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Rainbow Connectivity and Proper Rainbow Connectivity of $\theta(1, s, t)$

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Abstract

A connected graph G is rainbow connected with respect to an edge coloring of G if each pair of distinct vertices of G are joined by a rainbow path—a path with no color appearing on more than one edge of the path. G is strongly rainbow connected if each pair of distinct vertices of G are joined by a rainbow geodesic, a shortest path in G between the vertices. The (strong) rainbow connection number of G, denoted (s)rc(G), is the smallest number of colors in an edge coloring of G with respect to which G is (strongly) rainbow connected.

We consider two recently introduced parameters, prc and psrc, defined as rc and src were, with the additional requirement that the edge colorings be proper. We mention some relations among the 4 parameters and evaluate them on some classes of graphs, including some of the theta graphs.

1 Introduction

Rainbow connectivity and strong rainbow connectivity were introduced by Gary Chartrand, Garry L. Johns, Kathleen A. McKeon, and Ping Zhang in 2006 [2]. In recent years, the rainbow connectivity and strong rainbow connectivity parameters (defined below) have been heavily researched [3]. Two

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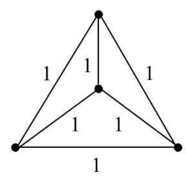


Figure 1: $\mathrm{rc}(K_4) = \mathrm{src}(K_4) = 1$

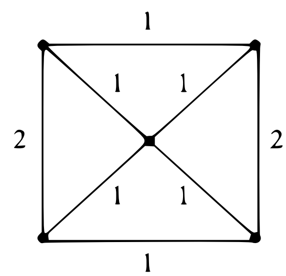


Figure 2: $\mathrm{rc}(W_4) = \mathrm{src}(W_4) = 2$

other parameters–proper rainbow connectivity and proper strong rainbow connectivity–were introduced recently [1]. We will begin by extending some early results for the initial parameters to the more recent proper versions. We will then give the values of these parameters for a small class of theta graphs.

Definition 1.1. *A connected graph G is* rainbow connected *with respect to an edge coloring of G if each pair of distinct vertices of G are joined by a rainbow path–a path with no color appearing on more than one edge of the path.*

A connected graph G is strongly rainbow connected *with respect to an edge coloring of G if each pair of distinct vertices of G are joined by a rainbow geodesic.*

The rainbow connection number *of G, denoted $rc(G)$, is the smallest number of colors in an edge coloring of G with respect to which G is rainbow connected.*

The strong rainbow connection number *of G, denoted $src(G)$, is the smallest number of colors in an edge coloring of G with respect to which G is strongly rainbow connected.*

Of more recent interest are the proper versions of rainbow connectivity and strong rainbow connectivity, in which we require the edge coloring of the

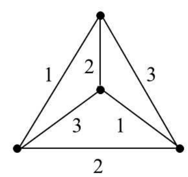


Figure 3: prc(K_4)= psrc(K_4) = 3

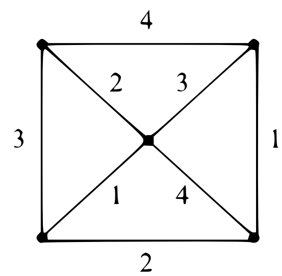


Figure 4: prc(W_4) = psrc(W_4) = 4

graph to be proper.

Definition 1.2. *A connected graph G is* proper rainbow connected *with respect to a proper edge coloring of G if each pair of distinct vertices of G are joined by a rainbow path.*

A connected graph G is proper strongly rainbow connected *with respect to a proper edge coloring of G if each pair of distinct vertices of G are joined by a rainbow geodesic.*

The proper rainbow connection number of *G, denoted prc(G), is the smallest number of colors in a proper edge coloring of G with respect to which G is proper rainbow connected.*

The proper strong rainbow connection number of *G, denoted psrc(G) is the smallest number of colors in a proper edge coloring of G with respect to which G is proper strongly rainbow connected.*

2 On the Rainbow Connectivity Parameters

Throughout, G will be a finite connected simple graph. The chromatic index, or edge chromatic number, of G is denoted $\chi'(G)$ and the diameter by diam(G). The inequalities in the following theorem are straightforward to see from the definitions. The first appears in [2].

Theorem 2.1. *$diam(G) \leq rc(G) \leq src(G) \leq |E(G)|$,
$\chi'(G) \leq prc(G) \leq psrc(G) \leq |E(G)|$, $rc(G) \leq prc(G)$,
and $src(G) \leq psrc(G)$.*

Theorem 2.2. *The following are equivalent.*

(a) $rc(G) = |E(G)|$;

(b) $src(G) = |E(G)|$;

(c) G is a tree.

The equivalence of (a) and (c), and thus (c) $\Rightarrow$ (b), is proven in [2]. A proof that (b) implies (c) is given in [1].

