$ and $\sigma \in Cohyp_G(\tau)$, then*

$$\hat{\sigma}(t[t_1, \ldots, t_n]) = \hat{\sigma}(t)[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)].$$

Proposition 2.3. [1] *For arbitrary coterms $t, t_0, \ldots, t_{n-1}$ and $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$, $(\hat{\sigma_1} \circ \sigma_2)\hat{} = \hat{\sigma_1} \circ \hat{\sigma_2}$.*

Define a binary operation $\circ_{CG} : Cohyp_G(\tau) \times Cohyp_G(\tau) \to Cohyp_G(\tau)$ on the set of all generalized cohypersubstitutions, $Cohyp_G(\tau)$, by $\sigma_1 \circ_{CG} \sigma_2 := \hat{\sigma_1} \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$ where $\circ$ is the usual composition of mappings. Let σ_{id} be the generalized cohypersubstitution such that $\sigma_{id}(f_i) := f_i$ for all $i \in I$. Then σ_{id} is an identity element in $Cohyp_G(\tau)$. So, $\underline{Cohyp_G(\tau)} := (Cohyp_G(\tau), \circ_{CG}, \sigma_{id})$ is a monoid and called *the monoid of generalized cohypersubstitutions of type* τ. The algebraic structural-properties of the monoid $\underline{Cohyp_G(\tau)}$, see [4].

Throughout this paper, we use the following notations:

$\sigma_t :=$ the generalized cohypersubstitution σ of type τ which maps f to the coterm t,

$e_j^n :=$ the injection symbol for all $0 \leq j \leq n-1, n \in \mathbb{N}$,

$E :=$ the set of all injection symbols i.e. $E := \{e_j^n \mid n, j \in \mathbb{N}\}$,

$E(t) :=$ the set of all injection symbols which occur in the coterm t.

3 Algebraic-structural Properties of Generalized Cohypersubstitutions

Definition 3.1. *A nonempty set R together with two binary operations, denoted by $+$ and $\cdot$ respectively, is said to be a left(right) seminear-ring if $(R, +)$ and $(R, \cdot)$ are semigroups and satisfy the left (right) distributive law; i.e., for all $a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c((a + b) \cdot c = a \cdot c + b \cdot c)$.*

Now, we define the binary operation $+_{CG}$ on the set $Cohyp_G(\tau)$ by

$$(\sigma_1 +_{CG} \sigma_2)(f_i) := (\sigma_2(f_i))[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}] \in CT_{(\tau)},$$

for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$. Then we have the following propositions:

Proposition 3.2. *For any* $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$ *and* $i \in I$,

$$\sigma_1 +_{CG} (\sigma_2 +_{CG} \sigma_3) = (\sigma_1 +_{CG} \sigma_2) +_{CG} \sigma_3.$$

Proof. Let $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$. We consider

$$\begin{aligned}
(\sigma_1 +_{CG} (\sigma_2 +_{CG} \sigma_3))(f_i) &= ((\sigma_2 +_{CG} \sigma_3)(f_i))[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}] \\
&= (\sigma_3(f_i))[\underbrace{\sigma_2(f_i), \ldots, \sigma_2(f_i)}_{n_i-terms}])[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}] \\
= (\sigma_3(f_i))[\sigma_2(f_i)[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}], &\ldots, \sigma_2(f_i)[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}]]) \\
&= (\sigma_3(f_i))[\underbrace{(\sigma_1 +_{CG} \sigma_2)(f_i), \ldots, (\sigma_1 +_{CG} \sigma_2)(f_i)}_{n_i-terms}] \\
&= ((\sigma_1 +_{CG} \sigma_2) +_{CG} \sigma_3)(f_i).
\end{aligned}$$

□

Then $(Cohyp_G(\tau), +_{CG})$ is a semigroup.

Proposition 3.3. *For any* $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$ *and* $i \in I$,

$$\sigma_1 \circ_{CG} (\sigma_2 +_{CG} \sigma_3) = (\sigma_1 \circ_{CG} \sigma_2) +_{CG} (\sigma_1 \circ_{CG} \sigma_3).$$

