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Ordered semigroups in which the radical of every quasi-ideal is a subsemigroup

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Abstract

Let $(S, \cdot, \leq)$ be an ordered semigroup and let A be a nonempty subset of S. The *radical* of A, denoted $\sqrt{A}$, is $\sqrt{A} = \{x \in S \mid x^n \in A$ for some positive integer $n\}$. The paper characterizes when the radical $\sqrt{Q}$ is a subsemigroup of S for all quasi-ideals Q of S.

1 Introduction and Preliminaries

Ćirić and Bogdanović [3] characterized when the radical $\sqrt{I}$ is a subsemigroup of S for every two-sided ideal I of S. Indeed, they studied left ideals, right ideals, two-sided ideals, and bi-ideals but not quasi-ideals. Therefore, we characterize when the radical $\sqrt{Q}$ is subsemigroup of S for every quasi-ideal Q of S.

An *ordered semigroup* $(S, \cdot, \leq)$ consists of a semigroup $(S, \cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation [4]; that is, for any a, b, c in S,

$$a \leq b \Longrightarrow ac \leq bc, ca \leq cb$$

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For a nonempty subset A of S, define

$$(A] = \{x \in S \mid \exists a \in A, x \leq a\}.$$

First, we observe that, for any nonempty subsets A and B of S,

(1) $A \subseteq B \Rightarrow (A] \subseteq (B]$; and

(2) $((A]] = (A]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *subsemigroup* of S if $AA \subseteq A$. We write $\langle a, b \rangle$ for the intersection of all subsemigroups of S containing the subset $\{a, b\}$. A nonempty subset A of S is called a *left* (respectively, *right*) ideal of S if

(1) $SA \subseteq A$ (respectively, $AS \subseteq A$); and

(2) $A = (A]$ (equivalently, for any $x \in A$ and $y \in S$, if $y \leq x$, then $y \in A$).

If A is both a left and a right ideal of S, then A is called an *ideal* (or *two-sided ideal* of S. A nonempty subset Q of S is said to be a *quasi-ideal* of S if

(1) $(QS] \cap (SQ] \subseteq Q$; and

(2) $Q = (Q]$.

A subsemigroup B of S is called a *bi-ideal* of S if

(1) $BSB \subseteq B$; and

(2) $B = (B]$.

Any one-sided ideal is a quasi-ideal and any quasi-ideal is a bi-ideal, [1], [2], [5].

2 Main Results

Hereafter, we deal with ordered semigroups with identity. For an ordered semigroup $(S, \cdot, \leq)$ with identity, and for $a, b \in S$, define

$$\begin{aligned} a\tau b &\iff \exists x, y \in S\ (b \leq xay); \\ a\tau_r b &\iff \exists x \in S\ (b \leq ax); \\ a\tau_l b &\iff \exists y \in S\ (b \leq ya); \\ a\tau_t b &\iff a\tau_r b \cap a\tau_l b; \\ a\eta b &\iff \exists n \in \mathbb{N}\ \exists x, y \in S\ (b^n \leq xay); \\ a\eta_r b &\iff \exists n \in \mathbb{N}\ \exists y \in S\ (b^n \leq ay); \\ a\eta_l b &\iff \exists n \in \mathbb{N}\ \exists x \in S\ (b^n \leq xa); \\ a\eta_t b &\iff a\eta_r b \cap a\eta_l b. \end{aligned}$$

A non-empty subset Q of an ordered semigroup $(S, \cdot, \leq)$ is a quasi-ideal of S if and only if Q is an intersection of a left and a right ideal of S, [1].

