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A bivariate distribution whose marginal laws are inverted beta and gamma

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Abstract

In this article, we define a four parameter bivariate distribution whose marginal distributions are inverted beta and gamma. For this distribution, we derive results such as product moments, correlation coefficient, marginal and conditional distributions, distributions of sum, product and quotient, information matrix, and entropies. We also deal with the problem of estimation of parameters.

1 Introduction

Bivariate distributions have attracted useful applications in several areas. They have been used for representing joint probabilistic properties of multivariate hydrological events such as floods and storms or in the modeling of rainfall at two nearby rain gauges, data obtained from rainmaking experiments, the dependence between annual stream flow and aerial precipitation, wind gust modeling, and the dependence between rainfall and runoff, reliability (see [4], [6], [12], [16], [10, 11] and references therein).

Several bivariate distributions have been proposed in the statistical literature. Variuos techniques to generate bivariate distributions have also been

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proposed in the scientific literature (e.g., see [3], [4], [9], [24]). The bivariate Rayleigh, Nakagami, Weibull, and lognormal distributions have been thoroughly studied in [2], [20], [21]. Nadarajah [10] defined gamma-exponential distribution whose margins have the gamma and the exponential distributions. By using two independent gamma variables, Nadarajah [11], constructed a bivariate distribution which has gamma and beta distributions as its marginals. By using the conditional approach (see Section 5.6 of [4]), Nagar, Nadarajah and Okorie [14] and Nagar, Zarrazola and Sánchez [15] have constructed bivariate distributions whose marginal laws are gamma and Macdonald/extended beta.

In this article, we will use conditional approach to construct a four parameter bivariate distribution which has inverted beta and gamma distributions as its marginals.

The gamma distribution has been defined by the probability density function (p.d.f.)

$$\frac{v^{a-1}\exp(-v/\sigma)}{\sigma^{a}\Gamma(a)}, \quad a>0, \quad \sigma>0, \quad v>0. \tag{1}$$

We will write $V \sim G(a,\sigma)$ if the density of V is given by (1). Here, a and σ determine the shape and scale of the distribution. The beta (type 1) distribution is defined by the density

$$\frac{u^{\alpha-1}(1-u)^{\beta-1}}{B(\alpha,\beta)}, \quad 0<u<1, \quad a>0, \quad b>0, \tag{2}$$

where $B(a,b)$ is the usual beta function. A notation to designate that U has the beta distribution defined by the density (2) is $U \sim B(\alpha,\beta)$. The random variable W is said to have a beta type 2 or inverted beta distribution with parameters α,β and σ, denoted as $X \sim IB(\alpha,\beta;\sigma)$ if its p.d.f is given by

$$f_{\text{IB}}(w;\alpha,\beta;\sigma)=\frac{w^{\alpha-1}(\sigma+w)^{-(\alpha+\beta)}}{\sigma^{-\beta}B(\alpha,\beta)}, \quad w>0, \quad a>0, \quad b>0. \tag{3}$$

The inverted beta is the most familiar statistical distribution in finance, economics and related areas. Moreover, by using the transformation $U = W/(\sigma+W)$ with the Jacobian $J(u\to w)=\sigma(\sigma+w)^{-2}$, the p.d.f of the inverted beta distribution can be derived.

Now, consider two random variables X and Y such that the conditional distribution of X given y is gamma with the shape parameter α and the scale parameter σ_1/y and the marginal distribution of Y is gamma with the shape parameter β and scale parameter σ_2. That is,

$$f(x|y) = \frac{x^{\alpha-1}\exp(-xy/\sigma_1)}{\sigma_1^{\alpha}\Gamma(\alpha)y^{-\alpha}}, \quad x>0, \quad \alpha>0, \quad \sigma_1>0$$

and

$$g(y) = \frac{y^{\beta-1}\exp(-y/\sigma_2)}{\sigma_2^{\beta}\Gamma(\beta)}, \quad y>0, \quad \beta>0, \quad \sigma_2>0.$$

