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Corrigendum to: On The Diophantine Equation $(132k)^x + (4355k)^y = (4357k)^z$

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Abstract

The Jeśmanowicz's conjecture written in 1956 states that for any primitive Pythagorean triple (a, b, c) with $a^2+b^2 = c^2$ and any positive integer k, the only solution of equation $(ak)^x+(bk)^y = (ck)^z$ in positive integers is $(x, y, z) = (2, 2, 2)$. In this paper, we show that the special Diophantine equation $(132k)^x + (4355k)^y = (4357k)^z$ has the only positive integer solution $(x, y, z) = (2, 2, 2)$ for every positive integer k.

1 Introduction

In 1956, Sierpiński [6] showed that the only positive integer solution of the Diophantine Equation

$$(ak)^x + (bk)^y = (ck)^z \tag{1.1}$$

is $(x, y, z) = (2, 2, 2)$, for $k = 1$ and $(a, b, c) = (3, 4, 5)$, and Jeśmanowicz [2] proved that the conjecture is true when $k = 1$ and $(a, b, c) \in \{(5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)\}$. Jeśmanowicz also conjectured that the Diophantine equation (1.1) has the only positive integer solution $(x, y, z) = (2, 2, 2)$ for any positive integer k. There are many special

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cases of Jeśmanowicz's conjecture solved for $k = 1$. In 2012, Yang and Tang [11] proved that the only solution of the Diophantine Equation

$$(8k)^x + (15k)^y = (17k)^z \tag{1.2}$$

is $(x, y, z) = (2, 2, 2)$, for $k \geqslant 1$. Several authors had shown that Jeśmanowicz's conjecture is true for $n \in \{2, 3, 4, 8\}$ where $(a, b, c) = (4n, 4n^2 - 1, 4n^2 + 1)$, see [9] and [12]. Yang and Jianxin [12] proved that the only solution of

$$(12k)^x + (35k)^y = (37k)^z \tag{1.3}$$

is $(x, y, z) = (2, 2, 2)$ for $k \geqslant 1$. In 2015, Ma and Wu [5] proved that the only solution of the Diophantine Equation

$$((4n^2 - 1)k)^x + (4nk)^y = ((4n^2 + 1)k)^z \tag{1.4}$$

is $(x, y, z) = (2, 2, 2)$ when $P(4n^2 - 1)|k$, where $P(m)$ denotes the product of distinct primes of m. They showed that if k is a positive integer and $P(k) \nmid (4n^2 - 1)$, then the only solution for equation (1.4) is $(x, y, z) = (2, 2, 2)$. In this case, they considered $n = p^m$, p prime and $m \geqslant 0$ with $p \equiv -1 (\mathrm{mod}\ 4)$. In 2017, Soydan, Demirci, Cangul, and Togbé [7] considered(1.1) with $(a, b, c) = (20, 99, 101)$ and they proved the Diophantine equation

$$(20k)^x + (99k)^y = (101k)^z \tag{1.5}$$

has only the solution $(x, y, z) = (2, 2, 2)$. In this paper, we consider the case $n = 33$ and $(a, b, c) = (4n, 4n^2 - 1, 4n^2 + 1)$ for (1.1). For other results, see for instance [10], [8], [3] and [1]. Our main result is the following theorem.

Theorem 1.1. *The only positive integer solution of the Diophantine equation*

$$(132k)^x + (4355k)^y = (4357k)^z \tag{1.6}$$

is $(x, y, z) = (2, 2, 2)$*, for every positive integer* k.

2 Proof Of Theorem 1.1

In this section, we begin with three useful results as follows:

Lemma 2.1. *(see [3]) If* (x, y, z) *is a solution of* (1.1) *with* $(x, y, z) \neq (2, 2, 2)$*, then* x, y *and* z *are distinct.*

Lemma 2.2. *(see [4]) The only positive integer solution of the Diophantine equation* $(4n^2-1)^x+(4n)^y=(4n^2+1)^z$ *is* $(x,y,z)=(2,2,2)$.

Lemma 2.3. *(see [1]) If* $z \geqslant \max\{x,y\}$*, then the Diophantine equation* $a^x+b^y=c^z$*, where* a,b *and* c *are any positive integers (not necessarily relatively prime) such that* $a^2+b^2=c^2$*, has no solution other than* $x=y=z=2$.

Proof. (Theorem 1.1)

When $k=1$, equation (1.6) becomes

$$(132)^x+(4355)^y=(4357)^z \tag{2.7}$$

from lemma 2.2, the Diophantine equation (2.7) has the only positive integer solution $(x,y,z)=(2,2,2)$. Suppose that (1.6) has at least another solution $(x,y,z)\neq(2,2,2)$. Then, by lemma 2.3, we have $z<\max\{x,y\}$ and, from lemma 2.1, $x\neq y, y\neq z$ and $x\neq z$. Thus, we consider two cases as follows:

Case 1 If $x<y$, then we consider two subcases $z<x<y$ and $x<z<y$.

Subcase 1.1 If $z<x<y$, then rewrite equation (1.6) as

$$k^{x-z}(132^x+4355^yk^{y-x})=4357^z. \tag{2.8}$$

So if $(k,4357)=1$, then $x=z$, where $k\geqslant 2$, which is a contradiction. In addition, if $(k,4357)=4357$, then we can write $k=4357^m n_1$, where $m\geqslant 1$, $n_1\geqslant 1$ and $(4357,n_1)=1$. Rewrite equation (2.8) as

$$4357^{m(x-z)}{n_1}^{x-z}(132^x+4355^y4357^{m(y-x)}{n_1}^{y-x})=4357^z. \tag{2.9}$$

Then $n_1^{x-z}\big|\,4357^z$ and so $n_1=1$. Therefore (2.9) becomes

$$132^x+4355^y4357^{m(y-x)}=4357^{z-m(x-z)} \tag{2.10}$$

which implies that $4357|\,132^x$ which is impossible.

Subcase 1.2 If $x<z<y$, then we rewrite (1.6) as

$$132^x+4355^yk^{y-x}=4357^zk^{z-x} \tag{2.11}$$

So if $(k,132)=1$, then $x=z$, where $k\geqslant 2$, which is a contradiction. In addition, if $(k,132)>1$, then we can write $k=2^r3^s11^qn_1$, where $r+s+q\geqslant 1$, $n_1\geqslant 1$ and $(66,n_1)=1$, So rewrite (2.11) as

$$132^x=2^{r(z-x)}3^{s(z-x)}11^{q(z-x)}{n_1}^{z-x}\left[\,4357^z-4355^y2^{r(y-z)}3^{s(y-z)}11^{q(y-z)}{n_1}^{y-z}\right] \tag{2.12}$$

Then we get seven cases as follows:

1. If $k = 2^r n_1$, where $r \geqslant 1$, $n_1 \geqslant 1$, $s = q = 0$ and $(66, n_1) = 1$, then (2.12) becomes
$$132^x = 2^{r(z-x)} {n_1}^{z-x} \left[\, 4357^z - 4355^y 2^{r(y-z)} {n_1}^{y-z} \right]. \tag{2.13}$$
Thus $2x = r(z-x)$ and $33^x = {n_1}^{z-x} \left[4357^z - 4355^y 2^{r(y-z)} {n_1}^{y-z}\right]$. Hence $n_1 = 1$ and
$$4357^z - 33^x = 5^y 13^y 67^y 2^{r(y-z)}, \tag{2.14}$$
where $(66, n_1) = 1$. By considering Equation (2.14) modulo 67, we obtain
$$2^z - 33^x \equiv 0 (\bmod 67). \tag{2.15}$$
Since 2 is a primitive root of 67, the congruence (2.15) becomes
$$z \equiv 32x (\bmod 66). \tag{2.16}$$
Therefore, z is even. Also, by considering Equation (2.14) modulo 13, we obtain
$$2^z - 7^x \equiv 0 (\bmod 13). \tag{2.17}$$
Since 2 is a primitive root of 13, the congruence (2.17) becomes
$$z \equiv 11x (\bmod 12). \tag{2.18}$$
Since $z - 11x$ and z are even, x is even. Assume that $z = 2z_1$ and $x = 2x_1$ with $z_1 > x_1$. Therefore, Equation (2.14) becomes
$$(4357^{z_1} - 33^{x_1})(4357^{z_1} + 33^{x_1}) = 5^y 13^y 67^y 2^{r(y-z)}. \tag{2.19}$$
Since
$$(4357^{z_1} - 33^{x_1}, 4357^{z_1} + 33^{x_1}) = 2,$$
based on Equation (2.19), we obtain
$$67^y | \, 4357^{z_1} - 33^{x_1} \ \textit{or} \ \ 67^y | \, 4357^{z_1} + 33^{x_1}. \tag{2.20}$$
But
$$\begin{aligned} 67^y > 67^z = 4489^{z_1} &> (4357 + 33)^{z_1}, \\ &> 4357^{z_1} + 33^{z_1}, \\ &> 4357^{z_1} + 33^{x_1}, \\ &> 4357^{z_1} - 33^{x_1}, \end{aligned}$$

and this contradicts (2.20).

2. If $k = 3^s n_1$ where $s \geqslant 1$, $n_1 \geqslant 1$, $r = q = 0$ and $(66, n_1) = 1$, then (2.12) becomes

$$\frac{2^{2x}3^x 11^x}{3^{s(z-x)}} = n_1^{z-x}\left[4357^z - 4355^y 3^{s(y-z)} n_1^{y-z}\right]. \tag{2.21}$$

Thus $x = s(z-x)$ and $44^x = n_1^{z-x}\left[4357^z - 4355^y 3^{s(y-z)} n_1^{y-z}\right]$. So $n_1 = 1$. Accordingly, Equation (2.21) becomes

$$4357^z - 44^x = 67^y 13^y 5^y 3^{s(y-z)}. \tag{2.22}$$

By considering Equation (2.22) modulo 3, we get

$$2^x \equiv 1 (\bmod 3).$$

Thus $x \equiv 0(\bmod 2)$. Similarly, by taking Equation (2.22) modulo 5, we obtain

$$2^z \equiv (-1)^x \equiv 1 (\bmod 5).$$

Thus $z \equiv 0(\bmod 4)$. Therefore, we can write $x = 2x_1$ and $z = 2z_1$ with $z_1 > x_1$. Accordingly, Equation (2.22) becomes

$$(4357^{z_1} - 44^{x_1})(4357^{z_1} + 44^{x_1}) = 67^y 13^y 5^y 3^{s(y-z)}. \tag{2.23}$$

Since

$$(4357^{z_1} - 44^{x_1}, 4357^{z_1} + 44^{x_1}) = 1,$$

based on Equation (2.23), we have

$$67^y | \, 4357^{z_1} - 44^{x_1} \;\; or \;\; 67^y | \, 4357^{z_1} + 44^{x_1}. \tag{2.24}$$

However,

$$\begin{aligned} 67^y > 67^z = 4489^{z_1} &> (4357 + 44)^{z_1}, \\ &> 4357^{z_1} + 44^{z_1}, \\ &> 4357^{z_1} + 44^{x_1}, \\ &> 4357^{z_1} - 44^{x_1}, \end{aligned}$$

and this contradicts (2.24).

3. If $k = 11^q n_1$, where $q \geqslant 1$, $n_1 \geqslant 1$, $r = s = 0$ and $(66, n_1) = 1$, then, from (2.12), we get

$$\frac{2^{2x}3^x 11^x}{11^{q(z-x)}} = n_1^{z-x}\left[4357^z - 4355^y 11^{q(y-z)} n_1^{y-z}\right]. \tag{2.25}$$

Thus, $x = q(z-x)$ and $12^x = n_1^{z-x}\left[4357^z - 4355^y 11^{q(y-z)} n_1^{y-z}\right]$. So $n_1 = 1$. Then Equation (2.25) becomes

$$4357^z - 12^x = 67^y 13^y 5^y 11^{q(y-z)}. \tag{2.26}$$

By considering Equation (2.26) modulo 13, we obtain

$$2^z - 12^x \equiv 0 (\operatorname{mod} 13). \tag{2.27}$$

Since 2 is a primitive root of 13, the congruence (2.27) becomes

$$z \equiv 6x (\operatorname{mod} 12). \tag{2.28}$$

Thus z must be even. Also, by considering Equation (2.26) modulo 5, we obtain

$$2^z - 2^x \equiv 0 (\operatorname{mod} 5). \tag{2.29}$$

Since 2 is a primitive root of 5, the congruence (2.29) becomes

$$z \equiv x (\operatorname{mod} 4), \tag{2.30}$$

and since $z - x$ and z are even, x is even. Therefore, we can write $x = 2x_1$ and $z = 2z_1$ with $z_1 > x_1$. Accordingly, Equation (2.26) becomes

$$(4357^{z_1} - 12^{x_1})(4357^{z_1} + 12^{x_1}) = 67^y 13^y 5^y 11^{q(y-z)}. \tag{2.31}$$

We have $(4357^{z_1} - 12^{x_1}, 4357^{z_1} + 12^{x_1}) = 1$. Thus, based on Equation (2.31), we obtain

$$67^y |\, 4357^{z_1} - 12^{x_1} \;\; or \;\; 67^y |\, 4357^{z_1} + 12^{x_1}. \tag{2.32}$$

But, from $x < z < y$, we have

$$\begin{aligned} 67^y > 67^z = 4489^{z_1} &> (4357 + 12)^{z_1}, \\ &> 4357^{z_1} + 12^{z_1}, \\ &> 4357^{z_1} + 12^{x_1}, \\ &> 4357^{z_1} - 12^{x_1}, \end{aligned}$$

and this contradicts (2.32).

4. If $k = 2^r 3^s n_1$, where $r \geqslant 1, s \geqslant 1, n_1 \geqslant 1, q = 0$ and $(66, n_1) = 1$, then, from (2.12), we get the equation

$$\frac{2^{2x}3^x 11^x}{2^{r(z-x)}3^{s(z-x)}} = n_1^{z-x}\left[4357^z - 4355^y 2^{r(y-z)}3^{s(y-z)}n_1^{y-z}\right]. \tag{2.33}$$

Thus,

$2x = r(z-x)$, $x = s(z-x)$ and $11^x = n_1^{z-x}\left[4357^z - 4355^y 12^{s(y-z)}n_1^{y-z}\right]$.

So, $n_1 = 1$ and Equation (2.33) becomes

$$4357^z - 11^x = 67^y 13^y 5^y 12^{s(y-z)}. \tag{2.34}$$

By considering Equation (2.34) modulo 3, we obtain $2^x \equiv 1 (\mathrm{mod}\, 3)$. Thus $x \equiv 0 (\mathrm{mod}\, 2)$. Also, by considering Equation (2.34) modulo 5, we get $2^z \equiv 1 (\mathrm{mod}\, 5)$. Hence, $z \equiv 0 (\mathrm{mod}\, 4)$. Therefore, we can write $x = 2x_1$ and $z = 2z_1$ with $z_1 > x_1$. Accordingly, Equation (2.34) becomes

$$(4357^{z_1} - 11^{x_1})(4357^{z_1} + 11^{x_1}) = 67^y 13^y 5^y 12^{s(y-z)}. \tag{2.35}$$

Observe that

$$(4357^{z_1} - 11^{x_1}, 4357^{z_1} + 11^{x_1}) = 2.$$

Thus, based on Equation (2.35), we obtain

$$67^y \,|\, 4357^{z_1} - 11^{x_1} \ \textit{or} \ 67^y \,|\, 4357^{z_1} + 11^{x_1}. \tag{2.36}$$

But

$$\begin{aligned} 67^y > 67^z = 4489^{z_1} &> (4357 + 11)^{z_1}, \\ &> 4357^{z_1} + 11^{z_1}, \\ &> 4357^{z_1} + 11^{x_1}, \\ &> 4357^{z_1} - 11^{x_1}, \end{aligned}$$

and this contradicts (2.36).