Theorem 2.1. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. Then the radical of every quasi-ideal of S is a subsemigroup of S if and only if*

$$\forall a, b \in S\ \forall i, j \in \mathbb{N}\ ((ab)^n \in (\{a^i, b^j\}S] \cap (S\{a^i, b^j\}]).$$

Proof. Assume first that the radical of every quasi-ideal of S is a subsemigroup of S. Let $a, b \in S$ and let $i, j \in \mathbb{N}$. Put

$$Q = (\{a^i, b^j\}S] \cap (S\{a^i, b^j\}].$$

Then Q is an intersection of a left ideal $(S\{a^i, b^j\}]$ and a right ideal $(\{a^i, b^j\}S]$ of S. Thus Q is a quasi-ideal of S. Observe that $a, b \in \sqrt{Q}$ because $a^i, b^j \in Q$. By assumption, $\sqrt{Q}$ is a subsemigroup of S and so $ab \in \sqrt{Q}$. As a result, $(ab)^n \in (\{a^i, b^j\}S] \cap (S\{a^i, b^j\}]$ for some $n \in \mathbb{N}$.

Conversely, assume that, for all $a, b \in S$ and $i, j \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$(ab)^n \in (\{a^i, b^j\}S] \cap (S\{a^i, b^j\}].$$

Let Q be a quasi-ideal of S and let $a, b \in \sqrt{Q}$. Then $a^i \in Q$ and $b^j \in Q$ for some $i, j \in \mathbb{N}$. By assumption, $(ab)^n \in (\{a^i, b^j\}S] \cap (S\{a^i, b^j\}]$ for some $n \in \mathbb{N}$. Since Q is a quasi-ideal of S,

$$(\{a^i, b^j\}S] \cap (S\{a^i, b^j\}] \subseteq (QS] \cap (SQ] \subseteq Q.$$

Thus $(ab)^n \in Q$. Therefore, $ab \in \sqrt{Q}$. Consequently, $\sqrt{Q}$ is a subsemigroup of S. □

Example 2.2. Let $S = \{a, b, c, d, 1\}$ be an ordered semigroup such that the multiplication and order relation are defined by:

$\cdot$	a	b	c	d	1
a	a	a	a	a	a
b	a	a	a	a	b
c	a	a	b	a	c
d	a	a	b	b	d
e	a	b	c	d	1

$$\leq = \{(a, a), (a, b), (b, b), (c, c), (d, d), (1, 1)\}.$$

The quasi-ideals of S is $\{\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, S\}$. Observe that $\sqrt{\{a\}} = \sqrt{\{a, b\}} = \sqrt{\{a, b, c\}} = \sqrt{\{a, b, d\}} = \sqrt{\{a, b, c, d\}} = \{a, b, c, d\}$ and $\sqrt{S} = S$. Then the radical of every quasi-ideal of S is a subsemigroup of S.

In general, the radical of a quasi-ideal of an ordered semigroup with identity need not be a subsemigroup as the following example shows:

Example 2.3. Let $S = \{a, b, c, d, f, 1\}$ be an ordered semigroup such that the multiplication and order relation are defined by:

$\cdot$	a	b	c	d	f	1
a	a	a	a	a	a	a
b	a	b	a	d	a	b
c	a	f	c	c	f	c
d	a	b	d	d	b	d
f	a	f	a	c	a	f
1	a	b	c	d	f	1

$$\leq = \{(a, a), (a, b), (a, c), (a, d), (a, f), (b, b), (c, c), (d, d), (f, f), (1, 1)\}.$$

The quasi-ideals of S is $\{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, f\}, \{a, b, d\}, \{a, c, d\}, \{a, b, f\}, \{a, c, f\}, \{a, b, c, d, f\}, S\}$. Observe that $\sqrt{\{a, c, d\}} = \{a, c, d, f\}$ is not a subsemigroup of S.

Theorem 2.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. Then the radical of every right ideal of S is a quasi-ideal of S if and only if*

$$\forall a, b, c \in S \ (a\tau_r c \wedge b\tau_l c \Longrightarrow \forall i, j \in \mathbb{N} \ (a^i \eta_r c \vee b^j \eta_r c)).$$

Proof. Assume that the radical of every right ideal of S is a quasi-ideal of S. Let $a, b, c \in S$ such that $a\tau_r c$ and $b\tau_l c$. Then $c \leq au$ and $c \leq vb$ for some $u, v \in S$. Let i, j in $\mathbb{N}$. It is easy to see that

$$R = (\{a^i, b^j\}S]$$

is a right ideal of S. By assumption, $\sqrt{R}$ is a quasi-ideal of S. From $a, b \in \sqrt{R}$, $c \leq au$ and $c \leq vb$, it follows that

$$c \in (\sqrt{R}S] \cap (S\sqrt{R}] \subseteq \sqrt{R}.$$

Then $c^n \in R$ for some $n \in \mathbb{N}$. Thus $a^i\eta_r c$ or $b^j\eta_r c$.