Then one can write

$$f_{\rm IB}(x,\alpha,\beta;\sigma_1/\sigma_2) = \int_0^\infty f(x|y)g(y)\,dy.$$

Thus, the product $f(x|y)g(y)$ can be used to create a bivariate density with inverted beta and gamma as marginal densities of X and Y, respectively. We, therefore, define the bivariate density of X and Y as

$$f(x,y;\alpha,\beta,\sigma_1,\sigma_2) = \frac{x^{\alpha-1}y^{\alpha+\beta-1}\exp\left(-xy/\sigma_1 - y/\sigma_2\right)}{\sigma_1^{\alpha}\sigma_2^{\beta}\Gamma(\alpha)\Gamma(\beta)}, \; x>0, \; y>0, \qquad (4)$$

where $\alpha>0, \beta>0, \sigma_1>0$ and $\sigma_2>0$. The bivariate distribution defined by the above density has many interesting features. For example, the marginal and the conditional distributions of X are inverted beta and gamma, respectively, the marginal distribution of Y is gamma, and the conditional distribution of Y given x is also gamma with scale parameter $\alpha+\beta$ and shape parameter $(x/\sigma_1 + 1/\sigma_2)^{-1}$. The gamma distribution has been used to model amounts of daily rainfall (Aksoy [1]). In neuroscience, the gamma distribution is often used to describe the distribution of inter-spike intervals (Robson and Troy [18]). The gamma distribution is widely used as a conjugate prior in Bayesian statistics. It is the conjugate prior for the precision (i.e. inverse of the variance) of a normal distribution. The inverted beta distribution is used in the analysis of carcinogenesis data, in the study of system availability or in measuring information in predictive distributions. Furthermore, the fact that marginal distributions are gamma and inverted beta makes this bivariate distribution a potential candidate for many real life problems.

In this article, we study distributions defined by the density (4), derive properties such as marginal and conditional distributions, moments, entropies and information matrix. We also deal with the problem of estimation of parameters.

2 Properties

First, we briefly discuss the shape of (4). The first order derivatives of $\ln f(x, y; \alpha, \beta, \sigma_1, \sigma_2)$ with respect to x and y are

$$f_x(x, y) = \frac{\partial \ln f(x, y; \alpha, \beta, \sigma_1, \sigma_2)}{\partial x} = \frac{\alpha - 1}{x} - \frac{y}{\sigma_1} \tag{5}$$

and

$$f_y(x, y) = \frac{\partial \ln f(x, y; \alpha, \beta, \sigma_1, \sigma_2)}{\partial y} = \frac{\alpha + \beta - 1}{y} - \frac{x}{\sigma_1} - \frac{1}{\sigma_2} \tag{6}$$

respectively. Setting (5) and (6) to zero, we note that (a, b), $a = \sigma_1(\alpha - 1)/\sigma_2\beta$, $b = \sigma_2\beta$ is a stationary point of (4). Computing second order derivatives of $\ln f(x, y; \alpha, \beta, \sigma_1, \sigma_2)$, from (5) and (6), we get

$$f_{xx}(x, y) = \frac{\partial^2 \ln f(x, y; \alpha, \beta, \sigma_1, \sigma_2)}{\partial x^2} = -\frac{\alpha - 1}{x^2}, \tag{7}$$

$$f_{xy}(x, y) = \frac{\partial^2 \ln f(x, y; \alpha, \beta, \sigma_1, \sigma_2)}{\partial x \partial y} = -\frac{1}{\sigma_1}, \tag{8}$$

and

$$f_{yy}(x, y) = \frac{\partial^2 \ln f(x, y; \alpha, \beta, \sigma_1, \sigma_2)}{\partial y^2} = -\frac{\alpha + \beta - 1}{y^2}. \tag{9}$$

In addition, from (7), (8) and (9), we get

$$f_{xx}(a, b) = -\frac{\sigma_2^2 \beta^2}{\sigma_1^2(\alpha - 1)}, \quad f_{yy}(a, b) = -\frac{(\alpha + \beta - 1)}{\sigma_2^2 \beta^2}$$

and

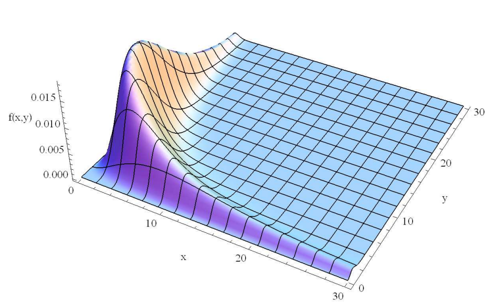
$$f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2 = \frac{\beta}{\sigma_1^2(\alpha - 1)}.$$

Now, observe that

- If $\alpha > 1$, then $f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$, $f_{xx}(a, b) < 0$ and $f_{yy}(a, b) < 0$, and therefore (a, b) is a maximum point.