Corollary 2.1. *If G is a tree then $prc(G) = psrc(G) = |E(G)|$.*

Neither converse of Corollary 2.1 holds, since $\text{prc}(K_3) = \text{psrc}(K_3) = 3 = |E(K_3)|$, which raises two extremal questions:

1. For which G is it true that $\text{prc}(G) = |E(G)|$?

2. For which G is it true that $\text{psrc}(G) = |E(G)|$?

It may be that K_3 is the only connected non-tree for which either equality holds.

Proposition 2.2. *If $diam(G) \leq 2$ then $prc(G) = psrc(G) = \chi'(G)$.*

Proof. We have $\chi'(G) \leq \text{prc}(G) \leq \text{psrc}(G)$. If G is properly edge-colored then every path in G of length 1 or 2 is rainbow, so, if $\text{diam}(G) \leq 2$ then every geodesic is rainbow in a proper coloring. Thus, $\text{diam}(G) \leq 2 \Rightarrow \text{psrc}(G) \leq \chi'(G)$. □

Let $\mathcal{P}$ denote the Petersen graph, and C_n the cycle on n vertices.

Theorem 2.3. *$rc(\mathcal{P}) = 3$, $src(\mathcal{P}) = prc(\mathcal{P}) = psrc(\mathcal{P}) = 4$, and, for $n > 3$, $rc(C_n) = src(C_n) = prc(C_n) = psrc(C_n) = \lceil \frac{n}{2} \rceil$.*

Proof. It is shown in [2] that $\text{rc}(\mathcal{P}) = 3$, $\text{src}(\mathcal{P}) = 4$, and $\text{rc}(C_n) = \text{src}(C_n) = \lceil \frac{n}{2} \rceil$ if $n > 3$. Since $\chi'(\mathcal{P}) = 4$ and $\mathcal{P}$ has diameter 2, $\text{prc}(\mathcal{P}) = \text{psrc}(\mathcal{P}) = 4$ follows from Proposition 2.1.

To see that $\text{prc}(C_n) = \text{psrc}(C_n) = \lceil \frac{n}{2} \rceil$ if $n > 3$, it suffices (since rc $\leq$ prc $\leq$ psrc) to see that C_n has a proper edge coloring with $\lceil \frac{n}{2} \rceil$ colors such that C_n is strongly rainbow connected with respect to this coloring. If $n > 3$ is even, color around the cycle with $1, 2, \ldots, \frac{n}{2}, 1, 2, \ldots, \frac{n}{2}$. If $n > 3$ is odd, color around the cycle with $1, 2, \ldots, \lceil \frac{n}{2} \rceil, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. □

Two facile conjectures exploded

It might seem reasonable to conjecture that if $\text{rc}(G) \geq \chi'(G)$ then $\text{prc}(G) = \text{rc}(G)$. In [1] this conjecture was disposed of by the example of a (large) clique with a (long) path attached. By making the clique large and the path yet longer, we can arrange to have $\text{prc}(G)$ - $\text{rc}(G)$ arbitrarily large while $\text{rc}(G) \geq \chi'(G)$. (In fact, $\text{rc}(G)$ - $\chi'(G)$ can be arbitrarily large). Here we note that the same class of examples dismisses another plausible conjecture, that $\text{rc}(G) \leq \chi'(G)$ might imply that $\text{prc}(G) = \chi'(G)$. By making the order of the clique much larger than the (large) order of the path we can arrange that $\chi'(G)$ - $\text{rc}(G)$ and $\text{prc}(G)$ - $\chi'(G)$ are simultaneously arbitrarily large.

However, in these examples, in the first case ($\text{rc}(G) \geq \chi'(G)$) we always have $\frac{\text{prc}(G)}{\text{rc}(G)} < 2$, and in the second ($\text{rc}(G) \leq \chi'(G)$) we always have $\frac{\text{prc}(G)}{\chi'(G)} < 2$. Verification is left to the reader. This observation raises two more extremal questions of a different sort from 1 and 2, above.

3. Is $\{\frac{\text{prc}(G)}{\text{rc}(G)} \mid G$ is finite, simple, connected, $|E(G)| > 0$, and $\text{rc}(G) \geq \chi'(G)\}$ unbounded, and, if not, what is its least upper bound?

4. Is $\{\frac{\text{prc}(G)}{\chi'(G)} \mid G$ is finite, simple, connected, $|E(G)| > 0$, and $\text{rc}(G) \leq \chi'(G)\}$ unbounded, and, if not, what is its least upper bound?