5. If $k = 2^r 11^q n_1$, where $r \geqslant 1, q \geqslant 1, n_1 \geqslant 1, s = 0$ and $(66, n_1) = 1$, then, from (2.12), we get the equation

$$\frac{2^{2x}3^x 11^x}{2^{r(z-x)}11^{q(z-x)}} = n_1^{z-x}\left[4357^z - 4355^y 2^{r(y-z)}11^{q(y-z)}n_1^{y-z}\right]. \tag{2.37}$$

Thus

$$2x = r(z-x), \;\; x = q(z-x) \text{ and } 3^x = n_1^{z-x}\left[4357^z - 4355^y 44^{q(y-z)} n_1^{y-z}\right].$$

So, $n_1 = 1$ and Equation (2.37) becomes

$$4357^z - 3^x = 67^y 13^y 5^y 44^{q(y-z)}. \tag{2.38}$$

By considering Equation (2.38) modulo 4, we obtain

$$1 \equiv (-1)^x (\bmod 4).$$

Thus x must be even. Similarly, by considering Equation (2.38) modulo 5, we obtain

$$2^z \equiv 3^x (\bmod 5). \tag{2.39}$$

Since 2 is a primitive root of 5, the congruence (2.39) becomes $z \equiv 3x(\bmod 4)$. Since $z-3x$ and x are even, z must be even. Therefore, we can write $x = 2x_1$ and $z = 2z_1$ with $z_1 > x_1$. Hence, Equation (2.38) becomes

$$(4357^{z_1} - 3^{x_1})(4357^{z_1} + 3^{x_1}) = 67^y 13^y 5^y 44^{q(y-z)}. \tag{2.40}$$

Since $(4357^{z_1} - 3^{x_1}, 4357^{z_1} + 3^{x_1}) = 2$,

$$67^y | \, 4357^{z_1} - 3^{x_1} \;\; or \;\; 67^y | \, 4357^{z_1} + 3^{x_1}. \tag{2.41}$$

But

$$\begin{aligned} 67^y > 67^z = 4489^{z_1} &> (4357+3)^{z_1}, \\ &> 4357^{z_1} + 3^{z_1}, \\ &> 4357^{z_1} + 3^{x_1}, \\ &> 4357^{z_1} - 3^{x_1}, \end{aligned}$$

and this contradicts (2.41).

6. If $k = 3^s 11^q n_1$, where $s \geqslant 1, q \geqslant 1, n_1 \geqslant 1, r = 0$ and $(66, n_1) = 1$, then, from (2.12), we get

$$\frac{2^{2x} 3^x 11^x}{3^{s(z-x)} 11^{q(z-x)}} = n_1^{z-x}\left[4357^z - 4355^y 3^{s(y-z)} 11^{q(y-z)} n_1^{y-z}\right]. \tag{2.42}$$

So

$$x = s(z-x) = q(z-x) \ \text{ and } \ 2^{2x} = n_1^{z-x}\left[4357^z - 4355^y 33^{s(y-z)} n_1^{y-z}\right].$$

Thus $n_1 = 1$ and Equation (2.42) becomes

$$4357^z - 2^{2x} = 67^y 13^y 5^y 33^{s(y-z)}. \tag{2.43}$$

By considering Equation (2.43) modulo 5, we obtain

$$2^z - 2^{2x} \equiv 0 (\text{mod}\, 5). \tag{2.44}$$

Since 2 is a primitive root of 5, the congruence (2.44) becomes

$$z \equiv 2x (\text{mod}\, 4), \tag{2.45}$$

and since $z - 2x$ is even, z is even. Put $z = 2z_1$. Hence Equation (2.43) becomes

$$(4357^{z_1} - 2^x)(4357^{z_1} + 2^x) = 67^y 13^y 5^y 33^{s(y-z)}. \tag{2.46}$$

Since

$$(4357^{z_1} - 2^x, 4357^{z_1} + 2^x) = 1,$$

$$67^y | \, 4357^{z_1} - 2^x \ \textit{or} \ \ 67^y | \, 4357^{z_1} + 2^x. \tag{2.47}$$

But

$$\begin{aligned} 67^y > 67^z = 4489^{z_1} &> (4357 + 2^2)^{z_1}, \\ &> 4357^{z_1} + 2^x, \\ &> 4357^{z_1} - 2^x, \end{aligned}$$

and this contradicts (2.47).

7. If $k = 2^r 3^s 11^q n_1$, where $r \geqslant 1, s \geqslant 1, q \geqslant 1, n_1 \geqslant 1$, and $(66, n_1) = 1$, then, from (2.12), we get the equation

$$n_1{}^{z-x}\left[4357^z - 4355^y 2^{r(y-z)} 11^{q(y-z)} 3^{s(y-z)} n_1{}^{y-z}\right] = 1 \tag{2.48}$$

Since $x \neq z$, $n_1 = 1$. Therefore,

$$4357^z - 1 = 4355^y 2^{r(y-z)} 11^{q(y-z)} 3^{s(y-z)} \tag{2.49}$$

Since $4357^z - 1 \equiv 2^z - 1 (\bmod 5)$, $z \equiv 0 (\bmod 4)$.
But

$$4357^2 \equiv 1 (\bmod 2179).$$

Thus

$$4357^z - 1 \equiv 0 (\bmod 2179).$$

From (2.49), we obtain

$$4355^y 2^{r(y-z)} 11^{q(y-z)} 3^{s(y-z)} \equiv 0 (\bmod 2179),$$

which is impossible. This completes the proof for the first case.

Case 2 If $x > y$, then we obtain two subcases $z < y < x$ and $y < z < x$.
Subcase 2.1 If $z < y < x$, then, rewrite Equation (1.6) as

$$k^{y-z}(132^x k^{x-y} + 4355^y) = 4357^z. \tag{2.50}$$

If $(k, 4357) = 1$, then $y = z$, where $k \geqslant 2$, which is a contradiction. In addition, if $(k, 4357) = 4357$, then we can write $k = 4357^m n_1$, where $m \geqslant 1, n_1 \geqslant 1$ and $(4357, n_1) = 1$. Rewrite Equation (2.50) as

$$4357^{m(y-z)} {n_1}^{y-z} (132^x 4357^{m(x-y)} {n_1}^{x-y} + 4355^y) = 4357^z \tag{2.51}$$

Since

$$(n_1, 4357) = (132^x 4357^{m(x-y)} {n_1}^{x-y} + 4355^y, 4357) = 1,$$
$${n_1}^{y-z} (132^x 4357^{m(x-y)} {n_1}^{x-y} + 4355^y) = 1$$

which is impossible.
Subcase 2.2 If $y < z < x$, then, rewriting (1.6) as

$$k^{z-y}(4357^z - 132^x k^{x-z}) = 4355^y, \tag{2.52}$$

we have if $(k, 4355) = 1$, then $y = z$, where $k \geqslant 2$, which is a contradiction. In addition, if $(k, 4355) > 1$, then we can write $k = 5^r 13^s 67^q n_1$, where $r + s + q \geqslant 1$, $n_1 \geqslant 1$ and $(4355, n_1) = 1$.