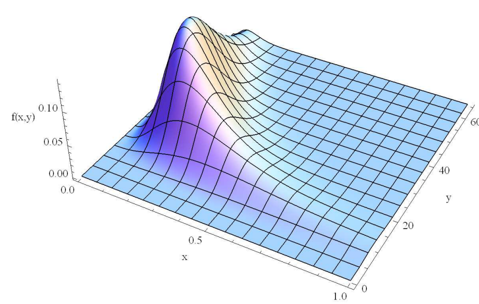
Figure 1: Graphs of the bivariate density (4)

Graph 1



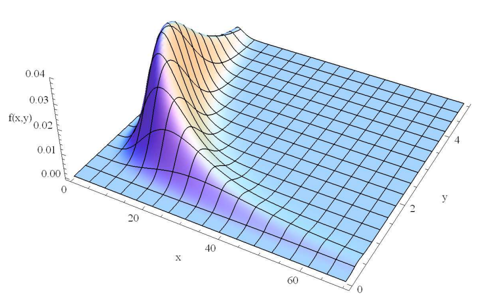
$\alpha = 3,\ \beta = 2,\ \sigma_1 = 8,\ \sigma_2 = 4$

Graph 2



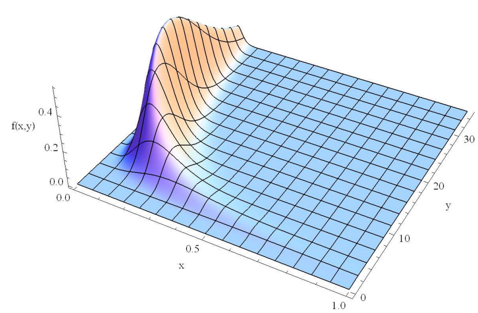
$\alpha = 4,\ \beta = 6,\ \sigma_1 = 1.5,\ \sigma_2 = 5$

Graph 3



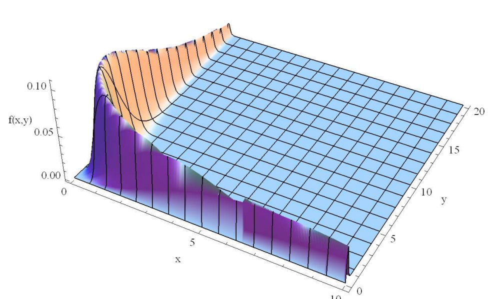
$\alpha = 6,\ \beta = 4,\ \sigma_1 = 3.5,\ \sigma_2 = 0.5$

Graph 4



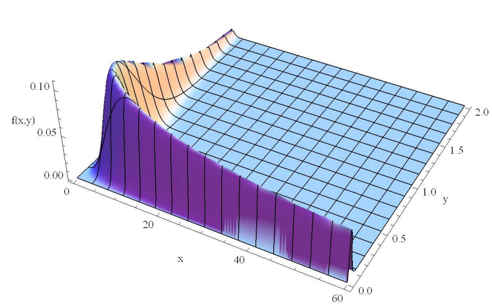
$\alpha = 6,\ \beta = 4,\ \sigma_2 = 0.25,\ \sigma_2 = 4$

Graph 5



$\alpha = 4,\ \beta = 0.5,\ \sigma_1 = 0.5,\ \sigma_2 = 6$

Graph 6



$\alpha = 4,\ \beta = 0.5,\ \sigma_2 = 0.5,\ \sigma_2 = 0.5$

- If $\alpha < 1$, then $f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 < 0$, and therefore (a,b) is a saddle point.

Figure 1 illustrates the shape of the p.d.f (4) for selected values of α, β, σ_1 and σ_2. Here one can appreciate the wide range of forms that result from the bivariate density defined by the density (4).

A distribution is said to be negatively likelihood ratio dependent (NLRD) if the density $f(x,y)$ satisfies

$$f(x_1,y_1)f(x_2,y_2) \le f(x_1,y_2)f(x_2,y_1) \tag{10}$$

for all $x_1 > x_2$ and $y_1 > y_2$ (Lehmann [7], Tong [23]). In the present case, (10) is equivalent to

$$y_1x_2 + x_1y_2 \le x_1y_1 + x_2y_2$$

which clearly holds. Thus the bivariate distribution defined by the density (4) is NLRD.