Conversely, assume that for all $a, b, c \in S$,

$$a\tau_r c \wedge b\tau_l c \Longrightarrow \forall i, j \in \mathbb{N}\ (a^i\eta_r c \vee b^j\eta_r c).$$

Let R be a right ideal of S. To show that $(\sqrt{R}S] \cap (S\sqrt{R}] \subseteq \sqrt{R}$, let $x \in (\sqrt{R}S] \cap (S\sqrt{R}]$. Then $x \leq au$ and $x \leq vb$ for some $u, v \in S$ and $a, b \in \sqrt{R}$. Since $a, b \in \sqrt{R}$, there exist $i, j \in \mathbb{N}$ such that $a^i, b^j \in R$. From $x \leq au$ and $x \leq vb$, it follows by assumption that $a^i\eta_r c$ or $b^j\eta_r c$. If $a^i\eta_r c$, then there exists $n \in \mathbb{N}$ such that $x^n \in (a^iS]$. Since

$$(a^iS] \subseteq (\{a^i, b^j\}S] \subseteq (RS] \subseteq (R] = R,$$

$x \in \sqrt{R}$. Now, let $x \in (\sqrt{R}]$. Then $x \leq y$ for some $y \in \sqrt{R}$. Since $y \in \sqrt{R}$, $y^n \in R$ for some $n \in \mathbb{N}$. Since $x \leq y$, $x^n \leq y^n$. Then $x^n \in R$ and $x \in \sqrt{R}$. Therefore, $\sqrt{R}$ is a quasi-ideal of S. □

Dually, we have the following theorem.

Theorem 2.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. Then the radical of every left ideal of S is a quasi-ideal of S if and only if*

$$\forall a, b, c \in S\ (a\tau_r c \wedge b\tau_l c \Longrightarrow \forall i, j \in \mathbb{N}\ (a^i\eta_l c \vee b^j\eta_l c)).$$

Theorem 2.6. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. Then the radical of every quasi-ideal of S is a right ideal of S if and only if*

$$\forall a, b \in S\ \forall k \in \mathbb{N}\ (a^k\eta_t ab).$$

Proof. Assume that the radical of every quasi-ideal of S is a right ideal of S. Let $a, b \in S$ and let $k \in \mathbb{N}$. Put

$$Q = (a^kS] \cap (Sa^k].$$

Then Q is a quasi-ideal of S because $(Sa^k]$ is a left ideal of S and $(a^kS]$ is a right ideal of S. Moreover, since $a^{k+1} \in (a^kS] \cap (Sa^k]$, we have $a \in \sqrt{Q}$. By assumption, $\sqrt{Q}$ is a right ideal of S. So

$$ab \in \sqrt{Q}S \subseteq \sqrt{Q}.$$

Consequently, $(ab)^n \in Q$ for some $n \in \mathbb{N}$. This implies $a^k \eta_t ab$.

Conversely, assume that $a^k \eta_t ab$ for every $a, b \in S$ and $k \in \mathbb{N}$. Let Q be a quasi-ideal of S and let $a \in \sqrt{Q}$. Then $a^k \in Q$ for some $k \in \mathbb{N}$. By assumption, $(ab)^n \in (a^kS] \cap (Sa^k]$ for some $n \in \mathbb{N}$. Observe that

$$(a^kS] \cap (Sa^k] \subseteq (QS] \cap (SQ] \subseteq Q.$$

That is, $ab \in \sqrt{Q}$. As in Theorem 2.4, $(\sqrt{Q}] = \sqrt{Q}$. Hence $\sqrt{Q}$ is a right ideal of S. □

Dually, we have the following.

