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Pasting Lemmas for $b-$Metric Preserving and Related Functions

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Abstract

Previously ([7], [8]), we established some relations between $b-$metrics and metric-preserving functions. In this article, we give pasting lemmas for those functions.

1 Introduction

It is well known that if $g : [a, b] \to \mathbb{R}$ and $h : [b, c] \to \mathbb{R}$ are continuous and $g(b) = h(b)$, then the function $f : [a, c] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} g(x), & \text{if } x \in [a, b); \\ h(x), & \text{if } x \in [b, c] \end{cases}$$

is also continuous. This is usually called a pasting lemma. A version of a pasting lemma for metric-preserving functions is given by Doboš [6, p. 26] but there is no pasting lemma for $b-$metric-preserving and other related functions in the literature. So we provide such a lemma in this article. Let us recall the definitions and useful results on $b-$metrics and metric-preserving functions which were previously given in [7, 8] as follows:

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Definition 1.1. *Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a b−metric if it satisfies the following three conditions:*

(B1) *for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,*

(B2) *for all $x, y \in X$, $d(x, y) = d(y, x)$,*

(B3) *there exists $s \geq 1$ such that*

$$d(x, y) \leq s(d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in X.$$

Definition 1.2. *The function $f : [0, \infty) \to [0, \infty)$ is called metric preserving if for all metric spaces (X, d), $f \circ d$ is a metric on X.*

The concept of b-metrics appears in many articles (for example in [3, 5, 7, 11]). We also refer the reader to [1, 2, 4, 6, 10] for more information on metric-preserving functions and to [9] for applications in fixed point theory. In connection with metric-preserving functions and b−metrics, Khemaratchatakumthorn and Pongsriiam [7] define the following notions:

Definition 1.3. *Let $f : [0, \infty) \to [0, \infty)$. We say that*

(i) *f is b−metric-preserving if for all b−metric spaces (X, d), $f \circ d$ is a b−metric on X,*

(ii) *f is metric-b−metric-preserving if for all metric spaces (X, d), $f \circ d$ is a b−metric on X, and*

(iii) *f is b−metric-metric-preserving if for all b-metric spaces (X, d), $f \circ d$ is a metric on X.*

We let $\mathcal{M}$ be the set of all metric-preserving functions, $\mathcal{B}$ the set of all b−metric-preserving functions, $\mathcal{MB}$ the set of all metric-b−metric-preserving functions, and $\mathcal{BM}$ the set of all b−metric-metric-preserving functions.

From [7, Theorem 15 and Example 16] and [8, Theorem 3.1], we have the following theorem.

Theorem 1.4. [7, 8] *We have $\mathcal{BM} \subseteq \mathcal{M} \subseteq \mathcal{B} = \mathcal{MB}$, $\mathcal{M} \not\subseteq \mathcal{BM}$, and $\mathcal{B} \not\subseteq \mathcal{M}$.*

2 Preliminaries and Lemmas

In order to prove our main theorem, we need to recall some basic definitions and results in [7].

Let $f : [0, \infty) \to [0, \infty)$ and let $I \subseteq [0, \infty)$. Then f is said to be increasing on I if $f(x) \leq f(y)$ for all $x, y \in I$ satisfying $x < y$, and f is said to be strictly increasing on I if $f(x) < f(y)$ for all $x, y \in I$ satisfying $x < y$. The notion of decreasing or strictly decreasing functions is defined similarly.

The function f is said to be amenable if $f^{-1}(0) = \{0\}$, and f is said to be tightly bounded on $(0, \infty)$ if there is $v > 0$ such that $f(x) \in [v, 2v]$ for all $x > 0$. We say that f is concave if $f((1-t)x_1 + tx_2) \geq (1-t)f(x_1) + tf(x_2)$ for all $x_1, x_2 \in [0, \infty)$ and $t \in [0, 1]$. In addition, we say that f is quasi-subadditive if there exists $s \geq 1$ such that $f(a + b) \leq s(f(a) + f(b))$ for all $a, b \in [0, \infty)$.

Definition 2.1. *A triangle triplet is a triple (a, b, c) of nonnegative real numbers for which*

$$a \leq b + c, \ b \leq a + c, \ \text{and } c \leq a + b,$$

or, equivalently,

$$|a - b| \leq c \leq a + b.$$

Let $s \geq 1$ and $a, b, c \geq 0$. A triple (a, b, c) is an s-triangle triplet if

$$a \leq s(b + c), \ b \leq s(a + c), \ \text{and } c \leq s(a + b).$$

Let Δ and Δ_s be the sets of all triangle triplets and $s-$triangle triplets, respectively.

Next, we recall results concerning $b-$metrics and metric-preserving functions. Again, we let $f : [0, \infty) \to [0, \infty)$ throughout.