3 Expected Values

By definition, the product moments of X and Y associated with (4) are given by

$$\begin{aligned} E(X^rY^s) &= \int_0^\infty \int_0^\infty \frac{x^{\alpha+r-1}y^{\alpha+\beta+s-1}\exp\left(-xy/\sigma_1 - y/\sigma_2\right)}{\sigma_1^\alpha \sigma_2^\beta \Gamma(\alpha)\Gamma(\beta)} \mathrm{d}x\,\mathrm{d}y \\ &= \frac{\sigma_1^r \Gamma(\alpha+r)}{\sigma_2^\beta \Gamma(\alpha)\Gamma(\beta)} \int_0^\infty y^{\beta+s-r-1}\exp\left(-\frac{y}{\sigma_2}\right)\,\mathrm{d}y \\ &= \frac{\sigma_1^r \sigma_2^{s-r}\Gamma(\alpha+r)\Gamma(\beta+s-r)}{\Gamma(\alpha)\Gamma(\beta)},\ \alpha+r>0,\ \beta+s-r>0, \end{aligned} \tag{11}$$

where both the lines have been derived by using the definition of gamma function.

Substituting appropriately in (11), the means of X and Y, denoted by μ_X and μ_Y, are computed as

$$\mu_X = E(X) = \frac{\sigma_1 \alpha}{\sigma_2(\beta-1)}, \quad \beta > 1, \quad \mu_Y = E(Y) = \sigma_2 \beta.$$

By using the definition of (i,j)-th central joint moment of (X,Y), namely,

$$\mu_{ij} = E[(X-\mu_X)^i(Y-\mu_Y)^j],$$

expressions for μ_{ij}, for different values of i and j, are computed as

$$\mu_{11} = \text{cov}(X, Y) = -\frac{\sigma_1 \alpha}{\beta - 1}, \quad \beta > 1,$$

$$\mu_{20} = \text{var}(X) = \frac{\sigma_1^2 \alpha(\alpha + \beta - 1)}{\sigma_2^2 (\beta - 1)^2 (\beta - 2)}, \quad \beta > 2,$$

$$\mu_{02} = \text{var}(Y) = \sigma_2^2 \beta,$$

$$\mu_{30} = \frac{2\sigma_1^3 \alpha(\alpha + \beta - 1)(2\alpha + \beta - 1)}{\sigma_2^3 (\beta - 1)^3 (\beta - 2)(\beta - 3)}, \quad \beta > 3,$$

$$\mu_{21} = -\frac{2\alpha(\alpha + \beta - 1)\sigma_1^2}{(\beta - 1)^2 (\beta - 2)\sigma_2}, \quad \beta > 2,$$

$$\mu_{12} = 0,$$

$$\mu_{03} = 2\beta\sigma_2^3,$$

$$\mu_{40} = \frac{3\alpha(\alpha + \beta - 1)[\alpha(\beta + 5)(\alpha + \beta - 1) + 2(\beta - 1)^2]\sigma_1^4}{(\beta - 1)^4 (\beta - 2)(\beta - 3)(\beta - 4)\sigma_2^4}, \quad \beta > 4,$$

$$\mu_{31} = -\frac{3\alpha(\alpha + \beta - 1)(\alpha\beta + \alpha + 2\beta - 2)\sigma_1^3}{(\beta - 1)^3 (\beta - 2)(\beta - 3)\sigma_2^2}, \quad \beta > 3,$$

$$\mu_{22} = \frac{\alpha(\beta^2 + 3\alpha\beta + \beta - 2\alpha - 2)\sigma_1^2}{(\beta - 1)^2 (\beta - 2)}, \quad \beta > 2,$$

$$\mu_{13} = -\frac{3\alpha\beta\sigma_1\sigma_2^2}{(\beta - 1)}, \quad \beta > 1,$$

$$\mu_{04} = 3\beta(\beta + 2)\sigma_2^4.$$

Moreover, using the above expressions, the correlation coefficient between X and Y is given by

$$\rho_{X,Y} = -\sqrt{\frac{\alpha(\beta - 2)}{\beta(\alpha + \beta - 1)}}, \quad \beta > 2.$$

4 Distributional Results

From the construction of the bivariate density (4), it is cleat that $X \sim IB(\alpha, \beta, \sigma_1/\sigma_2)$, $Y \sim G(\beta, \sigma_2)$, $Y|x \sim G(\alpha+\beta, \sigma_1/(\sigma_1/\sigma_2 + x))$ and $X|y \sim G(\alpha, \sigma_1/y)$. By using transformation of variables, it is easy to see that XY and Y are independently distributed with $XY \sim G(\alpha, \sigma_1)$.