Then we get seven cases as follows:

1. If $k = 5^r n_1$, where $r \geqslant 1$, $n_1 \geqslant 1$ and $s = q = 0$ with $(4355, n_1) = 1$, then rewrite Equation (2.52) as

$$\frac{4355^y}{5^{r(z-y)}} = \frac{5^y 13^y 67^y}{5^{r(z-y)}} = n_1^{z-y} \left[4357^z - 132^x 5^{r(x-z)} n_1^{x-z}\right]. \tag{2.53}$$

So, $5^y = 5^{r(z-y)}$ and $n_1 = 1$. Thus, Equation (2.53) becomes

$$4357^z - 871^y = 4^x 3^x 11^x 5^{r(x-z)}, \tag{2.54}$$

By considering Equation (2.54) modulo 33, we obtain

$$1 \equiv 13^y (\bmod 33). \tag{2.55}$$

Hence, we can write $y = 10m$. Similarly, by considering Equation (2.54) modulo 5, we obtain

$$2^z \equiv 1 (\bmod 5). \tag{2.56}$$

So, write $z = 4c$. Thus, assume

$$z = 2z_1 \quad \text{and} \quad y = 2y_1 \quad \text{with} \quad z_1 > y_1, \quad \text{where} \quad z_1 = 2c \quad \text{and} \quad y_1 = 5m. \tag{2.57}$$

Hence, Equation (2.54) becomes

$$(4357^{z_1} - 871^{y_1})(4357^{z_1} + 871^{y_1}) = 2^{2x} 3^x 11^x 5^{r(x-z)}. \tag{2.58}$$

Since

$$4357^{z_1} + 871^{y_1} \equiv 1 + 1 \equiv 2 (\bmod 3). \tag{2.59}$$

Hence, from (2.59), we get $3 \nmid 4357^{z_1} + 871^{y_1}$ and since

$$(4357^{z_1} - 871^{y_1}, 4357^{z_1} + 871^{y_1}) = 2,$$

based on Equation (2.58), we have two possibilities:

$$2^{2x-1} 3^x \mid 4357^{z_1} - 871^{y_1} \quad \text{and} \quad 2 \mid 4357^{z_1} + 871^{y_1}, \tag{2.60}$$

or

$$2(3^x) \mid 4357^{z_1} - 871^{y_1} \quad \text{and} \quad 2^{2x-1} \mid 4357^{z_1} + 871^{y_1}. \tag{2.61}$$

Taking (2.60), we observe that $4357^{z_1} \equiv 871^{y_1} (\bmod 4)$,
$1 \equiv 3^{y_1} (\bmod 4)$. Thus y_1 is even. From (2.57), we assume that
$y_1 = 5m = 10m_1$. Therefore

$$4357^{z_1} + 871^{y_1} \equiv 1 + 2^{y_1} \equiv 1 + 2^{10m_1} \equiv 1 + 1 \equiv 2 (\bmod 11).$$

Hence, $11 \nmid 4357^{z_1} + 871^{y_1}$. Consequently

$$2^{2x-1} 3^x 11^x \mid 4357^{z_1} - 871^{y_1}. \tag{2.62}$$

But

$$2^{2x-1}3^x11^x = \frac{132^x}{2} = \frac{(2^7+2^2)^x}{2} > 2^{7x-1} + 2^{2x-1} > 2^{7x-1} > 2^{13z_1}$$
$$> (4357+871)^{z_1}$$
$$> 4357^{z_1} + 871^{z_1},$$
$$> 4357^{z_1} + 871^{y_1},$$
$$> 4357^{z_1} - 871^{y_1},$$

and this contradicts (2.62). On the other hand, if we take (2.61), we observe that $4357^{z_1} + 871^{y_1} \equiv 0 (\bmod 4)$ and so $1 + 3^{y_1} \equiv 0 (\bmod 4)$. Therefore, y_1 is odd. From (2.57), we write $y_1 = 5m = 10m_1 + 5$. Similarly,

$$4357^{z_1} + 871^{y_1} \equiv 0 (\bmod 16),$$

thus

$$5^{z_1} + 7^{10m_1+5} \equiv 5^{z_1} + 7 \equiv 0 (\bmod 16).$$

Since $\mathrm{ord}_{16} 5 = 4$, $z_1 = 4s + 2$. Therefore,

$$4357^{z_1} - 871^{y_1} \equiv 1 - 2^{y_1} \equiv 1 - 2^{10m_1+5} \equiv 1 - 10 \equiv 2 (\bmod 11),$$

and

$$4357^{z_1} - 871^{y_1} \equiv 2^{z_1} - 1 \equiv 2^{4s+2} - 1 \equiv 3 (\bmod 5).$$

Hence $11 \nmid 4357^{z_1} - 871^{y_1}$ and $5 \nmid 4357^{z_1} - 871^{y_1}$. So, from (2.61), we have

$$2(3^x) \mid 4357^{z_1} - 871^{y_1} \quad \text{and} \quad 2^{2x-1}11^x5^{r(x-z)} \mid 4357^{z_1} + 871^{y_1}. \quad (2.63)$$

Then, from Equations (2.58) and (2.63), we obtain

$$4357^{z_1} - 871^{y_1} = 2(3^x). \quad (2.64)$$

Thus, by considering Equation (2.64) modulo 13, we obtain

$$2^{z_1} \equiv 2(3^x) (\bmod 13). \quad (2.65)$$

Since 2 is a primitive root of 13, the congruence (2.65) becomes

$$z_1 \equiv 1 + 4x (\bmod 12). \quad (2.66)$$

Thus z_1 is odd and this contradicts (2.57).

2. If $k = 13^s n_1$, where $s \geqslant 1$, $n_1 \geqslant 1$ and $r = q = 0$ with $(4355, n_1) = 1$, then rewrite Equation (2.52) as

$$\frac{4355^y}{13^{s(z-y)}} = \frac{5^y 13^y 67^y}{13^{s(z-y)}} = n_1^{z-y}\left[4357^z - 132^x 13^{s(x-z)} n_1^{x-z}\right]. \qquad (2.67)$$

It follows that $13^y = 13^{s(z-y)}$ and $n_1 = 1$. Therefore Equation (2.67) becomes

$$4357^z - 335^y = 4^x 3^x 11^x 13^{s(x-z)}. \qquad (2.68)$$

By considering Equation (2.68) modulo 99, we obtain

$$1 \equiv 38^y (\mathrm{mod}\, 99). \qquad (2.69)$$

Thus, we can write $y = 30m$. Similarly, by considering Equation (2.68) modulo 16, we obtain

$$5^z \equiv 15^y \equiv 15^{30m} \equiv 1(\mathrm{mod}\, 16). \qquad (2.70)$$

So, write $z = 4c$. Suppose

$$z = 2z_1 \quad \text{and} \quad y = 2y_1 \quad \text{with} \quad z_1 > y_1, \quad \text{where} \quad z_1 = 2c \quad \text{and} \quad y_1 = 15m. \qquad (2.71)$$

Hence Equation (2.68) becomes

$$(4357^{z_1} - 335^{y_1})(4357^{z_1} + 335^{y_1}) = 2^{2x} 3^x 11^x 13^{s(x-z)}. \qquad (2.72)$$

Since

$$4357^{z_1} + 335^{y_1} \equiv 1 + 5^{15m} \equiv 1 + 1 \equiv 2(\mathrm{mod}\, 11), \qquad (2.73)$$

from (2.73), we get $11 \nmid 4357^{z_1} + 335^{y_1}$. Since

$$(4357^{z_1} - 335^{y_1}, 4357^{z_1} + 335^{y_1}) = 2,$$

based on Equation (2.72), we have two possibilities:

$$2^{2x-1} 11^x \mid 4357^{z_1} - 335^{y_1} \quad \text{and} \quad 2 \mid 4357^{z_1} + 335^{y_1}, \qquad (2.74)$$

or

$$2(11^x) \mid 4357^{z_1} - 335^{y_1} \quad \text{and} \quad 2^{2x-1} \mid 4357^{z_1} + 335^{y_1}. \qquad (2.75)$$

