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On edge-weighted mean eccentricity of graphs

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Abstract

Let G be a connected edge-weighted graph of order n and size m. Let $w : E(G) \to \mathbb{R}^{\geq 0}$ be the weighting function. We assume that w is normalised; that is, $\sum_{e \in E(G)} w(e) = m$. The weighted distance $d_w(u, v)$ between any two vertices u and v is the least weight between them and the eccentricity $e_w(v)$ of a vertex v is the weighted distance from v to a vertex farthest from it in G. The mean (average) eccentricity of G, denoted as $avec(G, w)$, is the (weighted) mean of all eccentricities in G. We obtain upper and lower bounds on $avec(G, w)$ in terms of n, m or edge-connectivity λ for two cases: G is a tree and G is connected but not a tree. In addition, we obtain the Nordhaus-Gaddum-type results for edge-weighted average eccentricity.

1 Introduction

A graph is G is said to be *edge-weighted* if there is a weight function $w : E(G) \to \mathbb{R}^{\geq 0}$ which assigns to every edge $e \in E(G)$ a nonnegative function $w(e)$ called the *weight* of e. When w is integer-valued, an edge-weighted graph may be thought of as a special multigraph M with no loops, such that the weight on each edge of G can be interpreted as the multiplicity of the edge. The weighted distance $d_w(u, v)$ is the least weight of a path between vertices u and v and the eccentricity $e_w(v)$ of the vertex v is the weighted

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distance between v and a vertex farthest from v in G.

If a graph models a routing network, for example, with vertices representing routers, edges representing links between the routers, and weights representing the number of network packets on each link that can be forwarded from one router to the other simultaneously, then the eccentricity of a vertex could be an indicator for a router's best-case packet quantity; that is, the maximum possible packets that can be forwarded from the vertex to any other vertex in the network. A low average eccentricity indicates the routers carry relatively low numbers of packets compared to the total packets for distribution, whereas a high average eccentricity indicates that the routers carry high numbers of packets relative to the total number of packets. However, a redistribution of the network packets may increase or decrease an average eccentricity. Often, trees are the cheapest network to consider. We investigate how the distribution of weights affect the parameter when one considers trees with a given number of end vertices. In addition, it is desirable to monitor a network's average eccentricity and that of its alternative links for an optimal result. In particular, what is the minimum or maximum difference that can occur when one compares the average eccentricity for a graph and that of its complement?

Besides its use to modelling communication networks, the average eccentricity of graphs is also useful in chemistry as one of the main topological indices (chemical molecular descriptor) for modelling physio-chemical, biological, and other properties of chemical compounds [11]. Several other topological indices useful in chemistry have been independently studied by different authors. For example, Weiner Index, defined as the sum of distances between all pairs of vertices in the graph was studied by Dobrynin and Entringer [10].

The average eccentricity of unweighted graphs; that is, graphs whose edges have all unit length, was first introduced by Buckley and Harary in [4] under the name *eccentric mean* but only attracted much attention after its first systematic study by Dankelmann et.al. in [5], who, amongst other results, determined the maximum average eccentricity of an unweighted connected graph of given order.

Proposition 1.1. *[5] If G is a connected unweighted graph of order n, then*

$$\operatorname{avec}(G) \leq \frac{3n}{4} - \frac{1}{2},$$

with equality if and only if G is a path of even order.

In the same paper, a Nordhaus-Gaddum result for the unweighted average eccentricity was obtained.

Proposition 1.2. *[5] Assume both G and $\overline{G}$ are connected and $n \geq 5$. Then the following bounds are best possible.*

$$4 \leq \text{avec}(G) + \text{avec}(\overline{G}) \leq \frac{3n}{4} + \frac{3}{2},$$

$$4 \leq \text{avec}(G)\text{avec}(\overline{G}) \leq \frac{3n}{2} - 1.$$

Since then, the average eccentricity has been studied by several other authors (see [1],[2],[6],[7],[8],[13] for references). One notable result is the lower bound obtained by Tang and Zhou.

Proposition 1.3. *[13] Let G be a unicyclic connected graph of order $n \geq 5$. Then*

$$\text{avec}(G) \geq 2 - \frac{1}{n}.$$

Equality holds if and only if G is formed by adding one edge to the n-vertex star.