For $s = r$, the expression (11) reduces to

$$E(X^r Y^r) = \frac{\sigma_1^r \Gamma(\alpha + r)}{\Gamma(\alpha)},$$

which also shows that XY has a gamma distribution with shape parameter α and scale parameter σ_1. For $s = 2r$, from (11), the r-th moment of $XY^2/\sigma_1\sigma_2$ is derived as

$$E\left[\left(\frac{XY^2}{\sigma_1\sigma_2}\right)^r\right] = \frac{\Gamma(\alpha + r)\Gamma(\beta + r)}{\Gamma(\alpha)\Gamma(\beta)}.$$

From the above expression, it is clear that $XY^2/\sigma_1\sigma_2$ is distributed as the product of two independent standard gamma variables with shape parameters α and β. For $\beta = \alpha + 1/2$, one can write

$$E\left[\left(\frac{4XY^2}{\sigma_1\sigma_2}\right)^{r/2}\right] = \frac{\Gamma(2\alpha + r)}{\Gamma(2\alpha)}$$

which shows that $2Y\sqrt{X/\sigma_1\sigma_2}$ has a gamma distribution with shape parameter 2α. Similarly, for $\beta = \alpha - 1/2$, we have $2Y\sqrt{X/\sigma_1}\sigma_2 \sim G(2\beta)$. Furthermore, for arbitrary values of α and β, the density of $V = XY^2/\sigma_1\sigma_2$ is given as

$$f_V(v) = \frac{2}{\Gamma(\alpha)\Gamma(\beta)} v^{(\alpha+\beta)/2-1} K_{\alpha-\beta}(2\sqrt{v}), \quad v > 0,$$

where K_ν is the modified Bessel function of the second kind defined by the integral (Gradshteyn and Ryzhik [5, Eq. 3.471.9]),

$$\int_0^\infty \exp\left(-az - \frac{b}{z}\right) z^{\nu-1}\, dz = 2\left(\frac{b}{a}\right)^{\nu/2} K_\nu(2\sqrt{ab}), \quad a > 0. \quad b > 0.$$

Making the transformation $S = X + Y$ and $R = X/(X + Y)$ with the Jacobian $J(x, y \to r, s) = s$ in (4), the joint density of R and S is given by

$$f_{R,S}(r, s) = \frac{r^{\alpha-1}(1-r)^{\alpha+\beta-1} s^{2\alpha+\beta-1} \exp\left[-s(1-r)/\sigma_2 - s^2 r(1-r)/\sigma_1\right]}{\sigma_1^\alpha \sigma_2^\beta \Gamma(\alpha)\Gamma(\beta)},$$

where $0 < r < 1$ and $s > 0$. By integrating r, the marginal density of S is derived as

$$f_S(s) = \frac{s^{2\alpha+\beta-1}\ \exp(-s/\sigma_2)}{\sigma_1^\alpha \sigma_2^\beta \Gamma(\alpha)\Gamma(\beta)} \sum_{j=0}^{\infty} \frac{s^{2j}}{\sigma_1^j\, j!} \times \int_0^1 r^{\alpha+2j-1}(1-r)^{\alpha+\beta-1} \exp\left[\left(\frac{s}{\sigma_2} - \frac{s^2}{\sigma_1}\right) r\right] \mathrm{d}r.$$

Moreover, evaluating the above integral by using the definition of the confluent hypergeometric function (Luke [8, p. 115, Eq. (1)]), we get

$$f_S(s) = \frac{s^{2\alpha+\beta-1}\ \exp(-s/\sigma_2)}{\sigma_1^\alpha \sigma_2^\beta \Gamma(\alpha)\Gamma(\beta)} \sum_{j=0}^{\infty} \frac{s^{2j}}{\sigma_1^j\, j!} \frac{\Gamma(\alpha+2j)\Gamma(\alpha+\beta)}{\Gamma(2\alpha+\beta+2j)} \times {}_1F_1\left(\alpha+2j; 2\alpha+\beta+2j; \frac{s}{\sigma_2} - \frac{s^2}{\sigma_1}\right), \quad s > 0.$$

