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Biharmonic problem with indefinite asymptotically linear nonlinearity

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Abstract

In this paper, boundary fourth order semilinear elliptic problems involving nonnegative weight functions are investigated with indefinite asymptotically linear nonlinearities. Using the variational method, we prove the existence of nontrivial solutions without use of the Ambrosetti-Rabionovitz condition or any one of its replacements. When the nonlinearities are superlinear at infinity, a suitable condition is added in order to use the same techniques to prove the existence of solutions.

1 Introduction and main results

Let Ω be a regular bounded open domain in $\mathbb{R}^N$, $N \geq 2$. In this paper, we study the solvability of the following nonlinear elliptic equation

$$\begin{cases} \Delta^2 v - \operatorname{div}(\sigma(y)\nabla v) = h(y,v) & \text{in } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta^2 = \Delta(\Delta)$ is the bi-Laplace operator, $\sigma(y)$ is a nonnegative weight function and $h(y,t)$ is an indefinite nonlinearity that is a sign-changing function.

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This equation appears in physics as in describing the traveling waves in suspension bridge [16] as well as the static deflection of an elastic plate in fluid [5].
Weighted fourth partial differential equations arise also in micro-electro-mechanical systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, among others (see for example [3, 8, 10, 14, 20]).

We suppose that the function $h(y,t)$ is asymptotically linear at infinity; that is,

$$\lim_{|t|\to\infty} \frac{h(y,t)}{t} = \alpha \in (0,\infty). \tag{1.2}$$

Second order partial differential equations with positive asymptotically linear nonlinearities have been extensively studied. In [1, 11, 13, 15, 18, 19, 21], the nonlinearities were of the form $h(y,v) = \lambda g(v)$ with $g(v)$ positive, increasing and convex smooth functions satisfying

$$\lim_{v\to\infty} \frac{g(v)}{v} = a < \infty.$$

With the same type of nonlinearities and fourth order elliptic differential equations, we refer the reader to [1, 2, 6, 7, 22]. In [25], different conditions were assumed for the asymptotically nonlinearities and different type of results were proven under the assumptions:

(F1) $h(y,t)$ is continuous on $\overline{\Omega}\times\mathbb{R}$, non-negative and $h(y,t)\equiv 0$ for $t\le 0$ and $x\in\overline{\Omega}$.

(F2) $\lim_{t\to 0} \frac{h(y,t)}{t} = p(y)$ and $\lim_{t\to\infty} \frac{h(y,t)}{t} = \ell < \infty$ uniformly in y such that $0 \le p(y) \in L^\infty(\Omega)$, $\|p(y)\|_\infty < \lambda_1$, where $\lambda_1 > 0$ denotes the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

(F3) $\frac{h(y,t)}{t}$ is a nondecreasing function on $t>0$.

Later, in [23, 24], biharmonic problems have been investigated with these second type of conditions. In this paper, we extend these results to a weighted problem, where the weight is not positive. As a result, the norms will not be equivalent to those in [24]. In addition, the asymptotic linear nonlinearities change sign.

We suppose that:

(V1) $h(y,t)$ is in $C(\overline{\Omega}\times\mathbb{R})$, $h(y,t)t \geq 0$ for all $(y,t)\in\Omega\times\mathbb{R}$ and $h(y,0)=0$.

(V2) $\lim\limits_{t\to 0}\dfrac{h(y,t)}{t} < \nu_1$, uniformly for $y\in\Omega$, where ν_1 is the first eigenvalue associated to operator $\Delta^2 - div(\sigma(y)\nabla)$ with Navier boundary condition.

(V3) $\lim\limits_{|t|\to\infty}\dfrac{h(y,t)}{t} = \alpha$, uniformly for $y\in\Omega$ and $0<\alpha\leq\infty$.

(V4) $\lim\limits_{t\to\infty}\dfrac{h(y,t)}{t^{q-1}} = 0$, uniformly in $y\in\Omega$ for some $q\in(2,2_*)$, here and thereafter

$$2_* = \begin{cases} \frac{2N}{N-4} & if \quad N>4 \\ \infty & if \quad N\leq 4. \end{cases}$$

(V5) The function $\dfrac{h(y,t)}{t}$ is nondecreasing with respect to t in $(0,\infty)$, for a.e. $y\in\Omega$.

(V6) The function $\dfrac{h(y,t)}{t}$ is nonincreasing with respect to t in $(-\infty,0)$, for a.e. $y\in\Omega$.