Not much is known about the average eccentricity of graphs when the weight assigned to each edge is zero or exceeds one. Broere et al. [3] studied the average distance of edge-weighted graphs when some properties of the graph are known. It was shown in [9] by Djelloul and Kouider that if the collection of the weights to be assigned is fixed, then the problem of maximizing the average distance is NP-complete. They also showed that for λ-edge-connected multigraphs, the mean distance is bounded above by $\frac{2m}{3\lambda}$. In this paper, we establish similar bounds on the average eccentricity of an edge-weighted graph if the structure of the graph is known but not the actual weight function. We first obtain a lower bound for trees given that w is a normalised weight function and then for general graphs that are not trees. Finally, we establish Nordhaus-Gaddum results for edge-weighted average eccentricity.

1.1 Definitions and notations

Let $G = (V, E)$ be a connected graph with m edges and order n; and let $w : E(G) \to \mathbb{R}^{\geq 0}$ be a weight function. In this paper, we consider only graphs G that are finite, simple and connected. For $u, v \in V(G)$, we denote by $d_w(u,v)$ the minimum length of a path between u and v according to the valuation w. The w-eccentricity $e_w(v)$ of a vertex v in G is the largest weighted distance from v to any other vertex in G. Let $EX(G,w)$ be the sum over $V(G)$ of the w-eccentricities of vertices in G. The average eccentricity of a weighted graph G, denoted by $avec(G,w)$, is defined to be the average of all weighted eccentricities in G; that is,

$$avec(G,w) = \frac{1}{n}EX(G,w) = \frac{1}{n}\sum_{v\in V(G)} e_w(v).$$

A *normalised* weight function is a nonnegative weight function whose average weight of edges equals 1; that is,

$$\sum_{e\in E(G)} w(e) = m.$$

Every nonnegative weight function can be obtained from a normalised weight function by multiplying by a suitable real number. If $w(e) = 1$ for each $e \in E(G)$, we write $avec(G,1)$. Unless otherwise stated, every weight function used in this paper is normalised.

We define the *minimum average eccentricity* of a graph G by

$$avec_{\min}(G) = \inf_{w} avec(G,w),$$

and the *maximum average eccentricity* of G by

$$avec_{\max}(G) = \sup_{w} avec(G,w),$$

where the infimum and supremum are taken over all normalised weight functions w on G. Since $avec(G,w)$ is a linear, and therefore continuous, function of the values $w(e)$, $e \in E(G)$, and the normalised weighting functions on $E(G)$ form a closed and bounded subset of $\mathbb{R}^{|E(G)|}$, it follows that the *inf* and *sup* here are indeed the min and max, respectively. A normalised weight function for which $avec_{\max}(G) = avec(G,w)$ is called an *optimal* weight function of G.

If H is a subgraph of G, we write $w(H)$ for $\sum_{e\in E(H)} w(e)$. The degree of a vertex v is the number of vertices adjacent to v in G. Let T be a tree. A *leaf* or *end vertex* is a vertex of degree 1 in T. An *end edge* in T is an edge incident with a leaf of T and every edge that is not an end edge of T is an *internal edge* of T. An edge $e \in E(G)$ is called a *cut-edge* if the removal of e

disconnects G. If G is connected, a subset F of $E(G)$ is a *cut set* if deleting all edges in F from G disconnects G. The number of edges in a smallest cut set of G is the *edge-connectivity* of G, denoted as λ. A graph $\overline{G}$ is the *complement* of G on the same vertices in which two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. A graph is said to be *self-complementary* if it is isomorphic to its complement.

We define a ***single broom*** $B(3, n-3)$ or simply B_n, as the tree obtained from P_3 by appending a set of $n-3$ end vertices to an end vertex of the P_3.

2 Main Results

2.1 Edge-weighted trees

Let t be the number of leaves in a tree T with at least 3 vertices. The following theorem shows that for T, a weight function minimises the total and average eccentricity if and only if we assign a zero weight to all internal edges of T and weight $\frac{n-1}{t}$ to the end edges of T.