3 Rainbow Connectivity of Theta Graphs

Definition 3.1. *A* theta graph $\theta(m_1, m_2, \ldots, m_k)$ *is a set of two vertices* u *and* v *with* $k \geq 3$ *internally disjoint (simple) paths between them such that the paths have lengths* $m_1, m_2, \ldots, m_k$*, and at most one of the path lengths is 1.*

Chartrand et al. identified the rainbow connection number and strong rainbow connection number of $G = \theta(3, 3, 3)$: $\text{rc}(G) = \text{src}(G) = 4$ [2]. Their proof states that $\text{diam}(G) = 3$ and shows it is not possible to edge color G with 3 colors such that G is rainbow connected. It goes on to provide an

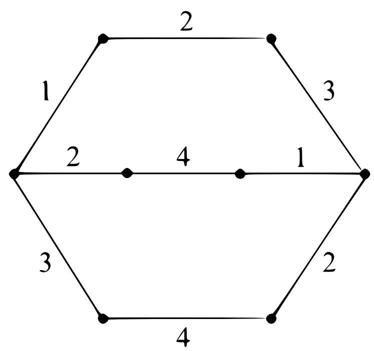


Figure 5: $\theta(3,3,3)$ with a proper strongly rainbow connective edge coloring with 4 colors.

edge coloring of G using 4 colors with respect to which the graph is strongly rainbow connected. As this edge coloring is proper, it follows that $\text{prc}(G) = \text{psrc}(G) = 4$. Our current goal is to identify the rainbow connection numbers and proper rainbow connection numbers of theta graphs with three paths in which $k = 3$, $m_1 = 1$, $m_2 = s = |E(S)|$, and $m_3 = t = |E(T)|$ for paths S and T. The common ends of the 3 paths will be u and v; they will also bear other names in the proofs.

Theorem 3.2. *Let $G = \theta(1,s,t)$, $s,t > 1$.*

I. If s and t are both odd, then rc(G) = src(G) = prc(G) = psrc(G) = diam(G).

II. If s and t are both even, then rc(G) = src(G) = diam(G).

- *(a) If $s = t = 2$, then prc(G) = psrc(G) = $\chi'(G)$ = 3.*
- *(b) If $s \geq 2$ and $t \geq 4$, then prc(G) = psrc(G) = diam(G).*

III. Suppose s is even and t is odd.

- *(a) If $s = 2$ or $t = 3$, then rc(G) = src(G) = diam(G) and prc(G) = psrc(G) = diam(G)+1.*
- *(b) If $s = 4$ and $t \geq 5$ then rc(G) = diam(G) and src(G) = prc(G) = psrc(G) = diam(G)+1.*
- *(c) If $s \geq 6$ and $t \geq 5$, then rc(G) = src(G) = prc(G) = psrc(G) = diam(G)+1.*

Proof. The edge uv will also be denoted e. In all cases, $\operatorname{diam}(G) = \lfloor \frac{s+t}{2} \rfloor$. If $s = t = 2$ or $s = 2$, $t = 3$ then $\operatorname{diam}(G) = 2$, so, by Proposition 2.3, $\operatorname{prc}(G) = \operatorname{psrc}(G) = \chi'(G) = 3$. In these cases it is easy to see that $\operatorname{rc}(G) = \operatorname{src}(G) = 2$. Thus, the claim in II(a) and one claim in III(a) are established. We will deal with the remaining claims below.

Before proceeding to the hard parts of the proof, we note that the claims for $\operatorname{rc}(G)$ and $\operatorname{prc}(G)$ have easy proofs for many pairs (s, t). Observe that $G - e = C_{s+t}$, so by Theorem 2.4, $A(G - e) = \lceil \frac{s+t}{2} \rceil$ for all $A \in \{\text{rc, src, prc, psrc}\}$.

If $s + t$ is even, we have $\operatorname{rc}(G - e) = \frac{s+t}{2} = \operatorname{diam}(G) = \operatorname{rc}(G)$; the last equality follows because if $G - e$ is rainbow connected with respect to an edge coloring with $\frac{s+t}{2}$ colors, then so is G if the coloring is extended to G by coloring e with any of the $\frac{s+t}{2}$ colors already appearing.

The same argument shows that $\operatorname{prc}(G) = \operatorname{rc}(G) = \frac{s+t}{2}$ when $s + t$ is even and $\frac{s+t}{2} \geq 5$; this last inequality would guarantee that if $G - e$ is properly edge-colored with $\frac{s+t}{2}$ colors so that $G - e$ is rainbow connected, then some color among the $\frac{s+t}{2}$ can be found which does not appear on any of the 4 edges of $G - e$ incident to either u or v. Then assigning such a color to e results in a proper edge coloring of G with respect to which G is rainbow connected. (However, our proofs below of the theorem's claims about $\operatorname{prc}(G)$ when $s + t$ is even apply even when $s + t \leq 8$.)

When $s + t$ is odd, consideration of $G - e$ gives us $\operatorname{rc}(G) \leq \lceil \frac{s+t}{2} \rceil = \operatorname{diam}(G)+1$ and, when $s + t \geq 9$, $\operatorname{prc}(G) \leq \lceil \frac{s+t}{2} \rceil = \operatorname{diam}(G)+1$. Therefore, if we show that $\operatorname{rc}(G) > \lfloor \frac{s+t}{2} \rfloor = \operatorname{diam}(G)$ when $s+t$ is odd, then the claims of III(c) for $\operatorname{rc}(G)$ and $\operatorname{prc}(G)$ will be established.

Consideration of $G-e$ is not much help in determining $\operatorname{src}(G)$ and $\operatorname{psrc}(G)$ because, for many pairs of vertices, the geodesics between them in G are quite different from the geodesics between them in $G - e$.