We shall prove the following two theorems:

Theorem 1.1. *Assume that* $(V1)$, $(V2)$ *and* $(V3)$ *are satisfied and* $\alpha\in(0,\infty)$. *Then,*

(i) If $0<\alpha<\nu_1$ *and the condition* $(V5)$ *(resp.* $(V6)$*) holds, then problem* (1.1) *does not have positive (resp. negative) solution.*

(ii) If $\alpha>\nu_1$, *then problem* (1.1) *has a nontrivial solution.*

(iii) If $\alpha=\nu_1$ *and* $(V5)$ *holds (resp.* $(V6)$ *holds), then problem* (1.1) *has a positive (resp. negative) solution* v *if and only if there exists a constant* $c_0>0$ *(resp.* $c_0<0$*) such that* $v=c_0v_1$ *and* $h(y,v)=\nu_1 v$, *where* v_1 *a positive eigenfunction associated to* ν_1.

Theorem 1.2. *Suppose that* $(V1)$, $(V2)$, $(V3)$ *and* $(V4)$ *are satisfied and* $\alpha=\infty$. *Then problem (1.1) has a positive solution (resp. negative solution) if* $(V5)$ *(resp.* $(V6)$*) holds.*

In the sequel, C is throughout used as a positive constant.

2 Variational Preliminaries

Let Ω be a regular bounded open domain in $\mathbb{R}^N$, $N \geq 2$. When $1 \leq p < \infty$ and $v \in L^p(\Omega)$, the L^p-norm of v is

$$\|v\|_p = \Big(\int_\Omega |v|^p \ dy\Big)^{\frac{1}{p}}.$$

Let $\sigma(y) \in L^1(\Omega)$ be a nonnegative function and set

$$\chi = \{u \in H^2(\Omega) \cap H_0^1(\Omega); \quad \int_\Omega [\Delta^2 u + \sigma(y)|\nabla u|^2] \ dy < \infty\}. \tag{2.3}$$

On the space χ, we have the inner product

$$< u, v >= \int_\Omega [\Delta u \Delta v + \sigma(y) \nabla u . \nabla v] \ dy.$$

The norm induced by the above inner product is

$$\|u\| = \Big(\int_\Omega [\Delta^2 u + \sigma(y)|\nabla u|^2] \ dy\Big)^{\frac{1}{2}}$$

Let $\psi : \chi \to \mathbb{R}$ be the C^1 functional given by

$$\psi(v) = \frac{1}{2}\int_\Omega [\Delta^2 v + \sigma(y)|\nabla v|^2] \ dy - \int_\Omega H(y, v) dy, \tag{2.4}$$

where

$$H(y, t) = \int_0^t h(y, s) ds.$$

We consider the following definition of a solution of problem (1.1).

Definition 2.1. *A function $v \in \chi$ is called a solution of the equation* (1.1) *if*

$$\int_\Omega [\Delta v \Delta \phi + \sigma(y) \nabla v \nabla \phi] \ dy = \int_\Omega h(y, v) \phi dy, \quad \forall \phi \in \chi. \tag{2.5}$$

So, in order to prove that problem (1.1) has a nontrivial solution, we will prove that the functional ψ has a nontrivial critical point.
To do this, we are going to use the Mountain Pass Theorem introduced by Ambrosetti and Rabionowitz in [4].

Definition 2.2. *Let χ be a real Banach space and $\psi \in C^1(\chi, \mathbb{R})$. We say that ψ satisfies the Palais-Smale $(PS)_d$ condition at level d, $d \in \mathbb{R}$, if any sequence $\{\upsilon_n\} \subset X$ satisfying*

$$\psi(\upsilon_n) \to d \quad in \quad \mathbb{R},$$

and

$$\psi'(\upsilon_n) \to \quad in \quad \chi',$$

where χ' is the dual space of χ, the sequence $\{\upsilon_n\}$ has a convergent subsequence.

Theorem 2.3. *(Mountain Pass Theorem [4]) Let χ be a Banach space and $\psi \in C^1(\chi, \mathbb{R})$ be a functional such that $\psi(0) = 0$ and satisfies the following conditions:*

(i) There exist δ, $\tau > 0$ such that $\psi(w) \geq \tau$, for all $w \in \partial B(0, \delta)$;

(ii) There exists $w_1 \in \chi$ such that $\|w_1\| > \delta$ and $\psi(w_1) < 0$;

(iii) ψ satisfies the $(PS)_d$ condition, at any level $d \in \mathbb{R}$.