Lemma 2.2. [7] *$f \in \mathcal{BM}$ if and only if f is amenable and tightly bounded.*

Lemma 2.3. [7] *If $f \in \mathcal{B}$, then f is amenable and quasi-subadditive.*

Lemma 2.4. [7, 8] *Suppose f is amenable. Then $f \in \mathcal{B}$ if and only if there exists $s \geq 1$ such that $(f(a), f(b), f(c)) \in \Delta_s$ for all $(a, b, c) \in \Delta$.*

Lemma 2.5. [6, p. 12] *Let f be amenable. Then f is concave if and only if for all $t \geq 0$ and $x, y, z \in [0, t]$ if $x + t = y + z$, then $f(x) + f(t) \leq f(y) + f(z)$.*

3 Main Results

We begin with a pasting lemma for functions in $\mathcal{B}$. We see that a slight modification from those in $\mathcal{M}$ is enough. In addition, by Theorem 1.4, this also gives a pasting lemma for functions in $\mathcal{MB}$ as follows.

Theorem 3.1. (A pasting lemma for functions in $\mathcal{B}$ and $\mathcal{MB}$) *Let $g, h \in \mathcal{B}$, $r > 0$, and $g(r) = h(r)$. Define $f : [0, \infty) \to [0, \infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Suppose that g is increasing, concave, and

$$\forall x, y \in [r, \infty), |x - y| \leq r \Rightarrow |h(x) - h(y)| \leq g(|x - y|).$$

Then $f \in \mathcal{B}$.

Proof. Since $g, h \in \mathcal{B}$, by Lemmas 2.3 and 2.4 there are $s_1, s_2 \geq 1$ such that

$$(g(a), g(b), g(c)) \in \Delta_{s_1} \text{ and } (h(a), h(b), h(c)) \in \Delta_{s_2} \text{ for every } (a, b, c) \in \Delta.$$

Let $s = \max\{s_1, s_2\}$ and let $(a, b, c) \in \Delta$. Without loss of generality, assume $0 \leq a \leq b \leq c \leq a + b$. If $a, b, c \in [0, r)$, then $(f(a), f(b), f(c)) = (g(a), g(b), g(c)) \in \Delta_{s_1} \subseteq \Delta_s$. If $a, b, c \in [r, \infty)$, then $(f(a), f(b), f(c)) = (h(a), h(b), h(c)) \in \Delta_{s_2} \subseteq \Delta_s$. So it remains to consider the cases where a, b, c are not in the same interval. If $c \in [0, r)$, then $a, b \in [0, r)$ too. So there are two cases left to consider as follows.

Case 1. $a, b \in [0, r)$ and $c \in [r, \infty)$. Then

$$f(a) = g(a) \leq g(b) = f(b) \leq f(b) + f(c) \leq s(f(b) + f(c)). \tag{3.1}$$

Since $|r - c| = c - r \leq a + b - r < r + r - r = r$,

$$|g(r) - h(c)| = |h(r) - h(c)| \leq g(|r - c|) = g(c - r).$$

Then

$$-g(c - r) \leq g(r) - h(c) \leq g(c - r). \tag{3.2}$$

Then $g(r) - g(c - r) \leq h(c)$. Since $c \leq a + b$, $c - r \leq a + b - r \leq a$. Since g is increasing, $g(c - r) \leq g(a)$ and therefore

$$\begin{aligned} f(b) = g(b) \leq g(r) &\leq g(r) + g(a) - g(c - r) = (g(r) - g(c - r)) + g(a) \\ &\leq h(c) + g(a) = f(c) + f(a) \\ &\leq s(f(c) + f(a)). \end{aligned} \tag{3.3}$$

Since g is concave, we can substitute $t = r$, $x = a + b - r$, $y = a$, $z = b$ in Lemma 2.5 to obtain $g(a + b - r) + g(r) \leq g(a) + g(b)$. By (3.2), $h(c) \leq g(r) + g(c - r)$. Therefore

$$\begin{aligned} f(c) = h(c) \leq g(r) + g(c-r) &\leq g(r) + g(a+b-r) \\ &\leq g(a) + g(b) = f(a) + f(b) \\ &\leq s(f(a) + f(b)). \end{aligned} \tag{3.4}$$

From (3.1), (3.3), and (3.4), we conclude that $(f(a), f(b), f(c)) \in \Delta_s$.

Case 2. $a \in [0, r)$ and $b, c \in [r, \infty)$. Since $r \leq b + c$, $b \leq c \leq c + r$, and $c \leq a + b \leq r + b$, we see that $(r, b, c) \in \Delta$. Then $(h(r), h(b), h(c)) \in \Delta_{s_2}$. Therefore

$$\begin{aligned} f(a) = g(a) \leq g(r) = h(r) &\leq s_2(h(b) + h(c)) \\ &\leq s(h(b) + h(c)) = s(f(b) + f(c)). \end{aligned} \tag{3.5}$$

Since $|b - c| = c - b \leq r$, $|h(b) - h(c)| \leq g(|b - c|) = g(c - b)$. Then $-g(c - b) \leq h(b) - h(c) \leq g(c - b)$ and therefore

$$\begin{aligned} f(b) = h(b) &\leq g(c-b) + h(c) \leq g(a) + h(c) \\ &= f(a) + f(c) \leq s(f(a) + f(c)), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} f(c) = h(c) &\leq g(c-b) + h(b) \leq g(a) + h(b) \\ &= f(a) + f(b) \leq s(f(a) + f(b)). \end{aligned} \tag{3.7}$$

From (3.5), (3.6), and (3.7), we obtain $(f(a), f(b), f(c)) \in \Delta_s$. In all cases, $(f(a), f(b), f(c))$ is in Δ_s, as required. Consequently, $f \in \mathcal{B}$ and the proof is complete. □

It remains to consider functions in $\mathcal{BM}$.