Now we will start the serious portion of this proof, with the proof of I.

Suppose that s and t are both odd, and $s \leq t$. Then $s, t \geq 3$. Let $k = \frac{s+1}{2}$ and $n = \operatorname{diam}(G) = \frac{s+t}{2}$. Let the vertices of G be as in Figure 6. Color the edges of G by the following instructions:

1. The diametral geodesic path $l_1, l_2, \ldots, l_n, r_n$ has its edges colored $1, 2, \ldots, n$. That is, $l_i l_{i+1}$ is colored i, $i = 1, \ldots, n-1$, and $l_n r_n$ is colored n. Note that $l_{k-1} l_k = l_{k-1} u$ is colored $k - 1$ and $l_k l_{k+1}$ is colored k; note that $n - k = \frac{s+t}{2} - \frac{s+1}{2} = \frac{t-1}{2} \geq 1$.

2. Color the edges of the path $r_n, \ldots, r_k$ with $k, \ldots, n-1$, in that order. That is, $r_j r_{j-1}$ is colored $k + n - j$, $j = k+1, \ldots, n$.

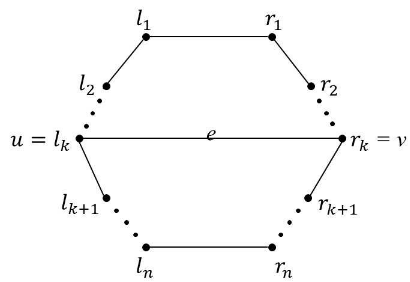


Figure 6: $\theta(1, s, t)$, s, t both odd

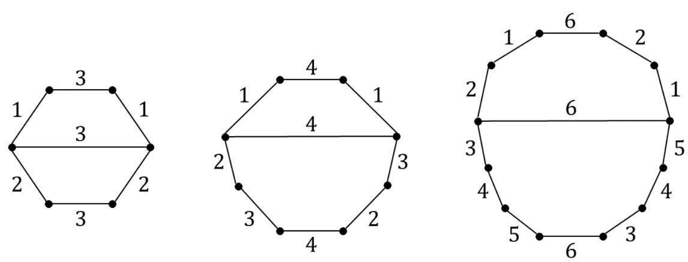


Figure 7: Proper strongly connecting edge coloring of $G = \theta(1, s, t)$ in the cases $(s, t) \in \{(3, 3), (3, 5), (5, 7)\}$

3. Color the edges of the path $r_1, \ldots, r_k$ with $k-1, \ldots, 1$, in that order. That is, $r_i r_{i+1}$ is colored $k - i$, $i = 1, \ldots, k-1$.

4. Color the edges e and $l_1 r_1$ with n.

This edge coloring is illustrated in Figure 7 for the cases $s = t = 3$; $s = 3$, $t = 5$; and $s = 5$, $t = 7$. Clearly this coloring is proper. We leave it to the reader to verify that G is strongly rainbow connected by this coloring.

This concludes the proof of I.

Next: The proof of II.

Suppose that $2 \leq s \leq t$ and that both s and t are even. The diameter of $G = \theta(1, s, t)$ is $\frac{s+t}{2} = n$. We have already disposed of the case $s = t = 2$. In all other cases, it suffices to show that $\mathrm{psrc}(G) \leq \mathrm{diam}(G) = n$.

Case 1: $s = 2$, $t = 4$. Then $n = 3$.

A proper edge 3-coloring of G with respect to which G is strongly rainbow connected is given in Figure 8.

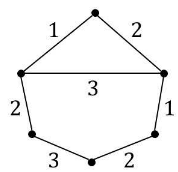


Figure 8: A proper strong rainbow connecting edge coloring of $\theta(1, 2, 4)$ with 3 colors

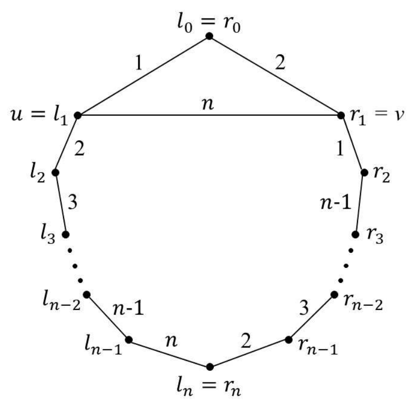


Figure 9: $\theta(1, 2, t)$, $t \geq 6$, even, with a proper strong rainbow connecting edge coloring with $n = \frac{t}{2} + 1$ colors

Case 2: $s = 2$, $t \geq 6$. Then $n = \frac{t}{2} + 1 \geq 4$.

Let the vertices of $G = \theta(1, 2, t)$ be named as in Figure 9. Included in Figure 9 is an indication of a proper edge coloring of G with $n = \text{diam}(G)$ colors with respect to which G is strongly rainbow connected. In case the color assignment is unclear, here it is in words.