Then the functional ψ has a critical point $\upsilon \in \chi$ such that $\psi(\upsilon) \geq \tau > 0$.

In the proof of the second geometric property for the functional ψ introduced by (2.4), we will use the function υ_1, where υ_1 denotes a normalised positive eigenfunction associated to the first eigenvalue ν_1; that is,

$$\begin{cases} \Delta^2 \upsilon_1 - \operatorname{div}(\sigma(y)\nabla \upsilon_1) & = & \nu_1 \upsilon_1 & \text{in} & \Omega \\ \upsilon_1 = \Delta \upsilon_1 & = & 0 & \text{on} & \partial\Omega \\ \|\upsilon_1\|_2 & = & 1. & & \end{cases} \tag{2.6}$$

At the end of this section, let us recall that weighted Sobolev spaces have been developed and an embedding theory has been studied in [9, 12].
For the space χ, the the embedding $\chi \hookrightarrow H^2(\Omega)$ is continuous; i.e., there exists a constant C such that $\|w\|_{H^2} \leq \|w\|$ for all $w \in \chi$, where $\|w\|_{H^2}$ is the standard norm on $H^2(\Omega)$. Also, the embedding

$$\chi \hookrightarrow L^q(\Omega)$$

is continuous for $q \in [2, 2_*]$ and compact if $q \in [2, 2_*)$. Finally, the space $(\chi, \|.\|)$ is a Hilbert space.

3 Proof of Theorem 1.1

To prove Theorem 1.1, we begin with some elementary results.

Lemma 3.1. *Assume that* $(V1)$-$(V3)$ *hold and* $\alpha \in (0, \infty)$*. Then there exist* $\tau, \delta > 0$ *such that*

$$\psi(v) > \tau, \quad \forall v \in \partial B(0, \delta).$$

Proof. From $(V2)$, there exist $\epsilon_0 \in (0, 1)$ and ρ_0 such that

$$H(y, t) \leq \frac{1}{2}\nu_1(1 - \epsilon_0)t^2, \quad \forall \quad |t| \leq \rho_0 \tag{3.7}$$

From $(V3)$, for any $1 \leq q \leq \frac{N+4}{N-4}$, there exists a constant $C > 0$ such that

$$H(y, t) \leq C|t|^{q+1}, \quad \forall \quad |t| \geq \rho_0. \tag{3.8}$$

Then

$$H(y, t) \leq \frac{1}{2}\nu_1(1 - \epsilon_0)t^2 + C|t|^{q+1}, \quad \forall \quad t \in \mathbb{R}.$$

Therefore

$$\psi(v) \leq \frac{1}{2}\|v\|^2 - \frac{1}{2}\nu_1(1 - \epsilon_0)\|v\|_2^2 - C\|v\|_{q+1}^{q+1}.$$

As $\nu_1\|v\|_2^2 \leq \|v\|^2$ and by using the continuous embedding result, we get

$$\psi(v) \geq \frac{1}{2}\epsilon_0\|v\|^2 - C_1\|v\|^{q+1}. \tag{3.9}$$

Now, we have to choose $\|v\| = \delta > 0$ and small enough in order to get $\psi(v) \geq \tau > 0$, since $2 < q + 1$. □

Next, we prove the second geometry property for the energy ψ.

Lemma 3.2. *Assume* $(V1)$ *and* $(V3)$ *and suppose that* $\nu_1 < \alpha < \infty$*. Then there exists* $w_1 \in \chi$ *such that* $\|w_1\| > \delta$ *and* $\psi(w_1) < 0$*.*

Proof. Let $t > 0$ and consider

$$\psi(tv_1) = \frac{t^2}{2}\int_\Omega [\Delta^2 v_1 + \sigma(y)|\nabla v_1|^2] \, dy - \int_\Omega H(x, tv_1) \, dy.$$

From (2.6), we have

$$\frac{\psi(t\upsilon_1)}{t^2} = \frac{\lambda_1}{2}\|\upsilon_1\|_2^2 - \int_\Omega \frac{H(y,t\upsilon_1)}{t^2}\,dy.$$

By $(V1)$, the function $H(y,t) \geq 0$ and from Fatou's Lemma and (2.6), we obtain

$$\lim_{t\to\infty} \frac{\psi(t\upsilon_1)}{t^2} \leq \frac{\nu_1}{2} - \int_\Omega \lim_{t\to\infty} \frac{H(y,t\upsilon_1)}{(t\upsilon_1)^2}\upsilon_1^2\,dy. \tag{3.10}$$