Theorem 2.1. *Let T be a tree with n vertices, $n \geq 3$, and t end vertices. If w is a normalised weight function on the edges of T, then*

$$avec(T, w) \geq \frac{(n-1)(n+t)}{nt}.$$

Equality holds if and only if every end edge has weight $\frac{n-1}{t}$, and all the other edges have weight 0.

Proof. Let $u_1, u_2, \dots, u_t$ be the end vertices of T. For $i = 1, 2, \dots, t$, let e_i be the end edge incident with u_i.
Let w be an arbitrary normalised weight function on T. We first show that for every internal vertex v of T.

$$e_w(v) \geq \frac{n-1}{t}. \tag{2.1}$$

Since each edge of T is on a (v, u_i)-path for some $i \in \{1, 2, \dots, t\}$, we have

$$\sum_{i=1}^{t} d(v, u_i) \geq \sum_{e \in E(T)} w(e) = n - 1. \tag{2.2}$$

Hence there exists $i \in \{1, 2, \dots, t\}$ such that

$$\frac{n-1}{t} \leq d(v, u_i) \leq e_w(v), \tag{2.3}$$

and so (2.1) follows. We now bound the sum of the eccentricities of the end vertices. The eccentricity of an end vertex u_i is clearly at least the average of the distances from u_i to the other end vertices u_j, $j \neq i$; that is,

$$e_w(u_i) \geq \frac{1}{t-1} \sum_{j:j\neq i} d(u_i, u_j).$$

Summing this over all $i \in \{1, 2, \ldots, t\}$ yields

$$\sum_{i=1}^{t} e_w(u_i) \geq \frac{1}{t-1} \sum_{i=1}^{t} \sum_{j:j\neq i} d(u_i, u_j). \tag{2.4}$$

For every edge e of T, let $c(e) = 2n_1n_2$, where n_1 and n_2 are the numbers of end vertices of T in the two components of $T - e$. Since $c(e)$ counts the number of ordered pairs (u_i, u_j) for which the (u_i, u_j)-path in T contains e, we have

$$\sum_{i=1}^{t} \sum_{j:j\neq i} d(u_i, u_j) = \sum_{e\ \in E(T)} c(e)w(e). \tag{2.5}$$

If e is an end edge, then $c(e) = 2(t - 1)$, and if e is not an end edge then $c(e) \geq 2(t - 1)$. So $c(e) \geq 2(t - 1)$. Since $\sum_{e\in E(T)} w(e) = n - 1$, it follows that

$$\sum_{e\in E(T)} c(e)w(e) \geq \sum_{e\in E(T)} 2(t-1)w(e) = 2(t-1)(n-1). \tag{2.6}$$

From (2.4), (2.5) and (2.6) we get that

$$\sum_{i=1}^{t} e_w(u_i) \geq 2(n-1). \tag{2.7}$$

Now further simplification from (2.1) and (2.7) yields

$$EX(T, w) \geq (n-t)\frac{n-1}{t} + 2(n-1) = \frac{(n-1)(n+t)}{t},$$

and division by n gives the bound in the theorem.

We prove the second part of the theorem. First suppose that $w(e) = 0$

for every internal edge e, and $w(e_i) = \frac{n-1}{t}$, $i = 1, 2, \ldots, t$. Then

$$\begin{aligned} avec(T, w) &= \frac{1}{n}\Big[\sum_{v,\text{ an internal vertex of } T} e_w(v) + \sum_{i=1}^{t} e_w(u_i).\Big] \\ &= \frac{1}{n}\Big[(n-t)\frac{n-1}{t} + 2t\frac{n-1}{t}\Big] \\ &= \frac{(n-1)(n+t)}{nt}. \end{aligned}$$