For $i = 1, \ldots, n$, the edge $l_{i-1}l_i$ is colored with i; that is, the edges of the "left" diametral geodesic from l_0 to l_n are colored $1, \ldots, n$, in that order. For $j = 3, \ldots, n$, the edge $r_{j-1}r_j$ is colored $n - j + 2$; that is, the edges of the path on the vertices $r_n, r_{n-1}, \ldots, r_2$ are colored $2, 3, \ldots, n - 1$, in that order. Then r_1r_2 is colored 1, r_0r_1 is colored 2, and $l_1r_1 = uv$ is colored n.

Note that the coloring in the case $s = 2$, $t = 4$ was a degenerate instance of this coloring. We judged that it would be clearer if dealt with as a separate case.

Case 3. $4 \leq s \leq t$. Then $n = \frac{s+t}{2} \geq 4$. Let $k = \frac{s}{2}$. We indicate in Figure 10 a naming of the vertices and a proper edge coloring of $G = \theta(1, s, t)$ with

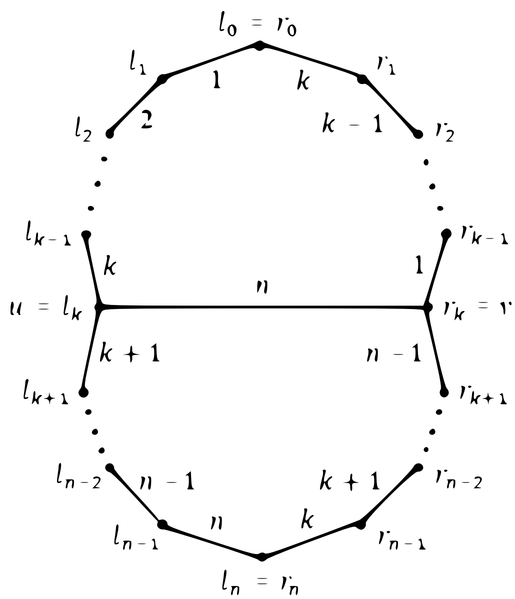


Figure 10: A proper coloring of $\theta(1, s, t)$, $4 \leq s \leq t$, s, t even, with $n = \frac{s+t}{2}$ colors, with respect to which the graph is strongly rainbow connected

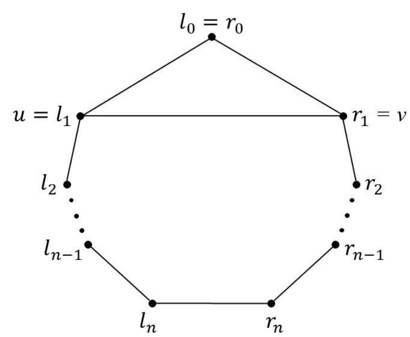
n colors with respect to which G is strongly rainbow connected, and that will finish the proof of II.

In the coloring above, $l_{i-1}l_i$ is colored i for $i = 1, \ldots, n$; $r_{j-1}r_j$ is colored $n - j + k$, $j = k+1, \ldots, n; r_{j-1}r_j$ is colored $k - j + 1$, $j = 1, \ldots, k$; and $l_k r_k = uv$ is colored n.

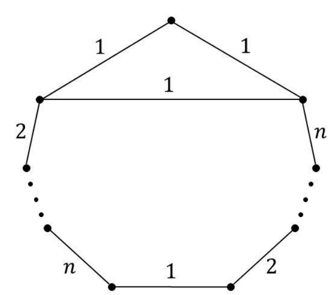
Finally, we undertake the proof of III. Throughout, $s \geq 2$ is even, $t \geq 3$ is odd, so $n = \text{diam}(G) = \lfloor \frac{s+t}{2} \rfloor = \frac{s+t-1}{2}$. The case $s = 2$, $t = 3$ has already been dealt with.

Case 1. $s = 2$, $t \geq 5$. In this case, $n = \frac{t+1}{2}$.

The vertices of $G = \theta(1, 2, t)$ will be named as follows:



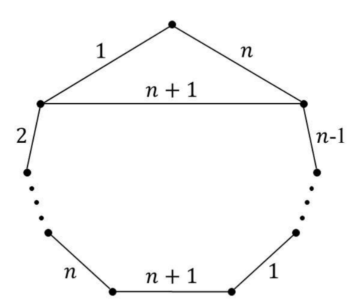
First we will show that $\mathrm{rc}(G) = \mathrm{src}(G) = n$ by giving an edge coloring of G with n colors appearing such that G is strongly rainbow connected with respect to this coloring.



We hope that this coloring is clear. In case it is not: The diametral geodesic $l_0, l_1, \ldots, l_n$ is edge-colored $1, \ldots, n$, and this coloring is repeated on $l_n, r_n, \ldots, r_1$. Then r_0r_1 and $r_1l_1 = uv$ are colored 1. This edge coloring is not proper, but the graph is strongly rainbow connected with respect to this edge coloring.