So, from $(V3)$, we get

$$\lim_{t\to\infty} \frac{\psi(t\upsilon_1)}{t^2} \leq \frac{\nu_1}{2} - \frac{\alpha}{2}\int_\Omega \upsilon_1^2\,dy;$$

that is,

$$\lim_{t\to\infty} \frac{\psi(t\upsilon_1)}{t^2} \leq \frac{\nu_1}{2} - \frac{\alpha}{2} < 0.$$

The proof of the lemma is complete. □

Proof of Theorem 1.1 (i) Assume that $(V5)$ holds, $0 < \alpha < \nu_1$ and problem (1.1) has a positive solution $v \in \chi$. By taking v as a test function in (2.5) and from conditions $(V1)$, $(V3)$ and $(V5)$, we get

$$\int_\Omega \sigma(y)|\nabla v|^2 dy = \int_\Omega h(y,v)v\,dy \leq \int_\Omega \alpha v^2 dy, \tag{3.11}$$

and so $\nu_1 \leq \alpha$. This gives a contradiction.
If we suppose that condition $(V6)$ holds and $v \in \chi$ is a negative solution of problem (1.1), we get the same formula (3.11) and the same contradiction.

(ii) In this part, we suppose that $\nu_1 < \alpha$. By using Theorem 2.3 and Lemmas 3.1 and 3.2, we have only to prove the compactness condition. Let $\{v_n\}$ be a $(PS)_d$ sequence of ψ, $d \in \mathbb{R}$. We have

$$\psi(v_n) = \frac{1}{2}\|v_n\|^2 - \int_\Omega H(y,v_n)\,dy \to d, \tag{3.12}$$

for some $d \in \mathbb{R}$ and

$$\|\psi'(v_n)\|_* \to 0 \quad \text{in} \quad \chi'. \tag{3.13}$$

In order to prove that $\{v_n\}$ is relatively compact, we prove that $\{v_n\}$ is bounded in χ and then it has a convergent subsequence. We begin by the second part in order to use some elementary results of the proof later.

Step 1. Suppose that $\{v_n\}$ is bounded in χ. By the compact embedding result, there exists $v \in \chi$ such that, up to subsequence,

$$v_n \rightharpoonup v, \quad \text{weakly in} \quad \chi,$$
$$v_n \to v, \quad \text{in} \quad L^2(\Omega)$$
and
$$v_n(y) \to v(y), \quad \text{a.e in} \quad \Omega.$$

From (3.13), it follows that

$$< \psi'(v_n), v_n >= \|v_n\|^2 - \int_\Omega h(y, v_n)u_n dy \to 0 \tag{3.14}$$

and, more generally,

$$\int_\Omega \sigma(y)\ \nabla v_n.\nabla\phi\ dy - \int_\Omega h(y, v_n)\phi dy \to 0, \quad \forall \phi \in \chi. \tag{3.15}$$

So

$$\Delta^2 v_n - \text{div}(\sigma(y)\nabla v_n) - h(y, v_n) \to 0 \quad \text{in} \quad \chi', \tag{3.16}$$

By exploiting $(V1)$, $(V2)$ and $(V3)$, we get $h(y, v_n) \to h(y, v)$ in $L^2(\Omega)$. The dual space of $L^2(\Omega)$ is itself and we have $L^2(\Omega) \hookrightarrow \chi'$. Then

$$\Delta^2 v_n - \text{div}(\sigma(y)\nabla v_n) \to h(y, v) \quad \text{in} \quad \chi'. \tag{3.17}$$

As in [17], we prove that the operator $L = \Delta^2 - \text{div}(\sigma(y)\nabla)$ is an isomorphism from χ, with the condition $v = \Delta v = 0$ on $\partial\Omega$, into the space χ' and so

$$v_n \to L^{-1}(h(y, v)) \quad \text{in} \quad \chi. \tag{3.18}$$

Step 2. Here we prove that the sequence $\{v_n\}$ is bounded in χ.
We argue by contradiction. Suppose that the sequence $\{v_n\}$ is not bounded. Then, up to subsequence,