Proof. Let $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$. Consider

$$\begin{aligned}
(\sigma_1 \circ_{CG} (\sigma_2 +_{CG} \sigma_3))(f_i) &= \hat{\sigma_1}((\sigma_2 +_{CG} \sigma_3)(f_i)) \\
&= \hat{\sigma_1}((\sigma_3)(f_i)[\underbrace{\sigma_2(f_i), \ldots, \sigma_2(f_i)}_{n_i-terms}]) \\
&= (\hat{\sigma_1}(\sigma_3)(f_i))[\underbrace{\hat{\sigma_1}(\sigma_2(f_i)), \ldots, \hat{\sigma_1}(\sigma_2(f_i))}_{n_i-terms}] \\
&= ((\sigma_1 \circ_{CG} \sigma_3)(f_i))[\underbrace{(\sigma_1 \circ_{CG} \sigma_2)(f_i), \ldots, (\sigma_1 \circ_{CG} \sigma_2)(f_i)}_{n_i-terms}] \\
&= ((\sigma_1 \circ_{CG} \sigma_2) +_{CG} (\sigma_1 \circ_{CG} \sigma_3))(f_i).
\end{aligned}$$

□

For any generalized cohypersubstitutions σ_1, σ_2 and σ_3, the right distributive law; i.e., $(\sigma_1 +_{CG} \sigma_2) \circ_{CG} \sigma_3 = (\sigma_1 \circ_{CG} \sigma_3) +_{CG} (\sigma_2 \circ_{CG} \sigma_3)$, is not true as the following example shows:

Example 3.4. *Let* $\tau = (2)$; *that is, there is a binary cooperation symbol* f, *and* $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$ *such that* $\sigma_1(f) = f[e_0^2, f[e_3^2, e_1^2]], \sigma_2(f) = f[f[e_1^2, e_4^2], e_2^2]$, *and* $\sigma_3(f) = f[e_1^2, e_2^2]$. *So,*

$$((\sigma_1 +_{CG} \sigma_2) \circ_{CG} \sigma_3)(f) = f[f[f[e_1^2, f[e_3^2, e_2^2]], e_4^2], e_2^2]$$

and

$$((\sigma_1 \circ_{CG} \sigma_3) +_{CG} (\sigma_2 \circ_{CG} \sigma_3))(f) = f[f[e_2^2, e_4^2], e_2^2].$$

Thus $(\sigma_1 +_{CG} \sigma_2) \circ_{CG} \sigma_3 \neq (\sigma_1 \circ_{CG} \sigma_3) +_{CG} (\sigma_2 \circ_{CG} \sigma_3)$.

Then we obtain the following lemma:

Lemma 3.5. $Cohyp_G^{+_{CG}}(\tau) := (Cohyp_G(\tau), \circ_{CG}, +_{CG})$ *is a left seminearring.*

Proof. Since $(Cohyp_G(\tau), \circ_{CG})$ is a monoid, the proof directly follows from Propositions 3.2 and 3.3. □

Next, we define another binary operation $\oplus_{CG}$ on the set $Cohyp_G(\tau)$ by

$$(\sigma_1 \oplus_{CG} \sigma_2)(f_i) := (\sigma_1(f_i))[\underbrace{\sigma_2(f_i), \ldots, \sigma_2(f_i)}_{n_i-terms}] \in CT_{(\tau)},$$

for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$. Then we have the following.

Proposition 3.6. *For any* $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$ *and* $i \in I$,

$$\sigma_1 \oplus_{CG} (\sigma_2 \oplus_{CG} \sigma_3) = (\sigma_1 \oplus_{CG} \sigma_2) \oplus_{CG} \sigma_3.$$

Proof. Let $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$. Consider

$$\begin{aligned}((\sigma_1 \oplus_{CG} \sigma_2) \oplus_{CG} \sigma_3)(f_i) &= ((\sigma_1 \oplus_{CG} \sigma_2)(f_i))[\underbrace{\sigma_3(f_i), \ldots, \sigma_3(f_i)}_{n_i-terms}] \\ &= (\sigma_1(f_i))[\underbrace{\sigma_2(f_i), \ldots, \sigma_2(f_i)}_{n_i-terms}])[\underbrace{\sigma_3(f_i), \ldots, \sigma_3(f_i)}_{n_i-terms}] \\ &= (\sigma_1(f_i))[\sigma_2(f_i)[\underbrace{\sigma_3(f_i), \ldots, \sigma_3(f_i)}_{n_i-terms}], \ldots, \\ &\qquad \sigma_2(f_i)[\underbrace{\sigma_3(f_i), \ldots, \sigma_3(f_i)}_{n_i-terms}]]) \\ &= (\sigma_1(f_i))[\underbrace{(\sigma_2 \oplus_{CG} \sigma_3)(f_i), \ldots, (\sigma_2 \oplus_{CG} \sigma_3)(f_i)}_{n_i-terms}] \\ &= (\sigma_1 \oplus_{CG} (\sigma_2 \oplus_{CG} \sigma_3))(f_i).\end{aligned}$$

□

Then $(Cohyp_G(\tau), \oplus_{CG})$ is a semigroup.