To finish the proof in this case we must do two things: (1) show that there is a proper strongly rainbow connective edge coloring of G with $n + 1$ colors, and (2) show that there is no proper rainbow connective edge coloring of G with n colors.

The picture following disposes of (1).

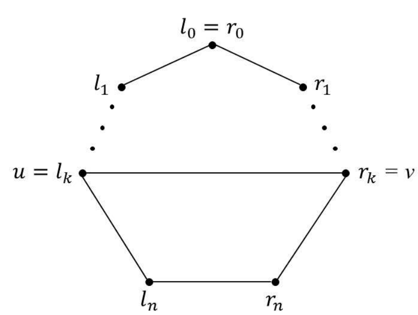


As for (2): Suppose that ϕ: $\mathrm{E}(G) \longrightarrow \{1,\ldots,n\}$ is a proper rainbow connective coloring. There is exactly one geodesic joining l_0 and l_n. Therefore we may assume that the edges of the path $l_0, l_1, \ldots, l_n$ are colored $1, 2, \ldots, n$, in that order.

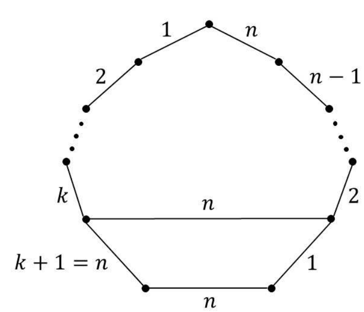
The unique geodesic in G from r_0 to r_n, which is $r_0, r_1, \ldots, r_n$ must be rainbow, and either the path $l_1, \ldots, l_n, r_n$ or the path $l_1, r_1, \ldots, r_n$ must be rainbow. But if the latter path is rainbow than the rainbowness of $r_0, \ldots, r_n$ and the assumption that there are only n colors appearing forces $\phi(r_0r_1) = \phi(l_1r_1)$, which would contradict the presumed properness of ϕ. Therefore the path $l_1, \ldots, l_n, r_n$ is rainbow. Therefore $\phi(l_nr_n) = 1$.

But applying the mirror-image argument to the rainbow geodesic path $r_0, \ldots, r_n$ forces us to the conclusion that $\phi(l_nr_n) = \phi(r_0r_1)$, which again contradicts the assumption that ϕ is proper. This completes the proof of III in this case.

Case 2. $s \geq 4$ and $t = 3$. Then $n = \frac{s}{2} + 1 = k + 1$. The vertices of $G = \theta(1, s, 3)$ will be as follows:



Here is an improper edge coloring of G with n colors with respect to which G is strongly rainbow connected.



Recoloring uv and l_nr_n with $n+1$ gives a proper strongly rainbow connective edge coloring with $n+1$ colors appearing.

It remains to be seen that there is no proper edge coloring of G with n colors with respect to which G is rainbow connected. Suppose, to the contrary, that ϕ: E$(G) \longrightarrow \{1, \ldots, n\}$ is a proper edge coloring of G with respect to which G is rainbow connected.

Both diametral geodesics $l_0, l_1, \ldots, l_n$ and $l_0 = r_0, r_1, \ldots, r_n$ must be rainbow. Let the colors be named so that $\phi(l_{i-1}l_i) = i$ and $\phi(r_{i-1}r_i) = i'$, $i = 1, \ldots, n$. Because ϕ is proper, $1 \neq 1'$.

At least one of the two l_1–r_n geodesics, either $l_1, \ldots, l_k, l_n, r_n$ or $l_1, \ldots, l_k, r_k, r_n$, must be rainbow. In either case, since the colors on $l_1, \ldots, l_k$ are $2, \ldots, k$, the colors on the last two edges of the path must be 1 and n. Because $\phi(l_kl_n) = n$, $\phi(l_kr_k) \neq n \neq \phi(l_nr_n)$. From these considerations we conclude that if $\phi(l_nr_n) \neq 1$ then $\phi(l_kr_k) = 1$ and $n' = \phi(r_kr_n) = n$.

Applying the same reasoning with l_1, r_n replaced by r_1, l_n, and the colors i replaced by the colors i', $i = 1, \ldots, n$, we conclude that if $\phi(l_nr_n) \neq 1'$ then $\phi(l_kr_k) = 1'$ and $n = \phi(l_kl_n) = n'$.

Since $\phi(l_nr_n)$ cannot equal both 1 and 1', we conclude that $n = n'$ and $\phi(l_kr_k) \in \{1, 1'\}$. Without loss of generality, we assume that $\phi(l_kr_k) = 1$.

There are two paths of length $\leq n$ with end vertices r_1 and l_k, namely, $r_1, r_2, \ldots, r_k, l_k$ and $r_1, r_0 = l_0, l_1, \ldots, l_k$. At least one of these must be rainbow. It cannot be the first, because $1 \notin \{1', n\}$ and $r_0, r_1, \ldots, r_n$ is rainbow, so 1 appears as a color somewhere on the path $r_1, \ldots, r_k$ as well as on r_kl_k.