$$\|v_n\| \to \infty \quad \text{as} \quad n \to \infty.$$

Without loss of generality, we suppose that $v_n \neq 0$ a.e. in Ω. Let

$$z_n = \frac{v_n}{\|v_n\|}, \quad \kappa_n = \|v_n\|. \tag{3.19}$$

Since the sequence $\{z_n\}$ is bounded in the Banach space χ, there exist a function $z \in \chi$ and a subsequence, which we still denote by $\{z_n\}$, such that $z_n \rightharpoonup z$ in χ,

$$z_n \to z \quad \text{in} \quad L^2(\Omega)$$

and $z_n(y) \to z(y)$, for y a.e in Ω.
By using conditions $(V2)$ and $(V3)$, we get a constant $C > 0$ such that

$$\frac{h(y,t)}{t} \leq C, \quad \forall (y,t) \in \Omega \times \mathbb{R}^\star \tag{3.20}$$

and then we get

$$\int_\Omega \frac{h(y,v_n)}{\|v_n\|^2} v_n dy = \int_\Omega \frac{h(y,v_n)}{v_n} z_n^2 dy \leq C \int_\Omega z_n^2 dy. \tag{3.21}$$

From (3.14) and (3.21), we obtain

$$z \not\equiv 0.$$

By formula (3.15), we get

$$\int_\Omega [\Delta^2 v_n + \sigma(y)\nabla z_n.\nabla\phi] \, dy - \int_\Omega \frac{h(y,v_n)}{v_n} z_n \phi dy \to 0, \quad \forall \phi \in \chi. \tag{3.22}$$

Referring to step 1, we have

$$\int_\Omega [\Delta^2 v_n + \sigma(y)\nabla z_n.\nabla\phi] \, dy \to \int_\Omega \sigma(y)\nabla z.\nabla\phi \, dy, \quad \forall \phi \in \chi. \tag{3.23}$$

Since t $v_n(y) = \|v_n\| z_n(y)$, $\lim_{n\to\infty} v_n(y) = \pm\infty$, whenever $z(y) \neq 0$. Set

$$h_n(y) = \begin{cases} \frac{h(y,v_n(y))}{v_n(y)} & if \quad v_n(y) \neq 0 \\ 0 & if \quad v_n(y) = 0. \end{cases}$$

From (3.20), it follows that the sequence $\{h_n\}$ is bounded on Ω and so, up to subsequence, it is weakly star convergent in $L^\infty(\Omega)$ to a function h.
The function $h(y) = \alpha$ a.e in Ω since we have $v_n(y) \neq 0$ a.e. in Ω and by using $(V3)$. Then

$$\int_\Omega \frac{h(y,v_n)}{v_n} z_n \phi dy = \int_\Omega h_n(y) z_n \phi dy;$$

that is,

$$\int_\Omega \frac{h(y,v_n)}{v_n} z_n \phi dy \to \alpha \int_\Omega z\phi dy \ \ \forall \phi \in \chi. \tag{3.24}$$

By using (3.22), (3.23) and (3.24), we get

$$\begin{cases} -\text{div}(\sigma(y)\nabla z) &= \alpha z \quad \text{in} \quad \Omega \\ z &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.25}$$

and so $z = cv_1$ and $\alpha = \nu_1$ which a contradiction. The sequence $\{v_n\}$ is bounded in χ and so, by step 1, it is relatively compact.

(iii) In this case, $\alpha = \nu_1$. Suppose that $(V5)$ holds and $v \in \chi$ is a positive solution of problem (1.1). Considering v_1 as a test function in (1.2), we get

$$\int_\Omega \sigma(y)[\Delta v \Delta v_1 + \sigma(y)\nabla v.\nabla v_1]\ dy = \int_\Omega h(y,v)\phi_1 dy. \tag{3.26}$$

If we take v as a test function for the equation (2.6), then

$$\int_\Omega [\Delta v \Delta v_1 + \sigma(y)\nabla v.\nabla v_1]\ dy = \alpha \int_\Omega v v_1 dy. \tag{3.27}$$

From (3.26) and (3.27), it follows that

$$\int_\Omega (h(y,v) - \alpha v)v_1 dy = 0.$$

From $(V3)$ and $(V5)$ and since $v_1 > 0$, we get $h(y,v) = \alpha v$ a.e. in Ω. That is, $h(y,v) = \nu_1 v$ and then v is an eigenfunction associated to the simple eigenvalue ν_1. Conversely, suppose that $\alpha = \nu_1$ and $v = c_0 v_1$ for some constant $c_0 \neq 0$ and the function $h(y,t)$ satisfies $h(y,v) = \nu_1 v$. Then v is a solution of problem (2.6) and then of (1.1) in this particular case.
We can construct a similar proof when we assume condition $(V6)$ and $v \in \chi$ is a negative solution for problem (1.1). □

4 Proof of Theorem 1.2

First, we prove the geometric properties for the functional ψ.