Proposition 3.7. *For any $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$ and $i \in I$,*

$$\sigma_1 \circ_{CG} (\sigma_2 \oplus_{CG} \sigma_3) = (\sigma_1 \circ_{CG} \sigma_2) \oplus_{CG} (\sigma_1 \circ_{CG} \sigma_3).$$

Proof. Let $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$. Consider

$$\begin{aligned}
(\sigma_1 \circ_{CG} (\sigma_2 \oplus_{CG} \sigma_3))(f_i) &= \hat{\sigma_1}((\sigma_2 \oplus_{CG} \sigma_3)(f_i)) \\
&= \hat{\sigma_1}((\sigma_2)(f_i)[\underbrace{\sigma_3(f_i), \ldots, \sigma_3(f_i)}_{n_i - terms}]) \\
&= (\hat{\sigma_1}(\sigma_2 3)(f_i))[\underbrace{\hat{\sigma_1}(\sigma_3(f_i)), \ldots, \hat{\sigma_1}(\sigma_3(f_i))}_{n_i - terms}] \\
&= ((\sigma_1 \circ_{CG} \sigma_2)(f_i))[\underbrace{(\sigma_1 \circ_{CG} \sigma_3)(f_i), \ldots, (\sigma_1 \circ_{CG} \sigma_3)(f_i)}_{n_i - terms}] \\
&= ((\sigma_1 \circ_{CG} \sigma_2) \oplus_{CG} (\sigma_1 \circ_{CG} \sigma_3))(f_i).
\end{aligned}$$

□

However, the equation $(\sigma_1 \oplus_{CG} \sigma_2) \circ_{CG} \sigma_3 = (\sigma_1 \circ_{CG} \sigma_3) \oplus_{CG} (\sigma_2 \circ_{CG} \sigma_3)$ is not true for all $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$ as the following example shows:

Example 3.8. *Let $\tau = (2)$; that is, there is a binary operation symbol f and $\sigma_1, \sigma_2, \sigma_3 \in Cohyp_G(\tau)$ such that $\sigma_1(f) = f[e_1^2, e_2^2], \sigma_2(f) = f[e_0^2, f[e_3^2, e_1^2]]$, and $\sigma_3(f) = f[f[e_1^2, e_0^2], e_2^2]$. So*

$$((\sigma_1 \oplus_{CG} \sigma_2) \circ_{CG} \sigma_3)(f) = f[f[f[f[e_1^2, f[e_3^2, e_0^2]]]f[e_3^2, e_2^2]], e_2^2]$$

and

$$((\sigma_1 \circ_{CG} \sigma_3) \oplus_{CG} (\sigma_2 \circ_{CG} \sigma_3))(f) = f[e_2^2, e_2^2].$$

Thus $(\sigma_1 \oplus_{CG} \sigma_2) \circ_{CG} \sigma_3 \neq (\sigma_1 \circ_{CG} \sigma_3) \oplus_{CG} (\sigma_2 \circ_{CG} \sigma_3)$.

So, we have the following lemma.

Lemma 3.9. *$Cohyp_G^{\oplus_{CG}}(\tau) := (Cohyp_G(\tau), \circ_{CG}, \oplus_{CG})$ is a left seminear-ring.*

Proof. Since $(Cohyp_G(\tau), \circ_{CG})$ is a monoid, the proof directly follows from Propositions 3.6 and 3.7. □

4 Submonoids of Generalized Cohypersubstitutions

Definition 4.1. *Let $\tau = (n_i)_{i\in I}$, $n_i \in \mathbb{N} \setminus \{0\}$ be a type with a cooperation symbol f_i of the arity n_i, for each $i \in I$.*

A generalized cohypersubstitution σ of type τ is called a ***projection generalized cohypersubstitution*** *if the coterm $\sigma(f_i)$ is the injection symbol for each $i \in I$. Let $P^{inj}_{CG}(\tau)$ be the set of all projection generalized cohypersubstitutions of type τ.*

A generalized cohypersubstitution σ of type τ is called a ***dual generalized cohypersubstitution*** *if the coterm $\sigma(f_i) = f[e^{n_i}_{\pi(0)}, \ldots, e^{n_i}_{\pi(n_i-1)}]$, where π is a permutation of the set $J = \{0, 1, \ldots, n_i - 1\}$. Let $D^{inj}_{CG}(\tau)$ be the set of all such dual generalized cohypersubstitutions of type τ.*