Theorem 3.2. (A pasting lemma for functions in $\mathcal{BM}$) *Let $g, h \in \mathcal{BM}$, $r > 0$, and $g(r) = h(r)$. Define $f : [0, \infty) \to [0, \infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Let $A = \sup_{x \in (0,\infty)} f(x)$ and $B = \inf_{x \in (0,\infty)} f(x)$. Then

(i) $A = \max\left\{\sup_{x\in(0,r)} g(x), \sup_{x\in[r,\infty)} h(x)\right\}$ *and*
$B = \min\left\{\inf_{x\in(0,r)} g(x), \inf_{x\in[r,\infty)} h(x)\right\}$,

and the following statements are equivalent

(ii) $f \in \mathcal{BM}$

(iii) $A \le 2B$

(iv) $\sup_{x\in(0,r)} g(x) \le 2\inf_{x\in[r,\infty)} h(x)$ *and* $\sup_{x\in[r,\infty)} h(x) \le 2\inf_{x\in(0,r)} g(x)$.

Proof. By Lemma 2.2, it follows that $\inf_{x\in(0,r)} g(x)$, $\sup_{x\in(0,r)} g(x)$, $\inf_{x\in[r,\infty)} h(x)$, and $\sup_{x\in[r,\infty)} h(x)$ exist. Then $\sup_{x\in(0,\infty)} f(x)$ and $\inf_{x\in(0,\infty)} f(x)$ exist, and the statement (i) is obvious. Next, assume that (ii) holds. By Lemma 2.2, there exists $v > 0$ such that $v \le f(x) \le 2v$ for all $x \in (0,\infty)$. Then $v \le B \le A \le 2v$. Therefore $2B \ge 2v \ge A$, which proves (iii). Now, suppose (iii) holds. Then for each $x \in (0,\infty)$, we have

$$B = \inf_{x\in(0,\infty)} f(x) \le f(x) \le \sup_{x\in(0,\infty)} f(x) = A \le 2B.$$

So f is tightly bounded. By Lemma 2.2, g and h are amenable. So f is also amenable. Applying Lemma 2.2 again, we obtain $f \in \mathcal{BM}$, as required. Hence (ii) and (iii) are equivalent. Next, we prove (iii) implies (iv). We have

$$\begin{aligned}\sup_{x\in(0,r)} g(x) &\le \max\left\{\sup_{x\in(0,r)} g(x), \sup_{x\in[r,\infty)} h(x)\right\} = A \le 2B \\ &= 2\min\left\{\inf_{x\in(0,r)} g(x), \inf_{x\in[r,\infty)} h(x)\right\} \le 2\inf_{x\in[r,\infty)} h(x),\end{aligned}$$

and, similarly,

$$\sup_{x\in[r,\infty)} h(x) \le A \le 2B \le 2\inf_{x\in(0,r)} g(x),$$

which proves (iv). Finally, assume that (iv) holds.

Case 1. $\sup_{x\in(0,r)} g(x) \ge \sup_{x\in[r,\infty)} h(x)$. Then $A = \sup_{x\in(0,r)} g(x)$. Since $g \in \mathcal{BM}$, we can use an argument similar to the prove of (ii)⇒(iii) to obtain

$$\sup_{x\in(0,r)} g(x) \le 2\inf_{x\in(0,r)} g(x).$$

By (iv),

$$\sup_{x\in(0,r)} g(x) \le 2\inf_{x\in[r,\infty)} h(x).$$

Therefore

$$\begin{aligned} A &\le \min\left\{2\inf_{x\in(0,r)} g(x), 2\inf_{x\in[r,\infty)} h(x)\right\} \\ &= 2\min\left\{\inf_{x\in(0,r)} g(x), \inf_{x\in[r,\infty)} h(x)\right\} = 2B. \end{aligned}$$

Case 2. $\sup_{x\in(0,r)} g(x) < \sup_{x\in[r,\infty)} h(x)$. Then $A = \sup_{x\in[r,\infty)} h(x)$. Similar to Case 1, since $h \in \mathcal{BM}$, we have $\sup_{x\in[r,\infty)} h(x) \le 2\inf_{x\in[r,\infty)} h(x)$. By (iv), $\sup_{x\in[r,\infty)} h(x) \le 2\inf_{x\in(0,r)} g(x)$. These imply $A \le 2B$.

In all cases, $A \le 2B$, which proves (iii). So the proof is complete. □

Pasting lemmas for other functions will be given in a future article.

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