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Extension of Polarity in a Signed Transformation Semigroup

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Abstract

Let $X_n = \{1, 2, 3, \cdots, n\}$. Let P_n and T_n denote the partial and full transformation semigroups on X_n. For $n = 1, 2, 3, 4$, we obtain the polarity in a signed order-preserving, order-decreasing and reversing partial transformation semigroups by the order $(n+1)^n$ of P_n.

1 Introduction

Let $X_n = \{1, 2, 3, \cdots, n\}$ and $X \subseteq N$. Let $\mathbb{Z}^* = \{\pm 1, \pm 2, \pm 3, \cdots, \pm n\}$. Suppose α is a transformation from X_n into $\mathbb{Z}$. Then, the signed (partial) transformation semigroup is defined as $\alpha : dom(\alpha) \subseteq X_n \to Im(\alpha) \subseteq Z^*$. If $dom(\alpha) = X_n$, then transformation said to full. In addition, let $Im(\alpha^-)$ represent the negative image and let $Im(\alpha^*)$ represent the non-zero image. A transformation is said to be order-preserving partial (SPO_n) if $i \leq j$, $|i\alpha| \leq |j\alpha|$ for all $i, j \in dom(\alpha)$; otherwise, it is called order-reversing if $|i\alpha| \geq |j\alpha|$ for all $i, j \in Dom(\alpha)$. Moreover, it is said to be order-decreasing partial (SPD_n) if $|i\alpha| \leq i$ or $i \geq |i\alpha| \forall i \in Dom(\alpha)$. The subsemigroup of all maps that are signed order-preserving or signed order-reversing can be represented by $SPOD_n$ for a partial signed transformation semigroup.

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Various subsemigroups of P_n andT_n were studied in [1], [3] and [4]. In [2], the semigroups of order-preserving or order-reversing partial transformation of X_n denoted by POD_n were established and $ORCP_n = OCP_n \cup RCP_n$ was defined as the semigroup of order-preserving or order-reversing contraction partial transformation of X_n. In [7], the study of signed symmetric group was initiated. In [5] and [6], SD_n, SO_n and SC_n the signed transformation order-decreasing were studied.

2 Main results

2.1 Extension of Signed order in a partial transformation

Let PSO_n, PSD_n and PSC_n be the polarity of signed order-preserving and signed order-decreasing full transformation semigroups defined on $\alpha : X_n \to X_n^*$. Let $PSPO_n, PSPD_n$ and $PSPOD_n$ be the polarities of signed order-preserving, signed order-decreasing and signed order-preserving or order-reversing partial transformation semigroups defined on $\alpha : X_n \to X_n^*$. The following theorem is very useful for our work:

Theorem 2.1. *[6] Let $S = PSC_n$. Then*

$$|S| = \frac{1}{n} \begin{pmatrix} 2n \\ n-1 \end{pmatrix} \left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} - 1 \right) \tag{2.1}$$

The semigroup of order-preserving or order-reversing partial transformation of X_n will be denoted by POD_n.

2.2 Signed order-preserving partial transformation semigroup $(PSPO_n)$

The following are the results obtained for polarity of elements in the signed order-preserving partial transformation semigroup $(PSPO_n)$.
When $n = 1$, PO_1 (order-preserving partial transformation) has the following two elements:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \phi \end{pmatrix}, \tag{2.2}$$

where ϕ is the empty element in $PSPO_1 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$$Im(\alpha^-) = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \; Im(\alpha^*) = \phi$$

When $n = 2$, PO_2 has the following eight elements:

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & \phi \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 \\ 2 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & \phi \end{pmatrix}$$

Thus,

$$PSPO_2 = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix},$$
$$\left. \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} \right\} \tag{2.3}$$

$$Im(\alpha^-) = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & \phi \end{pmatrix} \right.$$
$$\left. \begin{pmatrix} 1 & 2 \\ \phi & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -2 \end{pmatrix} \right\} \tag{2.4}$$

$$Im(\alpha^*) = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & \phi \end{pmatrix} \right.$$
$$\left. \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix} \right\} \tag{2.5}$$

When $n = 3$, PO_3 has 37 elements for with

$$|PSPO_3| = 133, |Im(\alpha^-)| = 37, |Im(\alpha^*)| = 96.$$

When $n = 4$ PO_4 has 191 elements with

$$|PSPO_4| = 1281, |Im(\alpha^-)| = 191, |Im(\alpha^*)| = 1090.$$

2.2 Polarity of elements in the signed order-decreasing partial transformation semigroup $(PSPD_n)$

When $n = 1$, the order-decreasing partial transformation PD_1 has one element

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{2.6}$$

with $PSPD_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Im(\alpha^-) = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, $Im(\alpha^*) = \phi$.

When $n = 2$, PD_2 has 6 elements:

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & \phi \end{pmatrix}$$

with

$$PSPD_2 = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & 2 \\ -1 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \right\} \tag{2.7}$$

$$Im(\alpha^-) = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -2 \end{pmatrix} \right\} \tag{2.8}$$

$$Im(\alpha^*) = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \right\} \tag{2.9}$$

When $n = 3$, PD_3 has 23 elements:

$$|PSPD_3| = 81, \ |Im(\alpha^-)| = 23, \ |Im(\alpha^*)| = 58.$$

When $n = 4$, PD_3 has 119 elements:

$$|PSPD_4| = 819, \ |Im(\alpha^-)| = 119, \ |Im(\alpha^*)| = 700.$$

2.3 Polarity of elements in signed order-preserving or order-reversing partial transformation semigroup $(PSPOD_n)$

When $n = 1$, POD_1 has 1 element: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with:

$$PSPOD_1 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \ Im(\alpha^-) = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \ Im(\alpha^*) = \phi.$$

When $n = 2$, POD_2 has 8 elements with:

$$PSPOD_2 = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -2 \end{pmatrix} \right\} \tag{2.10}$$

$$Im(\alpha^-) = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & \phi \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \phi & -2 \end{pmatrix}, \right\} \tag{2.11}$$

$$Im(\alpha^*) = \left\{ \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \right\} \tag{2.12}$$

When $n = 3$, POD_3 has 53 elements with $|PSPOD_3| = 209, \ |Im(\alpha^-)| = 53, \ |Im(\alpha^*)| = 156$.
When $n = 4$, POD_4 has 322 elements with $|PSPOD_4| = 2302, \ |Im(\alpha^-)| = 322, \ |Im(\alpha^*)| = 1980$.
In addition, the following tables summarize the values of elements obtained for $PSPO_n, PSPD_n$ and $PSPOD_n$, respectively.

n	$\|Im(\alpha^{-})\|$	$\|Im(\alpha^{*})\|$	$\|PSPO_n\|$
1	1	0	1
2	7	6	13
3	37	96	133
4	191	1090	1281

Table 1: The Values of Elements in $PSPO_n$

n	$\|Im(\alpha^{-})\|$	$\|Im(\alpha^{*})\|$	$\|PSPD_n\|$
1	1	0	1
2	5	4	9
3	23	58	81
4	191	700	819

Table 2: The Values of Elements in $PSPD_n$

n	$\|Im(\alpha^{-})\|$	$\|Im(\alpha^{*})\|$	$\|PSPOD_n\|$
1	1	0	1
2	8	8	16
3	53	156	209
4	322	1980	2302

Table 3: The Values of Elements in $PSPOD_n$

3 Conclusion

The formula for the sequence of $PSPO_n$, $PSPD_n$ and $PSPOD_n$ signed partial transformation semigroups are yet to be obtained. For future research, it is required alongside with signed contraction mapping full and partial transformations with subsemigroups of transformation.

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