Considering (2.74), observe that $4357^{z_1} \equiv 335^{y_1} (\mod 4)$. So $1 \equiv 3^{y_1} (\mod 4)$. Thus y_1 is even. Based on (2.71), we can write $y_1 = 15m = 30m_1$. Therefore

$$4357^{z_1} + 335^{y_1} \equiv 1 + 2^{y_1} \equiv 1 + 2^{30m_1} \equiv 1 + 1 \equiv 2 (\mod 3).$$

Hence $3 \nmid 4357^{z_1} + 335^{y_1}$. Consequently,

$$2^{2x-1} 3^x 11^x \mid 4357^{z_1} - 335^{y_1}. \tag{2.76}$$

However, we have $y < z < x$. Thus

$$\begin{aligned} 2^{2x-1} 3^x 11^x = \frac{132^x}{2} = \frac{(2^7 + 2^2)^x}{2} > 2^{7x-1} + 2^{2x-1} > 2^{7x-1} > 2^{13z_1}&, \\ > (4357 + 335)^{z_1}&, \\ > 4357^{z_1} + 335^{z_1}&, \\ > 4357^{z_1} + 335^{y_1}&, \\ > 4357^{z_1} - 335^{y_1}&, \end{aligned}$$

and this contradicts (2.76). On the other hand, if we consider (2.75), we see that $4357^{z_1} + 335^{y_1} \equiv 0 (\mod 4)$. Thus,

$$1 + 3^{y_1} \equiv 0 (\mod 4).$$

So, y_1 is odd. Based on (2.71), we can write $y_1 = 15m = 30m_1 + 15$. Therefore,

$$4357^{z_1} - 335^{y_1} \equiv 1 - 2^{y_1} \equiv 1 - 2^{30m_1+15} \equiv 1 - 2 \equiv 2 (\mod 3).$$

Hence,

$$3 \nmid 4357^{z_1} - 335^{y_1}. \tag{2.77}$$

Also, if $13 \mid 4357^{z_1} - 335^{y_1}$, then

$$2^{z_1} \equiv 335^{30m_1+15} \equiv 10^{15} \equiv 12 (\mod 13),$$

and since 2 is a primitive root of 13, we obtain $z_1 \equiv 6 (\mod 12)$. Thus $z_1 = 12c + 6$. It follows that

$$4357^{12c+6} + 335^{30m_1+15} \equiv 9 + 15 \equiv 24 \equiv 8 (\mod 16);$$

that is, $16 \nmid 4357^{z_1} + 335^{y_1}$ but this contradicts (2.75). Thus

$$13 \nmid 4357^{z_1} - 335^{y_1}. \tag{2.78}$$

Based on (2.68), (2.75), (2.77), and (2.78), we have

$$4357^{z_1} - 335^{y_1} = 2(11)^x.$$

Therefore, $2^{z_1} \equiv 2 \pmod 5$. Consequently, $z_1 = 4q + 1$; i.e., z_1 is odd and this contradicts (2.71).

3. If $k = 67^q n_1$, where $q \geqslant 1$, $n_1 \geqslant 1$ and $r = s = 0$ with $(4355, n_1) = 1$, then rewrite Equation (2.52) as

$$\frac{4355^y}{67^{q(z-y)}} = \frac{5^y 13^y 67^y}{67^{q(z-y)}} = n_1^{z-y}\left[4357^z - 132^x 67^{q(x-z)} n_1^{x-z}\right]. \tag{2.79}$$

So, $67^y = 67^{q(z-y)}$ and $n_1 = 1$. Thus Equation (2.79) becomes

$$4357^z - 65^y = 4^x 3^x 11^x 67^{q(x-z)}. \tag{2.80}$$

By considering Equation (2.80) modulo 3 and modulo 16, we obtain

$$1 \equiv 2^y \pmod 3), \quad \text{and} \quad 1 \equiv 5^z \pmod{16}.$$

Since $\mathrm{ord}_3 2 = 2$ and $\mathrm{ord}_{16} 5 = 4$, we can write

$$z = 4c = 2z_1 \quad \text{and} \quad y = 2y_1 \quad \text{with} \quad z_1 > y_1, \quad \text{where} \quad z_1 = 2c. \tag{2.81}$$

Hence Equation (2.80) becomes

$$(4357^{z_1} - 65^{y_1})(4357^{z_1} + 65^{y_1}) = 2^{2x} 3^x 11^x 67^{q(x-z)}. \tag{2.82}$$

Since $4357^{z_1} + 65^{y_1} \equiv 2 \pmod 4$, $4 \nmid 4357^{z_1} + 65^{y_1}$ and since $(4357^{z_1} - 65^{y_1}, 4357^{z_1} + 65^{y_1}) = 2$, from Equation (2.82), we get

$$2^{2x-1} \mid 4357^{z_1} - 65^{y_1}, \quad \text{and} \quad 2 \mid 4357^{z_1} + 65^{y_1}. \tag{2.83}$$

If $y_1 = 2m$, then

$$4357^{z_1} + 65^{2m} \equiv 1 + 1 \pmod 3), \quad \text{and} \quad 4357^{z_1} + 65^{2m} \equiv 1 + 1 \pmod{11}.$$

Hence $3 \nmid 4357^{z_1} + 65^{y_1}$ and $11 \nmid 4357^{z_1} + 65^{y_1}$. Thus, from (2.83), we have

$$2^{2x-1} 3^x 11^x \mid 4357^{z_1} - 65^{y_1}. \tag{2.84}$$

By $y < z < x$, we obtain

$$2^{2x-1}3^x11^x = \frac{132^x}{2} = \frac{(2^7+2^2)^x}{2} > 2^{7x-1} + 2^{2x-1} > 2^{7x-1} > 2^{13z_1},$$
$$> (4357+65)^{z_1},$$
$$> 4357^{z_1} + 65^{z_1},$$
$$> 4357^{z_1} + 65^{y_1},$$
$$> 4357^{z_1} - 65^{y_1},$$

and this contradicts (2.84). Otherwise, if $y_1 = 2m+1$, then

$4357^{z_1}+65^{2m+1} \equiv 1+2 \equiv 0(\mod 3)$ and $4357^{z_1}+65^{2m+1} \equiv 1+10 \equiv 0(\mod 11)$.

Thus, (2.83) becomes

$$2^{2x-1} \mid 4357^{z_1} - 65^{y_1} \text{ and } 2(3^x11^x) \mid 4357^{z_1} + 65^{y_1}. \tag{2.85}$$

So, we have two cases. The first case: if $67 \mid 4357^{z_1} - 65^{y_1}$, then from Equation (2.82)and (2.85), where

$$(4357^{z_1} - 65^{y_1}, 4357^{z_1} + 65^{y_1}) = 2,$$

we get

$$4357^{z_1} - 65^{y_1} = 2^{2x-1}67^{q(x-z)},$$

and

$$4357^{z_1} + 65^{y_1} = 2(3^x11^x). \tag{2.86}$$

Then

$$4357^{z_1} = 3^x11^x + 2^{2x-2}67^{q(x-z)},$$

and

$$65^{y_1} = 3^x11^x - 2^{2x-2}67^{q(x-z)}. \tag{2.87}$$

From Equation (2.87), we have $1 \equiv 33^x(\mod 64)$, since $\mathrm{ord}_{64}33 = 2$, hence x is even, and from (2.86), we have

$$2^{z_1} \equiv 2(3^x)(\mod 5). \tag{2.88}$$

Since 2 is a primitive root of 5, the congruence (2.88) becomes

$$z_1 \equiv 1 + 3x(\mod 4). \tag{2.89}$$

Since $z_1 - 1 - 3x$ and x are even, z_1 is odd and this contradicts (2.81). The second case: if $67 \mid 4357^{z_1} + 65^{y_1}$, then from Equation (2.82) and (2.85), where $(4357^{z_1} - 65^{y_1}, 4357^{z_1} + 65^{y_1}) = 2$, we have

$$4357^{z_1} - 65^{y_1} = 2^{2x-1}.$$

Therefore,

$$2^{z_1} \equiv 2^{2x-1} (\bmod 5), \tag{2.90}$$

and since 2 is a primitive root of 5, the congruence (2.90) becomes

$$z_1 \equiv 2x - 1 (\bmod 4). \tag{2.91}$$

Thus z_1 is odd and this contradicts (2.81).