Therefore the path $r_1, l_0, l_1, \ldots, l_k$ is rainbow. Therefore, because the path $l_0, \ldots, l_k, l_n$ is rainbow and there are only n colors, $1' = \phi(l_0r_1) = n$. But then the path $l_0 = r_0, r_1, \ldots, r_n$ is not rainbow. This contradiction finishes the proof in Case 2.

Case 3. $s \geq 4$, $t \geq 5$.

As before, let $k = \frac{s}{2}$ and $n = \text{diam}(G) = \frac{s+t-1}{2}$. The vertices of $G = \theta(1, s, t)$ will be named as in the diagram following Figure 11.

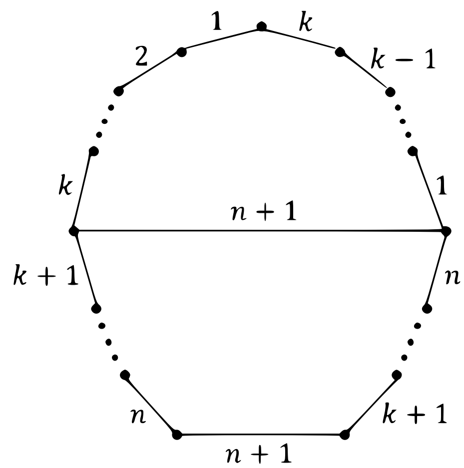
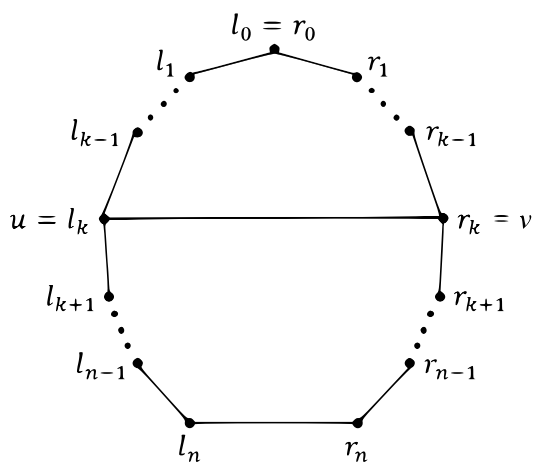


Figure 11: A proper strongly rainbow connective edge coloring of G with $n+1$ colors



The proper coloring with $n+1$ colors in Figure 11 shows that $\text{psrc}(G) \leq n+1$. It then remains to show that, when $s \geq 6$, $\text{rc}(G) > n$ and when $s = 4$, $\text{rc}(G) = n$ while $\text{src}(G)$, $\text{prc}(G) > n$.

Assuming only that $s \geq 4$, even, and $t \geq 5$, odd, suppose that ϕ: $\text{E}(G) \longrightarrow \{1, \dots, n\}$ is a coloring with respect to which G is rainbow connected. Our argument will show that this is impossible when $s > 4$, and, while possible when $s = 4$, is only achievable by improper edge colorings with respect to which G is *not* strongly rainbow connected. Both diametral geodesics $l_0, l_1, \dots, l_n$ and $l_0 = r_0, r_1, \dots, r_n$ must be rainbow. Let the colors along the first be $1, 2, \dots, n$, (i.e. $\phi(l_{j-1}l_j) = j$, $j = 1, \dots, n$), and let the colors along the second be $1', \ldots, n'$ (i.e., $\phi(r_{j-1}r_j) = j'$, $j = 1, \ldots, n$).

At least one of the paths $P_1 = [r_1, r_2, \ldots, r_n, l_n]$ and $P_2 = [r_1, r_2, \ldots, r_k, l_k, \ldots, l_n]$ is rainbow, since these are the only paths in G of length $\leq n$ from r_1 to l_n, and, for a similar reason, at least one of $P_3 = [l_1, \ldots, l_n, r_n]$ and $P_4 = [l_1, l_2, \ldots, l_k, r_k, \ldots, r_n]$ must be rainbow.

Suppose that P_2 is rainbow. Then, because $[l_0, \ldots, l_n]$ is rainbow and the colors on $[l_k, \ldots, l_n]$ are $k+1, \ldots, n$, it must be that $\{2', \ldots, k', \phi(l_k r_k)\} = \{1, \ldots, k\}$. We claim that this forces the conclusion that $\phi(l_k r_k) = 1$. Otherwise, $1 = \phi(r_{j-1}r_j) = j'$ for some $j \in \{2, \ldots, k\}$. Then neither path around $S \cup l_k r_k \simeq C_{s+1}$ from r_j to l_1 is rainbow: the color 1 appears twice on the "top" path, and the color $\phi(l_k r_k)$ appears twice on the "bottom" path. The only other path in G with end vertices r_j, l_1, the one with T as a subpath, is longer than n, and is therefore also not rainbow.

Therefore $1 = \phi(uv) = \phi(l_k r_k)$. Therefore $\{2', \ldots, k'\} = \{2, \ldots, k\}$. Since $[r_0, r_1, \ldots, r_n]$ is rainbow, it follows that $\{1', (k+1)', \ldots, n'\} = \{1, k+1, \ldots, n\}$.