Theorem 2.7. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. Then the radical of every quasi-ideal of S is a left ideal of S if and only if*

$$\forall a, b \in S \ \forall k \in \mathbb{N} \ (a^k \eta_t ba).$$

Theorem 2.8. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. Then the radical of every quasi-ideal of S is a quasi-ideal of S if and only if*

$$\forall a, b, c \in S \ (a\tau_r c \wedge b\tau_l c \Longrightarrow \forall i, j \in \mathbb{N} \ \exists n \in \mathbb{N} \ (c^n \in (\{a^i, b^j\}S] \cap (S\{a^i, b^j\}])).$$

Proof. Assume that the radical of every quasi-ideal of S is a quasi-ideal of S. Let $a, b, c \in S$ such that $a\tau_r c$ and $b\tau_l c$. Then $c \leq au$ and $c \leq vb$ for some $u, v \in S$. Let $i, j \in \mathbb{N}$. Put

$$Q = (\{a^i, b^j\}S] \cap (S\{a^i, b^j\}].$$

Then Q is a quasi-ideal of S such that $a, b \in \sqrt{Q}$. By assumption, $\sqrt{Q}$ is a quasi-ideal of S. From $c \leq au$ and $c \leq vb$, it follows that

$$c \in (\sqrt{Q}S] \cap (S\sqrt{Q}] \subseteq \sqrt{Q}.$$

Hence $c^n \in (\{a^i, b^j\}S] \cap (S\{a^i, b^j\}]$ for some $n \in \mathbb{N}$.

Conversely, assume that

$$\forall a, b, c \in S \ (a\tau_r c \wedge b\tau_l c \Longrightarrow \forall i, j \in \mathbb{N} \ \exists n \in \mathbb{N} \ (c^n \in (\{a^i, b^j\}S] \cap (S\{a^i, b^j\}])).$$

Let Q be a quasi-ideal of S. Let $x \in (\sqrt{Q}S] \cap (S\sqrt{Q}]$. Then $x \leq au$ and $x \leq vb$ for some $a, b \in \sqrt{Q}$ and $u, v \in S$. Therefore $a^i, b^j \in Q$ for some $i, j \in \mathbb{N}$. By assumption,

$$x^n \in (\{a^i, b^j\}S] \cap (S\{a^i, b^j\}]$$

for some $n \in \mathbb{N}$. Observe that

$$(\{a^i, b^j\}S] \cap (S\{a^i, b^j\}] \subseteq (QS] \cap (SQ] \subseteq Q.$$

Hence $x \in \sqrt{Q}$. Since $(\sqrt{Q}] = \sqrt{Q}$, $\sqrt{Q}$ is a quasi-ideal of S. □

Theorem 2.9. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. The radical of every ideal of S is a quasi-ideal of S if and only if*

$$\forall a, b, c \in S\ (a\tau_r c \wedge b\tau_l c \Longrightarrow \forall i, j \in \mathbb{N}\ (a^i \eta c \vee b^j \eta c)).$$

Proof. Assume that the radical of every ideal of S is a quasi-ideal of S. Let $a, b, c \in S$ be such that $a\tau_r c$ and $b\tau_l c$. Then $c \leq au$ and $c \leq vb$ for some $u, v \in S$. Let $i, j \in \mathbb{N}$. Put

$$A = (S\{a^i, b^j\}S].$$

Then A is an ideal of S such that $a, b \in \sqrt{A}$. By assumption, $\sqrt{A}$ is a quasi-ideal of S. Therefore $c \in (\sqrt{A}S] \cap (S\sqrt{A}] \subseteq \sqrt{A}$ and hence $c^n \in A$ for some $n \in \mathbb{N}$. Consequently, $a^i \eta c$ or $b^j \eta c$. The proof of the converse is similar to that of the converse of Therem 2.4. □

Theorem 2.10. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. Then the radical of every quasi-ideal of S is an ideal of S if and only if*

$$\forall a, b \in S\ \forall k \in \mathbb{N}\ (a^k \eta_t ab \wedge a^k \eta_t ba).$$