A generalized cohypersubstitution σ of type τ is called ***leftmost*** *if for every $i \in I$, the first injection symbol occurs in $\hat{\sigma}(f_i[e^{n_i}_0, \ldots, e^{n_i}_{n_i-1}])$ is $e^{n_i}_0$. Let $leftmost_{inj}(t)$ be the first injection symbol (from the left) which occur in the coterm t and $Left^{inj}_{CG}(\tau)$ be the set of all leftmost generalized cohypersubstitutions of type τ.*

A generalized cohypersubstitution σ of type τ is called ***rightmost*** *if, for every $i \in I$, the last injection symbol occur in $\hat{\sigma}(f_i[e^{n_i}_0, \ldots, e^{n_i}_{n_i-1}])$ is $e^{n_i}_{n_i-1}$. Let $rightmost_{inj}(t)$ be the last injection symbol which occur in the coterm t and $Right^{inj}_{CG}(\tau)$ be the set of all rightmost generalized cohypersubstitutions of type τ.*

A generalized cohypersubstitution σ of type τ is called ***outermost*** *if for every $i \in I$, the first injection symbol occurs in $\hat{\sigma}(f_i[e^{n_i}_0, \ldots, e^{n_i}_{n_i-1}])$ is $e^{n_i}_0$ and the last injection symbol is $e^{n_i}_{n_i-1}$. Let $Out^{inj}_{CG}(\tau)$ be the set of all outermost generalized cohypersubstitutions of type τ. Note that $Out^{inj}_{CG}(\tau) = Left^{inj}_{CG}(\tau) \cap Right^{inj}_{CG}(\tau)$.*

A generalized cohypersubstitution σ of type τ is called ***regular*** *if, for every $i \in I$, each of injection symbols $e^{n_i}_0, \ldots, e^{n_i}_{n_i-1}$ occurs in $\hat{\sigma}(f_i[e^{n_i}_0, \ldots, e^{n_i}_{n_i-1}])$. Let $Reg^{inj}_{CG}(\tau)$ be the set of all regular generalized cohypersubstitutions of type τ.*

A generalized cohypersubstitution σ of type τ is called ***pre-generalized cohypersubstitution*** *if the coterm $\sigma(f_i)$ is not the injection symbol. Let $Pre^{inj}_{CG}(\tau)$ be the set of all pre-generalized cohypersubstitutions of type τ.*

Proposition 4.2. *Let τ be a type of generalized cohypersubstitution. The sets $P^{inj}_{CG}(\tau) \cup \{\sigma_{id}\}$, $D^{inj}_{CG}(\tau)$, $Left^{inj}_{CG}(\tau)$, $Right^{inj}_{CG}(\tau)$, $Out^{inj}_{CG}(\tau)$, $Reg^{inj}_{CG}(\tau)$ and $Pre^{inj}_{CG}(\tau)$ are submonoids of $\underline{Cohyp_G(\tau)}$.*

Proof. Obviously σ_{id} belongs to all of the sets. We only show that all of the sets are closed under the binary operation $\circ_{CG}$.

Let $\sigma_1, \sigma_2 \in P_{CG}^{inj}(\tau) \cup \{0\}$. Consider the possible four cases as follows:

Case 1. If σ_1, σ_2 are not σ_{id}, then both $\sigma_1(f_i)$ and $\sigma_2(f_i)$ are injection symbols for each $i \in I$. Thus $(\sigma_1 \circ_{CG} \sigma_2)(f_i) = \hat{\sigma_1}(\sigma_2(f_i)) = \hat{\sigma_1}(e_j^{n_i}) = e_j^{n_i} \in E$.

Case 2. If $\sigma_1 = \sigma_2 = \sigma_{id}$, then

$$\begin{aligned}(\sigma_1 \circ_{CG} \sigma_2)(f_i) &= (\sigma_{id} \circ_{CG} \sigma_{id})(f_i) = \hat{\sigma}_{id}(\sigma_{id}(f_i))\\ &= \hat{\sigma}_{id}(f_i[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}])\\ &= (\sigma_{id}(f_i))[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}]\\ &= (f_i[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}])[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}]\\ &= f_i[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}] = \sigma_{id}(f_i).\end{aligned}$$

Case 3. If $\sigma_1 \in P_{CG}^{inj}(\tau)$ and $\sigma_2 = \sigma_{id}$, then

$$\begin{aligned}(\sigma_1 \circ_{CG} \sigma_2)(f_i) &= (\sigma_1 \circ_{CG} \sigma_{id})(f_i) = \hat{\sigma}_1(\sigma_{id}(f_i))\\ &= \hat{\sigma}_1(f_i[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}])\\ &= (\sigma_1(f_i))[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}].\end{aligned}$$

So, if $\sigma_1(f_i) = e_j^{n_i}; 0 \le j \le n_i - 1$, then $(\sigma_1 \circ_{CG} \sigma_2)(f_i) = e_j^{n_i}$ and if $\sigma_1(f_i) = e_k^{n_i}; k \ge n_i$, then $(\sigma_1 \circ_{CG} \sigma_2)(f_i) = e_k^{n_i}$.