Now, it cannot be that $1 = 1' = \phi(r_0 r_1)$, for, if it were, there would be no rainbow path in G from r_1 to l_1. The path of length 2, r_1, r_0, l_1, would have two 1's appearing, the other path around $S \cup uv$ would have $\phi(r_1 r_2) = 2' \in \{2, \ldots, k\}$ appearing twice, and the only other path in G from r_1 to l_1 is too long to be rainbow.

Therefore, $1 = j' = \phi(r_{j-1}r_j)$ for some $j \in \{k+1, \ldots, n\}$. If $j < n$ then there is no rainbow path in G from r_j to l_1. Therefore, $\phi(r_{n-1}r_n) = 1$ and the path $P_3 = [l_1, l_2, \ldots, l_n, r_n]$ must be rainbow. Because $[l_0, l_n]$ is rainbow, it follows that $\phi(l_n r_n) = \phi(l_0 l_1) = 1$.

From $\{1', (k+1)', \ldots, n'\} = \{1, k+1, \ldots, n\}$ and, as we now know, $n' = 1$, we conclude that $\{1', (k+1)', \ldots, (n-1)'\} = \{k+1, \ldots, n\}$. But now it follows that there is no rainbow path in G from r_{n-1} to l_n. One path from r_{n-1} to l_n around the cycle $T \cup uv$ has the color 1 appearing twice; the other has $\phi(r_{n-2}r_{n-1}) = (n-1)' \in \{k+1, \ldots, n\}$ appearing twice, and the only other path in G joining r_{n-1} to l_n is too long to be rainbow.

This shows, finally, that P_2 being rainbow means that G cannot be rainbow connected with respect to ϕ after all.

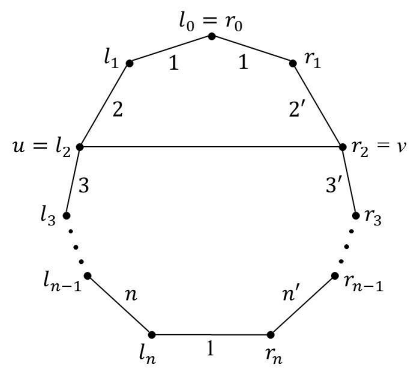
Therefore we may assume that neither P_2 nor P_4 are rainbow. Therefore both P_1 and P_3 are rainbow. By an inference previously applied, it follows that $1 = 1' = \phi(l_n r_n) = \phi(l_0 l_1) = \phi(r_0 r_1)$.

The path $P_2 - l_n = [r_1, \ldots, r_k, l_k, \ldots, l_{n-1}]$ must be rainbow, because of the other two paths in G with end vertices r_1 and l_{n-1}, one is too long to be rainbow and the other has the color 1 appearing twice. From the rainbowness of this path we conclude that $\{2', \ldots, k', \phi(l_k r_k)\} \subseteq \{1, \ldots, k, n\}$.

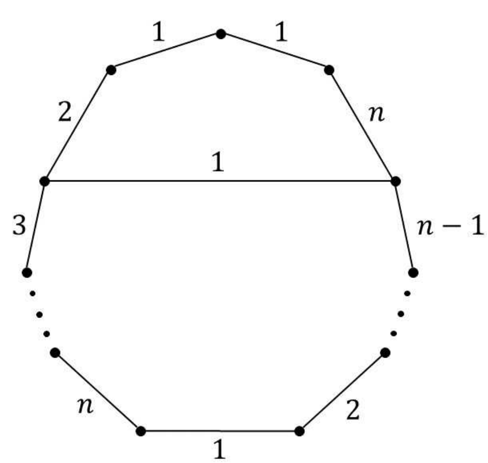
On the other hand, $\{2', \ldots, k', \phi(l_k r_k)\} \cap \{2, \ldots, k\} = \emptyset$, because, otherwise, $\phi(r_j w) = q \in \{2, \ldots, k\}$ for some $j \in \{1, \ldots, k\}$ and $w = r_{j+1}$ if $j < k$, $w = l_k$ if $j = k$. But then there is no rainbow path in G from r_j to l_1; on one of the possible paths the color q appears twice, on another 1 appears twice, and the third is too long to be rainbow.

Therefore, if $s \geq 6 \Rightarrow k = \frac{s}{2} \geq 3$, we have a proof-ending contradiction.

So assume that $s = 4$. The situation at present is depicted as follows.



The presence of a monochromatic path of length 2 in G, or in any edge-colored triangle-free graph, implies both that the coloring is not proper and that the graph is not strongly rainbow connected with respect to the coloring. Since we were forced to $l_1 l_0$ and $r_0 r_1$ having the same color by the assumption that ϕ is a rainbow connective coloring using only n colors, this proves, even in the case $s = 4$, that $\text{prc}(G)$, $\text{src}(G) > n$. That $\text{rc}(G) = n$ when $s = 4$ is proven by the following coloring.



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