Lemma 4.1. *Suppose that $(V1)$, $(V2)$, $(V3)$, $(V4)$ hold and $\alpha = \infty$. Then there exist δ, $\tau > 0$ such that*

$$\psi(v) > \tau, \quad \forall v \in \partial B(0,\delta).$$

Proof. From $(V1)$ and $(V4)$, there exist $C > 0$ and $t_0 \geq 1$ such that

$$|h(y,t)| \leq C|t|^{q-1}, \quad \forall |t| \geq t_0.$$

By using $(V1)$ and $(V2)$, there exist $\epsilon_0 \in (0,1)$ and $t_1 > 0$ such that

$$|h(y,t)| \leq \nu_1(1-\epsilon_0)t, \quad \forall |t| \leq t_1.$$

Since the function $h(y,t)$ is continuous, there exists a constant $C_1 > 0$ such that

$$|h(y,t)| \leq \nu_1(1-\epsilon_0)t + C_1|t|^{q-1}, \quad \forall t \in \mathbb{R}.$$

So

$$H(y,t) \leq \frac{1}{2}\nu_1(1-\epsilon_0)t^2 + C_1|t|^r, \quad \forall (y,t) \in \Omega \times \mathbb{R}. \tag{4.28}$$

We have

$$\psi(v) \geq \frac{1}{2}\|v\|^2 - \frac{1}{2}\nu_1(1-\epsilon_0)\|v\|_2^2 - C_1\|v\|_q^q.$$

By the continuous embedding result and the equation (2.6), we get

$$\psi(v) \geq \frac{1}{2}\|v\|^2 - \frac{1}{2}(1-\epsilon_0)\|v\|^2 - C\|v\|^r,$$

for some positive constant C.
We then have

$$\psi(v) \geq \frac{1}{2}\epsilon_0\|v\|^2 - C\|v\|^r, \tag{4.29}$$

and so we can choose $\|v\| = \delta > 0$ small enough and get $\psi(v) \geq \tau$ for some $\tau > 0$ since $2 < r$. □

In the next lemma, we prove the second geometric property for the functional ψ.

Lemma 4.2. *Suppose that* $(V1)$, $(V3)$, $(V4)$ *and* $(V5)$ *hold and* $\alpha = \infty$. *Then,* $\psi(tv_1) \to -\infty$ *as* $t \to \infty$.

Proof. Let $t > 0$. From (2.6) and the regularity of the function v_1, we have

$$\psi(tv_1) = \frac{t^2}{2}\|v_1\|^2 - \int_\Omega H(y,tv_1)\,dy,$$

and so

$$\psi(tv_1) = \frac{t^2}{2}\nu_1 - \int_\Omega H(y,tv_1)\,dy. \tag{4.30}$$

From $(V5)$, we have

$$0 \leq 2H(y,t) \leq th(y,t), \quad \forall (y,t) \in \Omega \times \mathbb{R}^+. \tag{4.31}$$

Then, the function $\dfrac{H(y,t)}{t^2}$ is nondecreasing with respect to $t > 0$ and it follows from $(V3)$ that

$$\lim_{t\to\infty} \frac{H(y,t)}{t^2} = \infty.$$

As a consequence, there exist two constants $C_1 > \lambda_1$ and $C_2 > 0$ such that

$$H(y,t) \geq \frac{C_1}{2}t^2 + C_2, \quad \forall t > 0.$$

From (4.30), it follows that

$$\psi(tv_1) \leq \frac{t^2}{2}\nu_1 - \frac{b}{2}t^2\|v_1\|_2^2 - C|\Omega|, \forall t > 0.$$

Then

$$\psi(tv_1) \leq t^2\frac{\nu_1 - b}{2} < 0, \quad \forall t > 0.$$

So, Lemma 4.2 follows. □

Lemma 4.3. *Assume that* $(V5)$ *holds and let* $\{v_n\}$ *be a sequence in* χ *satisfying*

$$< \psi'(v_n), v_n >\to 0.$$

Then, up to a subsequence, we have

$$\psi(tv_n) \leq \frac{1+t^2}{2n} + \psi(v_n), \quad \forall t > 0. \tag{4.32}$$