4. If $k = 5^r 13^s n_1$, where $r \geqslant 1$, $s \geqslant 1$, $n_1 \geqslant 1$ and $q = 0$ with $(4355, n_1) = 1$, then rewrite Equation (2.52) as

$$\frac{5^y 13^y 67^y}{5^{r(z-y)} 13^{s(z-y)}} = n_1^{z-y} \left[4357^z - 132^x 5^{r(x-z)} 13^{s(x-z)} n_1^{x-z} \right]. \tag{2.92}$$

Hence $n_1 = 1$, $5^y = 5^{r(z-y)}$ and $13^y = 13^{s(z-y)}$ and so $r = s$. Thus, Equation (2.92) becomes

$$4357^z - 67^y = 4^x 3^x 11^x 65^{r(x-z)}. \tag{2.93}$$

By considering Equation (2.93) modulo 4, we obtain
$1 \equiv 3^y (\bmod 4)$. Hence y is even and we can write $y = 2y_1$. Also, by considering Equation (2.93) modulo 8, we obtain
$5^z \equiv 3^{2y_1} \equiv 1(\bmod 8)$. So z is even; say $z = 2z_1$. Hence Equation (2.93) becomes

$$(4357^{z_1} - 67^{y_1})(4357^{z_1} + 67^{y_1}) = 2^{2x} 3^x 11^x 65^{r(x-z)}. \tag{2.94}$$

Since

$$4357^{z_1} + 67^{y_1} \equiv 2 (\bmod 3), \quad \text{and} \quad 4357^{z_1} + 67^{y_1} \equiv 2 (\bmod 11),$$

it follows that $3 \nmid 4357^{z_1} + 67^{y_1}$ and $11 \nmid 4357^{z_1} + 67^{y_1}$. Since

$$(4357^{z_1} - 67^{y_1}, 4357^{z_1} + 67^{y_1}) = 2.$$

Thus, from Equation (2.94), we obtain

$$2^{2x-1}3^x11^x \mid 4357^{z_1} - 67^{y_1} \quad \text{and} \quad 2 \mid 4357^{z_1} + 67^{y_1}, \tag{2.95}$$

or

$$2(3^x11^x) \mid 4357^{z_1} - 67^{y_1} \quad \text{and} \quad 2^{2x-1} \mid 4357^{z_1} + 67^{y_1}. \tag{2.96}$$

Considering (2.95), observe that $2^{2x-1}3^x11^x \mid 4357^{z_1} - 67^{y_1}$. But $y < z < x$. So

$$\begin{aligned} 2^{2x-1}3^x11^x = \frac{132^x}{2} = \frac{(2^7+2^2)^x}{2} > 2^{7x-1} + 2^{2x-1} > 2^{7x-1} > 2^{13z_1}, \\ > (4357+67)^{z_1}, \\ > 4357^{z_1} + 67^{y_1}, \\ > 4357^{z_1} - 67^{y_1}, \end{aligned}$$

and this contradicts (2.95). On the other hand, if we consider (2.96), we observe that

$$4357^{z_1} + 67^{y_1} \equiv 1 + 3^{y_1} \equiv 0 (\bmod 4).$$

Thus y_1 is odd. So, if $5 \mid 4357^{z_1} - 67^{y_1}$, then

$$2^{z_1} \equiv 2^{y_1} (\bmod 5).$$

Since y_1 is odd,

$$(z_1, y_1) \in \{(4c_1+1, 4c_2+1), (4c_1+3, 4c_2+3)\}.$$

Hence

$$4357^{z_1} + 67^{y_1} \equiv 5^{4c_1+1} + 3^{4c_2+1} \equiv 8 (\bmod 16),$$

and

$$4357^{z_1} + 67^{y_1} \equiv 5^{4c_1+3} + 3^{4c_2+3} \equiv 8 (\bmod 16).$$

We conclude that $16 \nmid 4357^{z_1} + 67^{y_1}$ but this contradicts (2.96). Thus

$$5 \mid 4357^{z_1} + 67^{y_1}. \tag{2.97}$$

Similarly, if $13 \mid 4357^{z_1} - 67^{y_1}$, then $2^{z_1} \equiv 2^{y_1} (\bmod 13)$ and since y_1 is odd and $\text{ord}_{13}2 = 12$,

$$(z_1, y_1) \in \left\{ \begin{array}{l} (12c_1+1, 12c_2+1), (12c_1+3, 12c_2+3), (12c_1+5, 12c_2+5), \\ (12c_1+7, 12c_2+7), (12c_1+9, 12c_2+9), (12c_1+11, 12c_2+11) \end{array} \right\}.$$

Hence,

$$\begin{aligned}
4357^{z_1}+67^{y_1} &\equiv 5^{12c_1+1}+3^{12c_2+1}\equiv 5+3\equiv 8(\bmod 16),\\
4357^{z_1}+67^{y_1} &\equiv 5^{12c_1+3}+3^{12c_2+3}\equiv 5^3+3^3\equiv 8(\bmod 16),\\
4357^{z_1}+67^{y_1} &\equiv 5^{12c_1+5}+3^{12c_2+5}\equiv 5^5+3^5\equiv 8(\bmod 16),\\
4357^{z_1}+67^{y_1} &\equiv 5^{12c_1+7}+3^{12c_2+7}\equiv 5^7+3^7\equiv 8(\bmod 16),\\
4357^{z_1}+67^{y_1} &\equiv 5^{12c_1+9}+3^{12c_2+9}\equiv 5^9+3^9\equiv 8(\bmod 16),\\
4357^{z_1}+67^{y_1} &\equiv 5^{12c_1+11}+3^{12c_2+11}\equiv 5^{11}+3^{11}\equiv 8(\bmod 16).
\end{aligned}$$

It follows that $16 \nmid 4357^{z_1}+67^{y_1}$, but this contradicts (2.96). Thus

$$13 \mid 4357^{z_1}+67^{y_1}. \tag{2.98}$$

Since $8 \mid 4357^{z_1}+67^{y_1}$ and y_1 is odd,

$$4357^{z_1}+67^{y_1}\equiv 5^{z_1}+3^{y_1}\equiv 5^{z_1}+3(\bmod 8).$$

Therefore,

$$z_1\equiv 1(\bmod 2). \tag{2.99}$$

From (2.94), (2.96), (2.97), and (2.98), we have

$$4357^{z_1}+67^{y_1}=2^{2x-1}5^{r(x-z)}13^{r(x-z)},$$

and

$$4357^{z_1}-67^{y_1}=2(3^x11^x).$$

Then,

$$4357^{z_1}=2^{2x-2}5^{r(x-z)}13^{r(x-z)}+3^x11^x, \tag{2.100}$$

and

$$67^{y_1}=2^{2x-2}5^{r(x-z)}13^{r(x-z)}-3^x11^x.$$

Since y and z are even and $y<z<x$, then from (2.100), we obtain

$$5^{z_1}\equiv 33^x(\bmod 64). \tag{2.101}$$

If x even, then (2.101) becomes $5^{z_1}\equiv 1(\bmod 64)$ and since $\mathrm{ord}_{64}5=16$, hence z_1 is even and this contradicts (2.99).
Similarly, if x is odd, we obtain

$$5^{z_1}\equiv 33(\bmod 64). \tag{2.102}$$

By substituting values $z_1=16k+r$, where $0\le r<15$ into the congruence(2.102), we find only

$$z_1=16k+8,$$

satisfies the congruence. So z_1 is even and this contradicts (2.99).