Case 4. If $\sigma_2 \in P_{CG}^{inj}(\tau)$ and $\sigma_1 = \sigma_{id}$, then $(\sigma_1 \circ_{CG} \sigma_2)(f_i) = (\sigma_{id} \circ_{CG} \sigma_2)(f_i) = \hat{\sigma}_{id}(\sigma_2(f_i)) = \hat{\sigma}_{id}(e_j^{n_i}) = e_j^{n_i} \in E$. Hence $\sigma_1 \circ_{CG} \sigma_2 \in P_{CG}^{inj}(\tau) \cup \{\sigma_{id}\}$.

Let $\sigma_1, \sigma_2 \in D_{CG}^{inj}(\tau)$. Then $\sigma_1(f_i) = f_i[e_{\pi(0)}^{n_i}, \ldots, e_{\pi(n_i-1)}^{n_i}]$ and $\sigma_2(f_i) = f_i[e_{\pi'(0)}^{n_i}, \ldots, e_{\pi'(n_i-1)}^{n_i}]$, where π, π' are permutations of the set $J = \{0, 1, \ldots, n_i - 1\}$. Consider

$$\begin{aligned}(\sigma_1 \circ_{CG} \sigma_2)(f_i) &= \hat{\sigma}_1(\sigma_2(f_i))\\ &= \hat{\sigma}_1(f_i[e_{\pi'(0)}^{n_i}, \ldots, e_{\pi'(n_i-1)}^{n_i}])\\ &= (\sigma_1(f_i))[e_{\pi'(0)}^{n_i}, \ldots, e_{\pi'(n_i-1)}^{n_i}]\\ &= (f_i[e_{\pi(0)}^{n_i}, \ldots, e_{\pi(n_i-1)}^{n_i}])[e_{\pi'(0)}^{n_i}, \ldots, e_{\pi'(n_i-1)}^{n_i}]\\ &= f_i[e_{\pi'(\pi(0))}^{n_i}, \ldots, e_{\pi'(\pi(n_i-1))}^{n_i}]\\ &= f_i[e_{\pi'\circ\pi(0)}^{n_i}, \ldots, e_{\pi'\circ\pi(n_i-1)}^{n_i}].\end{aligned}$$

Hence, $\sigma_1 \circ_{CG} \sigma_2 \in D_{CG}^{inj}(\tau)$.

Let $\sigma \in Out_{CG}^{inj}(\tau)$ and $t \in CT_{(\tau)}$. We will prove by induction on the complexity of the coterm t that the first and last injection symbols occurring in $\hat{\sigma}(t)$ agree with the first and last injection symbols, respectively, occurring in t. If $t = e_j^{n_i}$ is an injection symbol, then $\hat{\sigma}(t) = \hat{\sigma}(e_j^{n_i}) = e_j^{n_i}$. If $t = f_i[t_0, \ldots, t_{n_i-1}]$ is a composed coterm where the first and last injection symbol occurring in $\hat{\sigma}(t_l)$ agree with the first and last injection symbol occurring in $t_l; 0 \leqslant l \leqslant n_i-1$, respectively. Suppose that the first injection symbol in $\hat{\sigma}(t_0)$ is $e_0^{n_i}$ and the last injection symbol in $\hat{\sigma}(t_{n_i-1})$ is $e_{n_i-1}^{n_i}$. Then the first and last injection symbols in t is $e_0^{n_i}$ and $e_{n_i-1}^{n_i}$, respectively. Since $\sigma \in Out_{CG}^{inj}(\tau)$, the first and last injection symbol in $\hat{\sigma}(t) = (\sigma(f_i))[\hat{\sigma}(t_0), \ldots, \hat{\sigma}(t_{n_i-1})]$ is $e_0^{n_i}$ and $e_{n_i-1}^{n_i}$, respectively.

Now, we can show that $Left_{CG}^{inj}(\tau), Right_{CG}^{inj}(\tau)$ and $Out_{CG}^{inj}(\tau)$ are closed under the operation $\circ_{CG}$. Let σ_1, σ_2 be generalized cohypersubstitutions, both either leftmost, rightmost or outermost. Then

$$\begin{aligned}(\sigma_1 \circ_{CG} \sigma_2)\hat{}(f_i[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}]) &= (\hat{\sigma}_1 \circ \sigma_2)\hat{}(f_i[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}]) \\ &= \hat{\sigma}_1(\hat{\sigma}_2(f_i[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}])),\end{aligned}$$

and it follows from the previous reasoning that this product has the corresponding property.