Proof of Theorem 1.2 We suppose $(V1) - (V5)$ hold and $\alpha = \infty$. From Lemmas 4.1 and 4.2, we have to prove that the functional ψ satisfies the compactness condition.
Let $\{v_n\}$ be a $(PS)_d$ sequence in χ at a fixed level $d \in \mathbb{R}$. That is, the sequence satisfies (3.12) and (3.13). If we prove that the sequence $\{v_n\}$ is bounded in χ, then it will be relatively compact as in step 1 of the proof of Theorem 1.1 (ii).
Suppose that $\{v_n\}$ is not bounded and then, up to a subsequence, $\|v_n\| \to \infty$. Consider the two sequences

$$z_n = \frac{v_n}{c\|v_n\|} \quad \text{and} \quad s_n = c\|v_n\|, \tag{4.33}$$

where $c > 0$.
The sequence $\{z_n\}$ is bounded in χ. Thus there exists $z \in \chi$ such that, up to a subsequence,

$$z_n \rightharpoonup z \qquad \text{in} \ \ \chi \ \ \text{as} \ \ n \to \infty,$$

$$z_n \to z \qquad \text{in} \ \ L^2(\Omega) \ \ \text{as} \ \ n \to \infty,$$

$$z_n(y) \to z(y) \qquad \text{for a.e} \ \ y \ \ \text{in} \ \ \Omega.$$

We have

$$z_n^+(y) \to z^+(y) \qquad \text{a.e. in} \quad \Omega, \tag{4.34}$$

and

$$z_n^+ \to z^+ \quad \text{in} \quad L^2(\Omega). \tag{4.35}$$

The convergence in (4.35) follows from the fact that, for all $v \in L^2(\Omega)$, we can write $v^+ = sup(v, 0)$ or $v^+ = \dfrac{v + |v|}{2}\cdot$
Let

$$\Omega_+ = \{y \in \Omega; w^+(y) > 0\}.$$

As $z_n(y) = c\|v_n\|z_n(y)$ and $c > 0$, we get $v_n^+(y) \to \infty$ a.e.in Ω_+. From (4.34) and since $\alpha = \infty$, we obtain that for all $M > 0$, there exists n_0 such that for all $n \geq n_0$

$$\frac{h(y, v_n^+(y))}{v_n^+(y)}(z_n^+(y))^2 \geq M(z^+(y))^2. \tag{4.36}$$

Since

$$\langle \psi'(v_n), v_n\rangle = \|v_n\|^2 - \int_\Omega h(y, v_n)v_n \ dy \to 0,$$

multiplying it by $\frac{1}{c^2\|v\|^2}$ we get

$$\frac{1}{c^2} - \int_\Omega \frac{h(y, v_n)}{v_n}(z_n)^2 \ dy \to 0. \tag{4.37}$$

But

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega \frac{h(y, v_n)}{v_n}(z_n)^2 \ dy &\geq \lim_{n\to\infty} \int_{\Omega_+} \frac{h(y, v_n)}{v_n}(z_n)^2 \ dy \\ &\geq \lim_{n\to\infty} \int_{\Omega_+} \frac{h(y, v_n^+)}{v_n^+}(z_n^+)^2 \ dy \\ &\geq \int_{\Omega_+} \lim_{n\to\infty} \frac{h(y, v_n^+)}{v_n^+}(z_n^+)^2 \ dy. \end{aligned}$$

From (4.36) and (4.37), it follows that

$$\frac{1}{c^2} \geq M \int_{\Omega_+} (z^+)^2 \, dy,$$

for all $M > 0$ and so $|\Omega_+| = 0$. Then

$$z^+ \equiv 0 \ \text{ in } \ \Omega.$$

So

$$\lim_{n\to\infty} \int_\Omega H(y, z_n^+(y))dy = 0.$$

As

$$\psi(z_n) = \frac{1}{2c^2} - \int_\Omega H(y, z_n) \, dy,$$

we get

$$\psi(z_n) \to \frac{1}{2c^2} \ \text{ as } \ n \to \infty. \tag{4.38}$$

From Lemma 4.3 we have, up to subsequence,

$$J(w_n) \leq \frac{1}{2n}(1 + t_n^2) + J(u_n), \tag{4.39}$$

where $t_n = \dfrac{1}{s_n}$. From (4.39), (4.38) and (3.12) we obtain $\dfrac{1}{2c^2} \leq d$, for all $c > 0$. This is impossible and so the proof of Theorem 1.2 is complete. □

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References

[1] I. Abid, M. Dammak, I. Douchich, *Stable solutions and bifurcation problem for asymptotically linear Helmholtz equations,* Nonlinear Functional Analysis and Applications, **21,** no. 1, (2016), 15–31.