5. If $k = 13^s 67^q n_1$, where $s \geqslant 1$, $q \geqslant 1$, $n_1 \geqslant 1$ and $r = 0$ with $(4355, n_1) = 1$, then rewrite Equation (2.52) as

$$\frac{5^y 13^y 67^y}{13^{s(z-y)} 67^{q(z-y)}} = n_1^{z-y}\left[4357^z - 132^x 13^{s(x-z)} 67^{q(x-z)} n_1^{x-z}\right]. \quad (2.103)$$

It follows that $n_1 = 1$, $13^y = 13^{s(z-y)}$ and $67^y = 67^{q(z-y)}$, so $s = q$. Thus, Equation (2.103) becomes

$$4357^z - 5^y = 4^x 3^x 11^x 13^{s(x-z)} 67^{s(x-z)}. \quad (2.104)$$

By considering Equation (2.104) modulo 33, we obtain
$1 \equiv 5^y (\bmod 33)$ and since $\mathrm{ord}_{33}5 = 10$, hence, we can write $y = 10m = 2y_1$, with $y_1 = 5m$. Also, by considering Equation (2.104) modulo 8, we obtain $5^z \equiv 5^{2y_1} \equiv 1 (\bmod 8)$. So, z must be even. We write $z = 2z_1$. Hence, Equation (2.104) becomes

$$(4357^{z_1} - 5^{y_1})(4357^{z_1} + 5^{y_1}) = 2^{2x} 3^x 11^x 13^{s(x-z)} 67^{s(x-z)}. \quad (2.105)$$

Since

$$4357^{z_1} + 5^{y_1} \equiv 2 (\bmod 4), \text{ and } 4357^{z_1} + 5^{y_1} \equiv 1 + 5^{5m} \equiv 2 (\bmod 11),$$

$$4 \nmid 4357^{z_1} + 5^{y_1} \quad \text{and} \quad 11 \nmid 4357^{z_1} + 5^{y_1}.$$

Since $(4357^{z_1} - 5^{y_1}, 4357^{z_1} + 5^{y_1}) = 2$, from Equation (2.105), we have

$$2^{2x-1} 11^x \mid 4357^{z_1} - 5^{y_1}, \quad \text{and} \quad 2 \mid 4357^{z_1} + 5^{y_1}. \quad (2.106)$$

If y_1 is even, then $4357^{z_1} + 5^{y_1} \equiv 1 + 1 \equiv 2 (\bmod 3)$. So, from (2.106), we observe that

$$2^{2x-1} 3^x 11^x \mid 4357^{z_1} - 5^{y_1}. \quad (2.107)$$

But, from $y < z < x$, we get

$$\begin{aligned} 2^{2x-1} 3^x 11^x = \frac{132^x}{2} = \frac{(2^7 + 2^2)^x}{2} > 2^{7x-1} + 2^{2x-1} > 2^{7x-1} > 2^{13z_1}&, \\ > (4357 + 5)^{z_1}&, \\ > 4357^{z_1} + 5^{z_1}&, \\ > 4357^{z_1} - 5^{y_1}&, \end{aligned}$$

and this contradicts (2.107). Otherwise, if y_1 is odd, then $4357^{z_1} + 5^{y_1} \equiv 1 + 2 \equiv 0 (\bmod 3)$. So, from (2.106), we obtain

$$2^{2x-1}11^x \mid 4357^{z_1} - 5^{y_1}, \quad \text{and} \quad 2(3^x) \mid 4357^{z_1} + 5^{y_1}. \tag{2.108}$$

Thus, $4357^{z_1} \equiv 5^{y_1} \equiv 5 (\bmod 8)$; that is, $5^{z_1} \equiv 5 (\bmod 8)$, implies $5^{z_1+1} \equiv 1 (\bmod 8)$. Therefore, $z_1 + 1$ must be even. Thus z_1 is odd. Therefore, if $67 \mid 4357^{z_1} + 5^{y_1}$, then

$$2^{z_1} \equiv (-5)^{y_1} \equiv 62^{y_1} (\bmod 67). \tag{2.109}$$

Since 2 is a primitive root of 67, the congruence (2.109) becomes $z_1 \equiv 48y_1 (\bmod 66)$. Thus z_1 is even and this contradicts z_1 is odd. So $67 \nmid 4357^{z_1} + 5^{y_1}$ and (2.108) becomes

$$2^{2x-1}11^x 67^{s(x-z)} \mid 4357^{z_1} - 5^{y_1}, \quad \text{and} \quad 2(3^x) \mid 4357^{z_1} + 5^{y_1}. \tag{2.110}$$

Similarly, if $13 \mid 4357^{z_1} - 5^{y_1}$, then from (2.105) and (2.110), we observe that

$$4357^{z_1} - 5^{y_1} = 2^{2x-1}11^x 67^{s(x-z)} 13^{s(x-z)}, \quad \text{and} \quad 4357^{z_1} + 5^{y_1} = 2(3^x).$$

Thus

$$4357^{z_1} = 3^x + 2^{2x-2}11^x 13^{s(x-z)} 67^{s(x-z)}, \tag{2.111}$$

and

$$5^{y_1} = 3^x - 2^{2x-2}11^x 13^{s(x-z)} 67^{s(x-z)}. \tag{2.112}$$

From (2.111), we obtain $1 \equiv (-1)^x (\bmod 4)$. It follows that x is even. So, from (2.112), where y_1 is odd, we have $5 \equiv 1 (\bmod 8)$, which is impossible. So $13 \mid 4357^{z_1} + 5^{y_1}$. Then, from (2.105) and (2.110), we observe that

$$4357^{z_1} - 5^{y_1} = 2^{2x-1}11^x 67^{s(x-z)}, \quad \text{and} \quad 4357^{z_1} + 5^{y_1} = 2(3^x 13^{s(x-z)}).$$

So

$$4357^{z_1} = 3^x 13^{s(x-z)} + 2^{2x-2}11^x 67^{s(x-z)}, \tag{2.113}$$

and

$$5^{y_1} = 3^x 13^{s(x-z)} - 2^{2x-2}11^x 67^{s(x-z)}. \tag{2.114}$$

From (2.113), we obtain $1 \equiv (-1)^x (\bmod 4)$. It follows that x is even. From (2.114), where y_1 is odd and $x - z$ is even, we get $5 \equiv 1 (\bmod 8)$, which is impossible.