Now suppose that

$$avec(T, w) = avec_{min}(T) = \frac{(n-1)(n+t)}{nt} \tag{2.8}$$

We show that (2.8) implies that $w(e) = 0$ for every internal edge e, and $w(e_i) = \frac{n-1}{t}$, $i = 1, 2, \ldots, t$. Let $I(T) = \{v \in V(T) | v \text{ an internal vertex of } T\}$. The inequality

$$avec(T, w) \geq \frac{(n-1)(n+t)}{nt} = \frac{1}{n}\Big[(n-t)\frac{n-1}{t} + 2(n-1)\Big],$$

arose from two inequalities,

$$\sum_{v \in I(T)} e_w(v) \geq (n-t)\frac{n-1}{t}, \tag{2.9}$$

and

$$\sum_{i=1}^{t} e_w(u_i) \geq 2(n-1) \tag{2.10}$$

Therefore (2.8) implies equality in both of these. Inequality (2.9) arose from (2.1). Thus, equality in (2.9) implies equality in (2.1) for every $v \in I(T)$. Then, (2.2) and $d(v, u_i) \leq e_w(v) = \frac{n-1}{t}$ imply that $d(v, u_i) = \frac{n-1}{t}$ for every $i = 1, 2, \ldots, t$ and $v \in I(t)$.

But then: suppose that $e = uv$ is an internal edge of T, which implies that both u and v are internal vertices of T. If $w(e) > 0$, then $d(u, u_i) \neq d(v, u_i)$ for at least 2 values of i, contradicting $d(u, u_i) = \frac{n-1}{t} = d(v, u_i)$, $i = 1, 2, \ldots, t$. So $w(e) = 0$ for every internal edge e of T. Since $d(v, u_i) = \frac{n-1}{t}$ for $i = 1, 2, \ldots, n$ and $v \in I(T)$, it must be that $w(e_i) = \frac{n-1}{t}$, $i = 1, 2, \ldots, t$. □

Theorem 2.2. *Let T be a tree of order $n \geq 2$ and w a normalised weight function on $E(T)$. Then*

$$avec(T, w) \leq n - 1.$$

Equality holds if there exists an edge e of T with $w(e) = n - 1$.

Proof. Let w be an arbitrary normalised weight function on T. The total weight of all edges of T is $n-1$. Hence no distance between two vertices in T is greater than $n-1$, and thus

$$e_w(v) \leq n - 1 \quad \text{for all } v \in V(T).$$

Summing over all $v \in V(T)$ and dividing by n yields

$$avec(T, w) \leq n - 1.$$

To prove the second part of the theorem, assume that w is a normalised weight function on $E(T)$ for which $avec(T, w) = n - 1$. Then $e_w(v) = n - 1$ for all $v \in V(T)$. We show that there exists an edge of T of weight $n-1$. Suppose not. Then there exist two edges e_1, e_2 of positive weight in T. Let $e_1 = uv$, and assume without loss of generality that u is closer to e_2 than v. Let u' be an eccentric vertex of u and let P be the (u, u')-path in T. Then P cannot contain both, e_1 and e_2. If it does not contain e_1, then

$$d(u, u') = \sum_{e \in E(P)} w(e) \leq n - 1 - w(e_1) < n - 1.$$

Similarly, if P does not contain e_2, then

$$d(u, u') = \sum_{e \in E(P)} w(e) \leq n - 1 - w(e_2) < n - 1.$$

In both cases we obtain the contradiction $e_w(u) < n - 1$. Hence there exists an edge e with $w(e) = n - 1$. On the other hand, if w is a weight function that assigns weight $n-1$ to one edge and 0 to all other edges, then clearly every vertex has eccentricity $n-1$, and so $\text{avec}(T, w) = n - 1$. □

The following corollary is a consequence of Theorems 2.1 and 2.2.

Corollary 2.3. *Let T be a tree of order $n \geq 3$. Then*

$$2 - \frac{1}{n} \leq avec_{\min}(T) \leq \frac{n+1}{2} - \frac{1}{n},$$

and

$$avec_{\max}(T) = n - 1.$$

Proof. Since the tree T with the minimum number of end vertices is the path P_n with $t = 2$ and T with maximum number of end vertices is the star S_n with $t = n - 1$, it follows by Theorem 2.1 that the upper bound for $avec_{min}(T)$ is attained when $t = 2$ and the lower bound is attained when $t = n-1$. By substituting for $t = 2$ and $t = n-1$, respectively, the first part of the Corollary 2.3 follows.
To prove the second part of Corollary 2.3, we maximize $EX(T, w)$ by weighting one edge with $n - 1$ and all the other edges with 0, as in the proof of Theorem 2.2. Then $e_w(v) = n - 1$, for every vertex v in T and thus $EX_{max}(T) = n(n - 1)$. Therefore, $avec_{max}(T) = n - 1$. $\square$