[2] I. Abid, M. Jleli, N. Trabelsi, *Weak solutions of quasilinear biharmonic problems with positive increasing and convex nonlinearities,* Analysis and applications, **6,** no. 3, (2008), 213–227.

[3] L. E. Andersson, T. Elfving, G. H. Golub, *Solution of biharmonic equations with application to radar imaging,* J. Comp. and Appl. Math., **94,** no. 2, (1998), 153–180.

[4] A. Ambrosetti, P. H. Rabionowitz, *Dual variational methods in critical points theory and applications*, J. Funct. Anal.,**14,** (1973), 349–381.

[5] Y. Chen, P. McKenna, *Traveling waves in a nonlinear suspension beam: theoretical results and numerical observations,* J. Differ. Eq., **137,** (1997), 325–355.

[6] M. Dammak, M. El Ghord, *Bifurcation problem for biharmonic asymptotically linear elliptic equations,* Theory of approximation and applications, **11,** no. 1, (2016), 13–37.

[7] M. Dammak, R. Jaidane, C. Jerbi, *Positive solutions for an asymptotically linear biharmonic problem,* Nonlinear Functional Analysis and Applications, **22,** no. 1, (2017), 59–78.

[8] C. P. Danet, *Two maximum principles for a nonlinear fourth order equation from thin plate theory,* Electronic J. Qualitative Theory of Diff. Eq., **31,** (2014), 1–9.

[9] P. Drabek, A. Kufner, F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*, Walter de Gruyter, Berlin, 1997.

[10] A. Ferrero, G. Warnault, *On a solutions of second and fourth order elliptic with power type nonlinearities,* Nonlinear Anal. T. M. A., **70,** no. 8, (2009), 2889–2902.

[11] M. Ghergu, V. Rădulescu, *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis, Oxford Lecture Series in Mathematics and its Applications,* **37,** Oxford University Press, 2008.

[12] A. Kufner, *Weighted Sobolev spaces,* John Wiley and Sons Ltd, 1985.

[13] G. Li, Y. Huang, *Positive solutions for generalized quasilinear Schrödinger equations with asymptotically linear nonlinearities,* Applicable Analysis, (2019), DOI: 10.1080/00036811.2019.1634256

[14] M. C. Lai, H. C. Liu, *Fast direct solver for the biharmonic equation on a disk and its application to incompressible fows,* Appl. Math. Comput., (2005), 164–679.

[15] Y. Martel, *Uniqueness of weak solution for nonlinear elliptic problems,* Houston J. Math., **23,** (1997), 161–168.

[16] P. McKenna, W. Walter, *Traveling waves in a suspension bridge,* SIAM J. Appl. Math., **50,** (1990), 703–715.

[17] N. Meyers, *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa, **17,** (1963), 189–206.

[18] P. Mironescu, V. Rădulescu, *A bifurcation problem associated to a convex, asymtotically linear function,* C. R. Acad. Sci. Paris Ser. I, **316,** (1993), 667–672.

[19] P. Mironescu, V. Rădulescu, *The study of a bifurcation problem associated to an asymptotically linear function,* Nonlinear Anal., **26,** (1996), 857–875.

[20] T.G. Myers, *Thin films with high surface tension,* SIAM Review, **40,** no. 3, (1998), 441–462.

[21] S. Sâanouni, N. Trabelsi, *A bifurcation problem associated to an asymptotically linear function*, Acta Mathematica Scientia, **36 B,** no. 6, (2016), 1–16.

[22] S. Sâanouni, N. Trabelsi, *Bifurcation for elliptic forth-order problems with quasilinear source term,* Electronic Journal of Differential Equations, **92,** (2016), 1–16.

[23] L. Yue, W. Zhengping, *Biharmonic equations with asymptotically linear nonlinearities*, Acta Mathematica Scientia, **27 B,** no. 3, (2007), 549–560.

[24] H. Zahed, *Existence investigation of a fourth order semi-linear weighted problem,* International Journal of Mathematics and Computer Science, **16,** no. 2, (2021), 687–704.

[25] H. Zhou, *An application of a Mountain Pass Theorem*, Acta Mathematica Scientia, **18,** (2002), 27–36.