6. If $k = 5^r 67^q n_1$, where $r \geqslant 1$, $q \geqslant 1$, $n_1 \geqslant 1$ and $s = 0$ with $(4355, n_1) = 1$, then rewrite Equation (2.52) as

$$\frac{5^y 13^y 67^y}{5^{r(z-y)} 67^{q(z-y)}} = n_1^{z-y}\left[4357^z - 132^x 5^{r(x-z)} 67^{q(x-z)} n_1^{x-z}\right]. \qquad (2.115)$$

Hence $n_1 = 1$, $5^y = 5^{r(z-y)}$ and $67^y = 67^{q(z-y)}$ so $r = q$. Thus, Equation (2.115) becomes

$$4357^z - 13^y = 4^x 3^x 11^x 5^{r(x-z)} 67^{r(x-z)}. \qquad (2.116)$$

By considering Equation (2.116) modulo 11, we obtain
$1 \equiv 2^y (\mathrm{mod}\, 11)$ and since $\mathrm{ord}_{11} 2 = 10$, we can write $y = 10m = 2y_1$, where $y_1 = 5m$. Also, by considering Equation (2.116) modulo 8, we obtain $5^z \equiv 5^{2y_1} \equiv 1 (\mathrm{mod}\, 8)$.
So z must be even; say $z = 2z_1$. Hence, Equation (2.116) becomes

$$(4357^{z_1} - 13^{y_1})(4357^{z_1} + 13^{y_1}) = 2^{2x} 3^x 11^x 5^{r(x-z)} 67^{r(x-z)}. \qquad (2.117)$$

Since

$$4357^{z_1} + 13^{y_1} \equiv 2 (\mathrm{mod}\, 4) \text{ and } 4357^{z_1} + 13^{y_1} \equiv 2 (\mathrm{mod}\, 3),$$

$$4 \nmid 4357^{z_1} + 13^{y_1}, \quad \text{and} \quad 3 \nmid 4357^{z_1} + 13^{y_1}.$$

Since $(4357^{z_1} - 13^{y_1}, 4357^{z_1} + 13^{y_1}) = 2$, from Equation (2.117), we have

$$2^{2x-1} 3^x \mid 4357^{z_1} - 13^{y_1}, \quad \text{and} \quad 2 \mid 4357^{z_1} + 13^{y_1}. \qquad (2.118)$$

If y_1 is even, then we can write $y_1 = 5m = 10m_1$. Thus,
$4357^{z_1} + 13^{y_1} \equiv 1 + 2^{10m_1} \equiv 2 (\mathrm{mod}\, 11)$; that is,
$11 \nmid 4357^{z_1} + 13^{y_1}$. So, from (2.118), we observe that

$$2^{2x-1} 3^x 11^x \mid 4357^{z_1} - 13^{y_1}. \qquad (2.119)$$

But, from $y < z < x$, we observe that

$$\begin{aligned} 2^{2x-1} 3^x 11^x = \frac{132^x}{2} = \frac{(2^7 + 2^2)^x}{2} > 2^{7x-1} + 2^{2x-1} > 2^{7x-1} &> 2^{13z_1}, \\ &> (4357 + 13)^{z_1}, \\ &> 4357^{z_1} + 13^{z_1}, \\ &> 4357^{z_1} - 13^{y_1}, \end{aligned}$$

and this contradicts (2.119). On the other hand, if y_1 is odd, then we can write $y_1 = 5m = 10m_1 + 5$. Thus

$$4357^{z_1} + 13^{y_1} \equiv 1 + 2^{10m_1+5} \equiv 1 + 2^5 \equiv 1 + 10 \equiv 0 (\bmod 11);$$

that is, $11 \mid 4357^{z_1} + 13^{y_1}$. So, from (2.118), we obtain

$$2^{2x-1}3^x \mid 4357^{z_1} - 13^{y_1}, \quad \text{and} \quad 2(11^x) \mid 4357^{z_1} + 13^{y_1}. \tag{2.120}$$

Thus, $4357^{z_1} \equiv 13^{y_1} (\bmod 8)$; that is, $5^{z_1} \equiv 5^{y_1} (\bmod 8)$,
implies that $5^{z_1+y_1} \equiv 1 (\bmod 8)$. It follows that $z_1 + y_1$ is even. But y_1 is odd. So z_1 is odd. Now, if $67 \mid 4357^{z_1} + 13^{y_1}$, then

$$2^{z_1} \equiv (-13)^{y_1} \equiv 54^{y_1} (\bmod 67). \tag{2.121}$$

Since 2 is a primitive root of 67, the congruence (2.121) becomes $z_1 \equiv 52y_1 (\bmod 66)$. Thus z_1 must be even. This contradicts z_1 is odd and therefore $67 \nmid 4357^{z_1} + 13^{y_1}$ and (2.120) becomes

$$2^{2x-1}3^x 67^{r(x-z)} \mid 4357^{z_1} - 13^{y_1}, \quad \text{and} \quad 2(11^x) \mid 4357^{z_1} + 13^{y_1}. \tag{2.122}$$

Similarly, if $5 \mid 4357^{z_1} - 13^{y_1}$, then from (2.117) and (2.122), we observe that

$$4357^{z_1} - 13^{y_1} = 2^{2x-1}3^x 67^{r(x-z)} 5^{r(x-z)}, \text{ and } 4357^{z_1} + 13^{y_1} = 2(11^x).$$

Thus

$$4357^{z_1} = 11^x + 2^{2x-2}3^x 5^{r(x-z)} 67^{r(x-z)}, \tag{2.123}$$

and

$$13^{y_1} = 11^x - 2^{2x-2}3^x 5^{r(x-z)} 67^{r(x-z)}. \tag{2.124}$$

From (2.123), we obtain $1 \equiv (-1)^x (\bmod 4)$. It follows that x is even. So, from (2.124), where y_1 is odd, we get $5 \equiv 1 (\bmod 8)$, which is impossible. So $5 \mid 4357^{z_1} + 13^{y_1}$. Then, from (2.117) and (2.122), we observe that

$$4357^{z_1} - 13^{y_1} = 2^{2x-1}3^x 67^{r(x-z)}, \text{ and } 4357^{z_1} + 13^{y_1} = 2(11^x 5^{r(x-z)}).$$

So

$$4357^{z_1} = 11^x 5^{r(x-z)} + 2^{2x-2}3^x 67^{r(x-z)}, \tag{2.125}$$

and

$$13^{y_1} = 11^x 5^{r(x-z)} - 2^{2x-2}3^x 67^{r(x-z)}. \tag{2.126}$$

Thus, from (2.125), we obtain $1 \equiv (-1)^x (\bmod 4)$. It follows that x is even. So, from (2.126), where y_1 is odd and $x-z$ is even, we have $5 \equiv 1(\bmod 8)$, which is impossible.

7. If $k = 5^r 13^s 67^q n_1$, where $r \geqslant 1$, $s \geqslant 1$, $q \geqslant 1$ and $n_1 \geqslant 1$ with $(4355, n_1) = 1$, then rewrite Equation (2.52) as

$$\frac{5^y 13^y 67^y}{5^{r(z-y)} 13^{s(z-y)} 67^{q(z-y)}} = n_1^{z-y} \left[4357^z - 132^x 5^{r(x-z)} 13^{s(x-z)} 67^{q(x-z)} n_1^{x-z}\right]. \tag{2.127}$$

So, $n_1 = 1$, $5^y = 5^{r(z-y)}$, $13^y = 13^{s(z-y)}$ and $67^y = 67^{q(z-y)}$. Thus $r = s = q$ and Equation (2.127) becomes

$$4357^z - 1 = 4^x 3^x 11^x 5^{r(x-z)} 13^{r(x-z)} 67^{r(x-z)}. \tag{2.128}$$

We conclude that $4357^z - 1 \equiv 2^z - 1 \equiv 0(\bmod 5)$. Hence $z \equiv 0(\mod 4)$ and since $4357^2 - 1 \equiv 0(\mod 2179)$, where 2179 is prime. Thus, $4357^z - 1 \equiv 0(\bmod 2179)$. Hence, from Equation (2.128), we obtain

$$4^x 3^x 11^x 5^{r(x-z)} 13^{r(x-z)} 67^{r(x-z)} \equiv 0(\bmod 2179),$$

which is impossible.

This completes the proof for the second case and consequently completes the proof of theorem (1.1). □

3 Conclusion

We have obtained a new Pythagorean triple for Jeśmanowicz's conjecture and proved that the special Diophantine equation $(132k)^x + (4355k)^y = (4357k)^z$ has the only positive integer solution $(x, y, z) = (2, 2, 2)$ for every positive integer k.

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