We remark that both inequalities of the first part of Corollary 2.3 are best possible, since equality is attained on the left hand side by the star $K_{1,n-1}$ and on the right hand side by the path P_n. Further, these extreme examples for $avec_{min}(T)$ are achieved only by $K_{1,n-1}$ and P_n, respectively.

2.2 Edge-weighted graphs

We now turn our attention to connected graphs that are not trees.

Theorem 2.4. *Let G be a connected graph but not a tree with m edges and edge-connectivity λ. Then,*

$$avec_{min}(G) = 0 \quad \text{and} \quad avec_{max}(G) = \frac{m}{\lambda}.$$

Proof. Clearly we have $avec_{min}(G) \geq 0$ since we consider only nonnegative weight functions for which every distance is nonnegative. To show that $avec_{min}(G) = 0$, it suffices to construct a weight function w with $EX(G, w) = 0$. If G is not a tree, then $m \geq n$. Let T be a spanning tree of G. Let w be a weight function that assigns weight 0 to all edges in T, and the remaining edges receive an arbitrary nonnegative weight so that the total weight is m. Since T contains a path of weight 0 between any two vertices of G, we have $e_w(v) = 0$ for all $v \in V(G)$, and thus $EX(G, w) = 0$. Hence the minimum possible average eccentricity is zero.

To prove the upper bound of the theorem, let w be an arbitrary normalised weight function on $E(G)$. By Menger's Theorem there are λ edge-disjoint $u - v$ paths in G for all $u, v \in V(G)$. Among these there exists a path of total weight at most $\frac{m}{\lambda}$. Hence $d_w(u, v) \leq \frac{m}{\lambda(G)}$ for all $u, v \in V(G)$, which implies that $e_w(v) \le \frac{m}{\lambda}$ for all $v \in V(G)$. Hence $EX_w(G) \le \frac{mn}{\lambda}$ and thus $avec_w(G) \le \frac{m}{\lambda}$. Since w was arbitrary, we conclude that

$$avec(G, w) \le \frac{m}{\lambda}.$$

To show that the bound is sharp, we show that $avec_{max}(G) = \frac{m}{\lambda}$. It suffices to give a weight function w for which $avec(G, w) = \frac{m}{\lambda}$. Let S be a minimum cut-edge of G. Define the weight function w by

$$w(e) = \begin{cases} \frac{m}{\lambda} & \text{if } e \in S, \\ 0 & \text{if } e \notin S \end{cases}$$

Clearly, w is normalised. Also, the distance between any two vertices in the same component of $G - S$ is 0, while the distance between two vertices in different components of $G - S$ is $\frac{m}{\lambda}$. Hence $e_w(v) = \frac{m}{\lambda}$ for all $v \in V(G)$ and thus $avec_w(G) = \frac{m}{\lambda}$, as desired. □

Remark: We have seen that the average eccentricity of a weighted graph can be much larger than that of the underlying unweighted graph. The above theorem is a generalisation of the corresponding result for trees, whose weighted maximum average eccentricity is always $n - 1$. It is important to note that the result for trees, which says that the weight functions realising the maximal average eccentricity are those in which all the weight is concentrated in one edge, does not generalise to general graphs. For example the maximum average eccentricity of an even cycle can also be realised, among others, by the all-1 function.

2.3 Nordhaus-Gaddum Type Results

As a consequence of the results obtained for avec(G, w), Nordhaus-Gaddum results for edge-weighted average eccentricity are easy to obtain. We consider cases for trees and connected graphs that are not trees. First, we prove the lower and upper bounds on the sum $avec(G, w) + avec(\overline{G}, w)$ when G is a tree and characterise the associated extremal conditions.