Let $\sigma \in Reg_{CG}^{inj}(\tau)$ and $t \in CT_\tau$. We will prove by induction on the complexity of the coterm t that the injection symbols occurring in t and $\hat{\sigma}(t)$ are the same. If $t = e_j^{n_i}$ is an injection symbol, then $\hat{\sigma}(t) = \hat{\sigma}(e_j^{n_i}) = e_j^{n_i}$. If $t = f_i[t_0, \ldots, t_{n_i-1}]$, where the injection symbol occurring in $\hat{\sigma}(t_l)$ and $t_l; 0 \leqslant l \leqslant n_i - 1$ are the same. Since $\hat{\sigma}(t) = (\sigma(f_i))[\hat{\sigma}(t_0), \ldots, \hat{\sigma}(t_{n_i-1})]$ and $\sigma \in Reg_{CG}^{inj}(\tau)$, the injection symbols occurring in t and $\hat{\sigma}(t)$ are the same. So, if $\sigma_1, \sigma_2 \in Reg_{CG}^{inj}(\tau)$, then $(\sigma_1 \circ_{CG} \sigma_2)\hat{}(f_i[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}]) = \hat{\sigma}_1(\hat{\sigma}_2(f_i[e_0^{n_i}, \ldots, e_{n_i-1}^{n_i}]))$. It follows from the previous reasoning that this product has the corresponding property.

Finally, let $\sigma_1, \sigma_2 \in Pre_{CG}^{inj}(\tau)$. It is clear that $\sigma_1 \circ_{CG} \sigma_2$ is again a pre-generalized cohypersubstitution.

Therefore, the sets $P_{CG}^{inj}(\tau) \cup \{\sigma_{id}\}, D_{CG}^{inj}(\tau), Left_{CG}^{inj}(\tau), Right_{CG}^{inj}(\tau)$, $Out_{CG}^{inj}(\tau), Reg_{CG}^{inj}(\tau)$, and $Pre_{CG}^{inj}(\tau)$ are submonoids of $\underline{Cohyp_G(\tau)}$. □

Proposition 4.3. *For any type τ, the following properties hold:*

(i) $D_{CG}^{inj}(\tau) \subset Pre_{CG}^{inj}(\tau)$,

(ii) $Reg_{CG}^{inj}(\tau) \subset Pre_{CG}^{inj}(\tau)$,

(iii) $Out_{CG}^{inj}(\tau) \subset Pre_{CG}^{inj}(\tau)$.

Proof. The proof is straightforward. □

Proposition 4.4. *For any type τ, the sets $P_{CG}^{inj}(\tau) \cup \{\sigma_{id}\}, Left_{CG}^{inj}(\tau)$, $Right_{CG}^{inj}(\tau), Out_{CG}^{inj}(\tau), Reg_{CG}^{inj}(\tau)$, and $Pre_{CG}^{inj}(\tau)$ are subsemigroups of $(Cohyp_G(\tau), +_{CG})$.*

Proof. We will prove that the sets $P_{CG}^{inj}(\tau) \cup \{\sigma_{id}\}, Left_{CG}^{inj}(\tau), Right_{CG}^{inj}(\tau)$, $Out_{CG}^{inj}(\tau), Reg_{CG}^{inj}(\tau)$, and $Pre_{CG}^{inj}(\tau)$ are closed under the operation $+_{CG}$.

Let $\sigma_1, \sigma_2 \in P_{CG}^{inj}(\tau)$. Then $\sigma_1(f_i)$ and $\sigma_2(f_i)$ are injection symbols for each $i \in I$. Since $(\sigma_1 +_{CG} \sigma_2)(f_i) := (\sigma_2(f_i))[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}]$ and both of the coterm $\sigma_1(f_i)$ and $\sigma_2(f_i)$ are injection symbols, this implies that the coterm $(\sigma_1 +_{CG} \sigma_2)(f_i)$ is an injection symbol. So $(\sigma_1 +_{CG} \sigma_2)(f_i) \in P_{CG}^{inj}(\tau)$.