Furthermore, integrating s, the marginal density of R is derived as

$$f_R(r) = \frac{\sigma_1^{\beta/2}\Gamma(2\alpha+\beta)}{2^{\alpha+\beta/2}\sigma_2^\beta \Gamma(\alpha)\Gamma(\beta)} \frac{(1-r)^{\beta/2-1} \exp\left[\sigma_1(1-r)/8\sigma_2^2 r\right]}{r^{\beta/2+1}} \times U\left(2\alpha+\beta-\frac{1}{2}, \sqrt{\frac{\sigma_1(1-r)}{2\sigma_2^2 r}}\right), \quad 0 < r < 1,$$

where the parabolic cylinder function $U(a, z)$ is defined by [22],

$$U(a, z) = \frac{\exp(-z^2/4)}{\Gamma(a+1/2)} \int_0^\infty \exp\left(-\frac{1}{2}t^2 - zt\right) t^{a-1/2}\, \mathrm{d}t.$$

Further, one can also write

$$D_\nu(z) \equiv U\left(-\nu - \frac{1}{2}, z\right) = 2^{(\nu-1)/2} \exp\left(-\frac{z^2}{4}\right) \Psi\left(\frac{1-\nu}{2}, \frac{3}{2}; \frac{z^2}{2}\right),$$

where $\Psi(a, b; w)$ is the confluent hypergeometric function.

5 Entropies

In this section, exact forms of Rényi and Shannon entropies are determined for the bivariate distribution defined in Section 1.

Let $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ be a probability space. Consider a p.d.f. f associated with $\mathcal{P}$, dominated by $\sigma-$finite measure μ on $\mathcal{X}$. Denote by $H_{SH}(f)$ the well-known Shannon entropy introduced in Shannon [19] and defined by

$$H_{SH}(f) = -\int_{\mathcal{X}} f(x) \ln f(x) \, d\mu. \tag{12}$$

One of the main extensions of the Shannon entropy was defined by Rényi [17]. This generalized entropy measure is given by

$$H_R(\gamma, f) = \frac{\ln G(\gamma)}{1-\gamma} \qquad (\text{for } \gamma > 0 \text{ and } \gamma \neq 1), \tag{13}$$

where

$$G(\gamma) = \int_{\mathcal{X}} f^{\gamma} d\mu.$$

The additional parameter γ is used to describe complex behavior in probability models and the associated process under study. Rényi entropy is monotonically decreasing in γ, while Shannon entropy (12) is obtained from (13) for $\gamma \uparrow 1$. For details see [13], [26] and [25].

Now, we derive the Rényi and the Shannon entropies for the bivariate density defined in (4).

Theorem 5.1. *For the bivariate distribution defined by the p.d.f. (4), the Rényi and the Shannon entropies are given by*

$$\begin{aligned} H_R(\gamma, f) = \frac{1}{1-\gamma}[&(1-\gamma)\ln\sigma_1 + \ln\Gamma[\gamma(\alpha-1)+1] + \ln\Gamma(\gamma\beta) \\ &- [\gamma(\alpha+\beta-1)+1]\ln\gamma - \gamma\ln\Gamma(\alpha) - \gamma\ln\Gamma(\beta)] \end{aligned} \tag{14}$$

and

$$H_{SH}(f) = \ln\sigma_1 - (\alpha-1)\psi(\alpha) - \beta\psi(\beta) + (\alpha+\beta) + \ln\Gamma(\alpha) + \ln\Gamma(\beta), \tag{15}$$

where $\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{1}{\Gamma(z)}\frac{d}{dz}\Gamma(z)$ *is the digamma function.*

Proof. For $\gamma > 0$ and $\gamma \neq 1$, using the p.d.f. of (X, Y) given by (4), we have