Corollary 2.5. *Let G be an n-vertex tree with $n \ge 4$. Let w be a normalised weighting of the edges of $K_n = G \cup \overline{G}$, whose restriction to $E(G)$ and $E(\overline{G})$ are normalised weightings of G and $\overline{G}$, respectively.*
(a) If $\overline{G}$ is also a tree, then $G = \overline{G} = P_4$. Thus,

$$\frac{9}{2} \le avec(G, w) + avec(\overline{G}, w) \le 6, \quad \textit{and} \quad \frac{81}{16} \le avec(G, w)avec(\overline{G}, w) \le 9.$$

(b) If $\overline{G}$ is connected but not a tree with $n \geq 5$, size $\overline{m}$ and edge-connectivity $\overline{\lambda}$, then

$$\frac{2(n-1)^2}{n(n-2)} \leq avec(G,w) + avec(\overline{G},w) \leq \frac{(n-1)[n+2(\overline{\lambda}-1)]}{2\overline{\lambda}}$$

and

$$0 \leq avec(G,w)avec(\overline{G},w) \leq \frac{\overline{m}(n-1)}{\overline{\lambda}}.$$

Both lower bounds can be achieved; the first is achievable if and only if G has $n-2$ end vertices. The upper bounds can be achieved when G is a single broom.

Proof:
(a) Since G is a tree, we have $m = |E(\overline{G})| = \binom{n}{2} - (n-1)$ which when simplified yields $\frac{(n-1)(n-2)}{2}$. Since $\overline{G}$ is also a tree with $n-1$ edges, solving for n in $\frac{(n-1)(n-2)}{2} = n-1$ yields that $n = 4$ or $n = 1$. It is easy to see that amongst all trees of order 4, the self-complementary path P_4 is the only tree whose complement is a tree. By Theorem 2.1, the lower bounds are attained with $n = 4$ and $t = 2$ and by Theorem 2.2, the upper bound is attained since $avec_{max}(P_4) = 3$.

(b) We first prove the lower bound. Since G is a tree, $\overline{m} = |E(\overline{G})| = \frac{(n-1)(n-2)}{2}$ by the same argument as in the proof of (a). Let t be the number of end vertices of G. It is easy to see that $2 \leq t \leq n-2$ since letting $t = n-1$ results in a $\overline{G}$ that is disconnected. Denote $\frac{(n-1)(n+t)}{nt}$ by $f(n,t)$. A straight forward minimisation shows that $f(n,t)$ is minimised when $t = n-2$. Hence, by Theorem 3.1,

$$\begin{aligned} avec(G,w) &\geq \frac{(n-1)(2n-2)}{n(n-2)} \\ &= \frac{2(n-1)^2}{n(n-2)}. \end{aligned}$$

Thus, since $avec(\overline{G},w) \geq 0$ by Theorem 2.4, it follows that

$$avec(G,w) + avec(\overline{G},w) \geq \frac{2(n-1)^2}{n(n-2)}.$$

To prove the upper bound, let $\overline{m}$ and $\overline{\lambda}$ be the size and edge-connectivity of

$\overline{G}$, respectively. We use the fact that $avec_{max}(G) = n-1$ from Corollary 2.3 and $avec_{max}(\overline{G}) = \frac{\overline{m}}{\overline{\lambda}}$ from Theorem 2.4. Recall that $\overline{m} = \binom{n}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$. It then follows that

$$\begin{aligned} avec(G,w) + avec(\overline{G},w) &\leq n-1+\frac{\overline{m}}{\overline{\lambda}} \\ &= (n-1)+\frac{(n-1)(n-2)}{2\overline{\lambda}} \\ &= \frac{(n-1)[n+2(\overline{\lambda}-1)]}{2\overline{\lambda}}. \end{aligned}$$

To see that the lower bounds can be achieved when G has $n-2$ end vertices, by Theorems 2.1 and 2.4 we can arrange the restrictions of w to $E(G)$ and $E(\overline{G})$ so that $avec(G,w) = \frac{2(n-1)^2}{n(n-2)}$ and $avec(\overline{G},w) = 0$.