Let $\sigma_1, \sigma_2 \in Left_{CG}^{inj}(\tau)$. Then $leftmost_{inj}(\sigma_1(f_i)) = leftmost_{inj}(\sigma_2(f_i)) = e_0^{n_i}$. By the definition of $+_{CG}$, we have $(\sigma_1 +_{CG} \sigma_2)(f_i) := (\sigma_2(f_i))[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}]$. Since $leftmost_{inj}(\sigma_1(f_i)) = leftmost_{inj}(\sigma_2(f_i)) = e_0^{n_i}$, if we substitute in the coterm $\sigma_2(f_i)$ by a coterm $\sigma_1(f_i)$, then we also have $leftmost_{inj}((\sigma_2(f_i))[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}]) = e_0^{n_i}$. So, $\sigma_1 +_{CG} \sigma_2 \in Left_{CG}^{inj}(\tau)$.

Similarly, we can show that $\sigma_1 +_{CG} \sigma_2 \in Right_{CG}^{inj}(\tau)$ and $\sigma_1 +_{CG} \sigma_2 \in Out_{CG}^{inj}(\tau)$.

Let $\sigma_1, \sigma_2 \in Reg_{CG}^{inj}(\tau)$. Then every injection symbols $e_0^{n_i}, \ldots, e_{n_i-1}^{inj}$ occurs in $\sigma_1(f_i)$ and $\sigma_2(f_i)$. Since $(\sigma_1 +_{CG} \sigma_2)(f_i) := (\sigma_2(f_i))[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}]$ and each of injection symbols $e_0^{n_i}, \ldots, e_{n_i-1}^{inj}$ occurs in $\sigma_1(f_i)$ and $\sigma_2(f_i)$, then every injection symbols $e_0^{n_i}, \ldots, e_{n_i-1}^{inj}$ are in $(\sigma_1 +_{CG} \sigma_2)(f_i)$. Thus $\sigma_1 +_{CG} \sigma_2 \in Reg_{CG}^{inj}(\tau)$.

Let $\sigma_1, \sigma_2 \in Pre_{CG}^{inj}(\tau)$. Then both $\sigma_1(f_i)$ and $\sigma_2(f_i)$ are not injection symbols. Since $(\sigma_1 +_{CG} \sigma_2)(f_i) := (\sigma_2(f_i))[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}]$ and $\sigma_1(f_i)$ and $\sigma_2(f_i)$ are not injection symbols, this implies that $(\sigma_1 +_{CG} \sigma_2)(f_i)$ is not an injection symbol. Hence $\sigma_1 +_{CG} \sigma_2 \in Pre_{CG}^{inj}(\tau)$.

Therefore, $P_{CG}^{inj}(\tau) \cup \{\sigma_{id}\}, Left_{CG}^{inj}(\tau)$, $Right_{CG}^{inj}(\tau), Out_{CG}^{inj}(\tau), Reg_{CG}^{inj}(\tau)$, and $Pre_{CG}^{inj}(\tau)$ are subsemigroups of $(Cohyp_G(\tau), +_{CG})$. □

Proposition 4.5. *Let τ be a type of generalized cohypersubstitution. The set $D_{CG}^{inj}(\tau)$ is not a subsemigroup of $(Cohyp_G(\tau), +_{CG})$.*

Proof. Let $\sigma_1, \sigma_2 \in D_{CG}^{inj}(\tau)$. Then $\sigma_1(f_i) = f_i[e_{\pi(0)}^{n_i}, \ldots, e_{\pi(n_i-1)}^{n_i}]$ and $\sigma_2(f_i) = f_i[e_{\pi'(0)}^{n_i}, \ldots, e_{\pi'(n_i-1)}^{n_i}]$ where π, π' are permutations of the set $J = \{0, \ldots, n_i -$

1}. Consider

$$\begin{aligned}
& (\sigma_1 +_{CG} \sigma_2)(f_i) \\
= \; & (\sigma_2(f_i))[\underbrace{\sigma_1(f_i), \ldots, \sigma_1(f_i)}_{n_i-terms}] \\
= \; & (f_i[e^{n_i}_{\pi'(0)}, \ldots, e^{n_i}_{\pi'(n_i-1)}])[\underbrace{f_i[e^{n_i}_{\pi(0)}, \ldots, e^{n_i}_{\pi(n_i-1)}], \ldots, f_i[e^{n_i}_{\pi(0)}, \ldots, e^{n_i}_{\pi(n_i-1)}]}_{n_i-terms}] \\
= \; & f_i[f_i[e^{n_i}_{\pi(0)}, \ldots, e^{n_i}_{\pi(n_i-1)}], \ldots, f_i[e^{n_i}_{\pi(0)}, \ldots, e^{n_i}_{\pi(n_i-1)}]].
\end{aligned}$$