$$\begin{aligned} G(\gamma) &= \int_0^\infty \int_0^\infty [f(x, y; \alpha, \beta, \sigma_1, \sigma_2)]^\gamma \, \mathrm{d}x \, \mathrm{d}y \\ &= \frac{1}{\left[\sigma_1^\alpha \sigma_2^\beta \Gamma(\alpha)\Gamma(\beta)\right]^\gamma} \int_0^\infty \int_0^\infty x^{\gamma(\alpha-1)} y^{\gamma(\alpha+\beta-1)} \exp\left(-\frac{\gamma y}{\sigma_2} - \frac{\gamma x y}{\sigma_1}\right) \mathrm{d}x \, \mathrm{d}y \\ &= \frac{\sigma_1^{-\gamma+1}}{\left[\sigma_2^\beta \Gamma(\alpha)\Gamma(\beta)\right]^\gamma} \frac{\Gamma[\gamma(\alpha-1)+1]}{\gamma^{\gamma(\alpha-1)+1}} \int_0^\infty y^{\gamma\beta-1} \exp\left(-\frac{\gamma y}{\sigma_2}\right) \mathrm{d}y \\ &= \frac{\sigma_1^{-\gamma+1} \Gamma[\gamma(\alpha-1)+1]\Gamma(\gamma\beta)}{\gamma^{\gamma(\alpha+\beta-1)+1} [\Gamma(\alpha)\Gamma(\beta)]^\gamma}, \end{aligned}$$

where, to evaluate above integrals, we have used the definition of gamma function. Now, taking the logarithm of $G(\gamma)$ and using (13), we get (14). The Shannon entropy (15) is obtained from (14) by taking $\gamma \uparrow 1$ and using L'Hopital's rule. □

6 Fisher Information Matrix

In this section, we calculate the Fisher information matrix for the bivariate distribution defined by the density (4). The information matrix plays a significant role in statistical inference in connection with estimation, sufficiency and properties of variances of estimators. For a given observation vector (x, y), the Fisher information matrix for the bivariate distribution defined by the density (4) is defined as

$$-\begin{bmatrix} E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \sigma_1 \partial \alpha}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \sigma_2 \partial \alpha}\right) \\ E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \beta \partial \alpha}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \beta^2}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \beta \partial \sigma_1}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \beta \partial \sigma_2}\right) \\ E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \sigma_1 \partial \alpha}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \sigma_1 \partial \beta}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \sigma_1^2}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \sigma_1 \partial \sigma_2}\right) \\ E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \sigma_2 \partial \alpha}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \sigma_2 \partial \beta}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \sigma_1 \partial \sigma_2}\right) & E\left(\frac{\partial^2 l(\alpha,\beta,\sigma_1,\sigma_2)}{\partial \sigma_2^2}\right) \end{bmatrix},$$

where $l(\alpha, \beta, \sigma_1, \sigma_2) = \ln L(\alpha, \beta, \sigma_1, \sigma_2) = \ln f(x, y; \alpha, \beta, \sigma_1, \sigma_2)$. From (4), the natural logarithm of $L(\alpha, \beta, \sigma_1, \sigma_2)$ is obtained as

$$\begin{aligned} l(\alpha, \beta, \sigma_1, \sigma_2) = {} & -\alpha \ln \sigma_1 - \beta \ln \sigma_2 - \ln \Gamma(\alpha) - \ln \Gamma(\beta) + (\alpha - 1) \ln x \\ & + (\alpha + \beta - 1) \ln y - \frac{y}{\sigma_2} - \frac{yx}{\sigma_1}, \end{aligned}$$

where $x > 0$ and $y > 0$. The second order partial derivatives of $l(\alpha, \beta, \sigma_1, \sigma_2)$ are given by

$$\frac{\partial^2 l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \alpha^2} = -\psi_1(\alpha),$$

$$\frac{\partial^2 l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \beta^2} = -\psi_1(\beta),$$

$$\frac{\partial^2 l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \sigma_1^2} = \frac{\alpha}{\sigma_1^2} - \frac{2xy}{\sigma_1^3},$$

$$\frac{\partial^2 l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \sigma_2^2} = \frac{\beta}{\sigma_2^2} - \frac{2y}{\sigma_2^3},$$

$$\frac{\partial^2 l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \alpha \, \partial \beta} = 0,$$

$$\frac{\partial^2 l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \alpha \, \partial \sigma_1} = -\frac{1}{\sigma_1},$$

$$\frac{\partial^2 l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \alpha \, \partial \sigma_2} = 0,$$

$$\frac{\partial^2 l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \beta \, \partial \sigma_1} = 0,$$

$$\frac{\partial^2 l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \beta \, \partial \sigma_2} = -\frac{1}{\sigma_2},$$

$$\frac{\partial^2 l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \sigma_1 \, \partial \sigma_2} = 0,$$

where $\psi_1(z)$ is the trigamma function defined as derivative of the digamma function, $\psi_1(z) = \frac{d}{dz}\psi(z)$.