By Theorem 2.1, again, if $t < n-2$ then $avec(G,w) > \frac{(n-1)(n+(n-2))}{n(n-2)} = \frac{2(n-1)^2}{n(n-2)}$, so the first lower bound can be achieved only when $t = n-2$. Since $\overline{G}$ is connected but not a tree, the second lower bound can always be achieved by allowing that $avec(\overline{G},w) = 0$, by Theorem 2.4.

To prove that the upper bounds are sharp, let P_3 be the path labelled with end vertices a and c, and center vertex b. Let B_n be the single broom obtained from P_3 by appending to c a set of $n-3$ end vertices labelled $A = v_1, v_2, \ldots, v_{n-3}$. Then B_n has $n-2$ end edges and an internal edge bc. Let $\overline{B}_n$ be the complement of B_n with size $\overline{m}$. Since c is a vertex of degree $n-2$ in B_n, $deg_{\overline{B}_n}(c) = 1$ so that the edge ac is a bridge in $\overline{B}_n$ and every other vertices in $\overline{B}_n$ is connected to at least $n-3$ vertices. Hence $\overline{B}_n$ is connected and contains a complete graph K_{n-3} as a subgraph.

To maximise $avec(B_n,w)$, we move all weights to any edge so that the eccentricity of every vertex becomes $n-1$. To maximise $avec(\overline{B}_n,w)$, we assign weight $\overline{m}$ to the edge ac (see Figure 2). Hence, $d_w(v,c) = \frac{(n-1)(n-2)}{2}$ for all $v \in V(G) - \{c\}$ implying that $e_w(v) = \frac{(n-1)(n-2)}{2}$ for all $v \in V(G)$. Thus $avec(\overline{B}_n,w) = \frac{(n-1)(n-2)}{2}$, and hence,

$$avec(G,w) + avec(\overline{G},w) = (n-1)+\frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2}.$$

This proof shows that equality can be attained, not only when $G = B_n$, but whenever $\overline{G}$ is connected and has a bridge. □

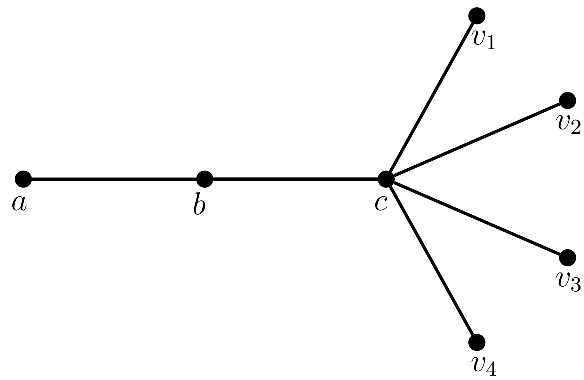

Figure 1: Single Broom, $B(3,4)$ with $n = 7$

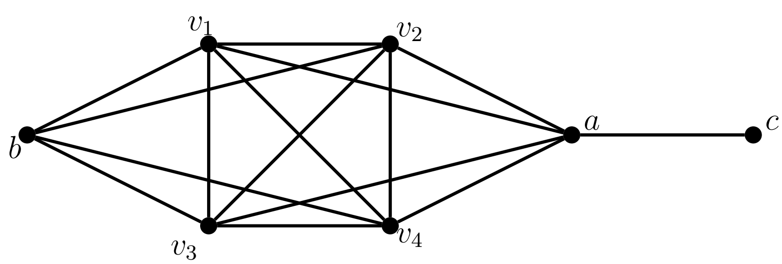

Figure 2: Complement of the Single Broom, $\overline{B}(3,4)$ with $n = 7$

Corollary 2.6. *Let G and $\overline{G}$ be connected non-tree graphs of edge-connectivities λ and $\overline{\lambda}$, respectively, on $n \geq 5$ vertices. Let w be as defined in the hypothesis of Corollary 2.5. Then*

$$0 \leq avec(G, w) + avec(\overline{G}, w) \leq \frac{m}{\lambda} + \frac{\overline{m}}{\overline{\lambda}},$$

and

$$0 \leq avec(G, w)avec(\overline{G}, w) \leq \frac{m\overline{m}}{\lambda\overline{\lambda}}.$$

All four bounds are sharp.

The proof of the corollary follows immediately from Theorem 2.4. □

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