Hence, $\sigma_1 \circ_{CG} \sigma_2 \notin D^{inj}_{CG}(\tau)$. Therefore, $D^{inj}_{CG}(\tau)$ is not a subsemigroup of $(Cohyp_G(\tau), +_{CG})$. □

Proposition 4.6. *For any type τ, the set $Pre^{inj}_{CG}(\tau)$ is a maximal subsemigroup of $(Cohyp_G(\tau), +_{CG})$.*

Proof. The proof directly follows from Propositions 4.3, 4.4, and 4.5. □

Proposition 4.7. *For any type τ, the sets $P^{inj}_{CG}(\tau) \cup \{\sigma_{id}\}, Left^{inj}_{CG}(\tau)$, $Right^{inj}_{CG}(\tau), Out^{inj}_{CG}(\tau), Reg^{inj}_{CG}(\tau)$, and $Pre^{inj}_{CG}(\tau)$ are subsemigroups of $(Cohyp_G(\tau), \oplus_{CG})$.*

Proof. The proof is similar to that of Proposition 4.4 by considering $(\sigma_1 \oplus_{CG} \sigma_2)(f_i) := (\sigma_1(f_i))[\underbrace{\sigma_2(f_i), \ldots, \sigma_2(f_i)}_{n_i-terms}]$. □

Proposition 4.8. *Let τ be a type of generalized cohypersubstitution. The set $D^{inj}_{CG}(\tau)$ is not a subsemigroup of $(Cohyp_G(\tau), \oplus_{CG})$.*

Proof. The proof is similar to that of Proposition 4.5. □

Proposition 4.9. *For any type τ, the set $Pre^{inj}_{CG}(\tau)$ is a maximal subsemigroup of $(Cohyp_G(\tau), \oplus_{CG})$.*

Proof. The proof directly follows from Propositions 4.3, 4.7, and 4.8. □

Now, we have the following theorem:

Theorem 4.10. *Let τ be a type of generalized cohypersubstitution. The sets $P^{inj}_{CG}(\tau) \cup \{\sigma_{id}\}, Left^{inj}_{CG}(\tau), Right^{inj}_{CG}(\tau), Out^{inj}_{CG}(\tau), Reg^{inj}_{CG}(\tau)$, and $Pre^{inj}_{CG}(\tau)$ are sub-left seminear-ring of $Cohyp^{+_{CG}}_G(\tau)$.*

Proof. The proof directly follows from Propositions 4.2 and 4.4. □

Theorem 4.11. *Let τ be a type of generalized cohypersubstitution. The sets $P_{CG}^{inj}(\tau) \cup \{\sigma_{id}\}, Left_{CG}^{inj}(\tau), Right_{CG}^{inj}(\tau), Out_{CG}^{inj}(\tau), Reg_{CG}^{inj}(\tau)$, and $Pre_{CG}^{inj}(\tau)$ are sub-left seminear-ring of $Cohyp_G^{\oplus_{CG}}(\tau)$.*

Proof. The proof directly follows from Propositions 4.2 and 4.7. ☐

Acknowledgment. The authors would like to thank the referees for their careful reading of the manuscript and their useful comments.

References

[1] J. Boonsol, Linear Cohypersubstitutions and Generalized Linear Cohypersubstitutions of type $\tau = (n)$, M. Sc. Thesis, Mahasarakham University, 2019.

[2] K. Denecke, K. Saengsura, Separation of clones of cooperations by cohyperidentities, Discrete Mathematics, **309,** (2009), 772–783.
doi: 10.1016/j.dise.2008.01.043

[3] K. Denecke, S. L. Wismath, Universal Algebra and Coalgebra, World Scientific Publishing Co., Singapore, 2009).

[4] S. Jermjitpornchai, N. Saengsura, Generalized Cohypersubstitutions of Type $\tau = (n_i)_{i \in I}$, International Journal of Pure and Applied Mathematics, **86,** no. 4, (2013), 745–755.
doi: http://dx.doi.org/10.12732/ijpam.v86i4.12

[5] S. Leeratanavalee, Structural Properties of Generalized Hypersubstitutions, Kyungpook Mathematical Journal, **44,** no. 2, (2004), 261–267.

[6] S. Leeratanavalee, Submonoids of Generalized Hypersubstitutions, Demonstratio Mathematica, **XL,** no. 1, (2007), 13–22.
doi: https://doi.org/10.1515/dema-2007-0103.

[7] N. Saengsura, S. Jermjitpornchai, Idempotent and Regular Generalized Cohypersubstitutions of Type $\tau = (2)$, International Journal of Pure and Applied Mathematics, **86,** no. 4, (2013), 757–766.
doi: http://dx.doi.org/10.12732/ijpam.v86i4.13