Now, noting that $E(Y) = \sigma_2 \beta$, $E(XY) = \sigma_1 \alpha$, and the expected value of a constant is the constant itself, we obtain the Fisher information matrix as

$$\begin{bmatrix} \psi_1(\alpha) & 0 & \frac{1}{\sigma_1} & 0 \\ 0 & \psi_1(\beta) & 0 & \frac{1}{\sigma_2} \\ \frac{1}{\sigma_1} & 0 & \frac{\alpha}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2} & 0 & \frac{\beta}{\sigma_2^2} \end{bmatrix}.$$

7 Estimation

The density given by (4) is parameterized by $(\alpha, \beta, \sigma_1, \sigma_2)$. Here, we consider estimation of these four parameters by the method of maximum likelihood.

Suppose $(x_1, y_1), \ldots, (x_n, y_n)$ is a random sample from (4). The log-likelihood can be expressed as:

$$l(\alpha, \beta, \sigma_1, \sigma_2) = -n\alpha \ln \sigma_1 - n\beta \ln \sigma_2 - n \ln \Gamma(\alpha) - n \ln \Gamma(\beta)$$
$$+ (\alpha - 1)\sum_{i=1}^{n} \ln x_i + (\alpha + \beta - 1)\sum_{i=1}^{n} \ln y_i - \sum_{i=1}^{n} \frac{y_i}{\sigma_2} - \sum_{i=1}^{n} \frac{x_i y_i}{\sigma_1},$$

The first-order derivatives of this with respect to the four parameters are:

$$\frac{\partial l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \alpha} = -n \ln \sigma_1 - n\psi(\alpha) + \sum_{i=1}^{n} (\ln x_i + \ln y_i),$$

$$\frac{\partial l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \beta} = -n \ln \sigma_2 - n\psi(\beta) + \sum_{i=1}^{n} \ln y_i,$$

$$\frac{\partial l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \sigma_1} = -\frac{n\alpha}{\sigma_1} + \frac{1}{\sigma_1^2} \sum_{i=1}^{n} x_i y_i$$

and

$$\frac{\partial l(\alpha, \beta, \sigma_1, \sigma_2)}{\partial \sigma_2} = -\frac{n\beta}{\sigma_2} + \frac{1}{\sigma_2^2} \sum_{i=1}^{n} y_i.$$

The maximum likelihood estimators of $(\alpha, \beta, \sigma_1, \sigma_2)$, say $(\hat{\alpha}, \hat{\beta}, \hat{\sigma}_1, \hat{\sigma}_2)$, are the simultaneous solutions of the above four equations. It is straightforward to see that

$$\hat{\sigma}_1 \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} x_i y_i, \quad \ln \hat{\sigma}_1 + \psi(\hat{\alpha}) = \frac{1}{n} \sum_{i=1}^{n} (\ln x_i + \ln y_i),$$

$$\hat{\sigma}_2 \hat{\beta} = \frac{1}{n} \sum_{i=1}^{n} y_i, \quad \ln \hat{\sigma}_2 + \psi(\hat{\beta}) = \frac{1}{n} \sum_{i=1}^{n} \ln y_i.$$

Thus, $\hat{\alpha}$ and $\hat{\beta}$ can be calculated by solving numerically the equations

$$\psi(\hat{\alpha}) - \ln \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} (\ln x_i + \ln y_i) - \ln \left(\frac{1}{n} \sum_{i=1}^{n} x_i y_i \right) = \ln(\tilde{x}\tilde{y}) - \ln \overline{xy}$$

and

$$\psi(\hat{\beta}) - \ln \hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} \ln y_i - \ln\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) = \ln \tilde{y} - \ln \bar{y},$$

where $\tilde{x} = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$, $\tilde{y} = \left(\prod_{i=1}^{n} y_i\right)^{1/n}$, $\bar{y} = \sum_{i=1}^{n} y_i/n$, $\overline{xy} = \sum_{i=1}^{n}(x_i y_i)/n$. Finally, for $\hat{\alpha}$ and $\hat{\beta}$ so obtained, $\hat{\sigma}_1$ and $\hat{\sigma}_2$ can be computed by using the equations

$$\hat{\sigma}_1 = \frac{\overline{xy}}{\hat{\alpha}}, \quad \hat{\sigma}_2 = \frac{\bar{y}}{\hat{\beta}}.$$

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