Proof. Suppose that the radical of every quasi-ideal of S is an ideal of S. Let $a, b \in S$ and let $k \in \mathbb{N}$. Put

$$A = (a^k S] \cap (Sa^k].$$

Then A is a quasi-ideal of S and $a \in \sqrt{A}$. By assumption, $\sqrt{A}$ is an ideal of S. We have $ab \in \sqrt{A}S \subseteq \sqrt{A}$ and $ba \in S\sqrt{A} \subseteq \sqrt{A}$. Therefore, $(ab)^m, (ba)^n \in A$ for some $m, n \in \mathbb{N}$. This implies that $ab, ba \in A$. This means that $a^k \eta_t ab$ and $a^k \eta_t ba$.

Conversely, assume that

$$\forall a, b \in S\ \forall k \in \mathbb{N}\ (a^k \eta_t ab \wedge a^k \eta_t ba).$$

Let Q be a quasi-ideal of S and let $a \in \sqrt{Q}$ and $b \in S$. By $a \in \sqrt{Q}$, $a^k \in Q$ for some $k \in \mathbb{N}$. By assumption, there exist $m, n \in \mathbb{N}$ such that $(ab)^m, (ba)^n \in (a^k S] \cap (Sa^k]$. Consider

$$(ab)^m, (ba)^n \in (a^k S] \cap (Sa^k] \subseteq (QS] \cap (SQ] \subseteq Q.$$

Thus $ab, ba \in \sqrt{Q}$. Since $(\sqrt{Q}] = \sqrt{Q}$, $\sqrt{Q}$ is a quasi-ideal of S. □

The following theorem gives a sufficient and necessary condition in order that the radical of every quasi-ideal of an ordered semigroup $(S, \cdot, \leq)$ is a bi-ideal of the ordered semigroup.

Theorem 2.11. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. The radical of every quasi-ideal of S is a bi-ideal of S if and only if*

$$\forall a, b, c \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ ((abc)^n \in (\{a^i, c^j\}S] \cap (S\{a^i, c^j\}]).$$

Proof. Assume that the radical of every quasi-ideal of S is a bi-ideal of S. Let $a, b, c \in S$ and let $i, j \in \mathbb{N}$. Put

$$Q = (\{a^i, c^j\}S] \cap (S\{a^i, c^j\}].$$

Then Q is a quasi-ideal of S such that $a, c \in \sqrt{Q}$. By assumption, $\sqrt{Q}$ is a bi-ideal of S. Consider

$$abc \in \sqrt{Q}S\sqrt{Q} \subseteq \sqrt{Q}.$$

Hence $(abc)^n \in Q$ for some $n \in \mathbb{N}$.

Conversely, assume that

$$\forall a, b, c \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ ((abc)^n \in (\{a^i, c^j\}S] \cap (S\{a^i, c^j\}]).$$

Let Q be a quasi-ideal of S, and let $a, c \in \sqrt{Q}$ and $b \in S$. Then $a^i, c^j \in Q$ for some $i, j \in \mathbb{N}$. By assumption, there is $n \in \mathbb{N}$ such that $(abc)^n \in (\{a^i, c^j\}S] \cap (S\{a^i, c^j\}]$. Then

$$(abc)^n \in (\{a^i, c^j\}S] \cap (S\{a^i, c^j\}] \subseteq (QS] \cap (SQ] \subseteq Q.$$

Therefore, $abc \in \sqrt{Q}$. Again, $(\sqrt{Q}] = \sqrt{Q}$ and thus $\sqrt{Q}$ is a bi-ideal of S. □

Theorem 2.12. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. Then the following conditions are equivalent:*

(1) *the radical of every a bi-ideal of S is a quasi-ideal of S;*

(2) $\forall a,b,c \in S\ (a\tau_r c \wedge b\tau_l c \Longrightarrow \forall i,j \in \mathbb{N} \exists n \in \mathbb{N}\ (c^n \in (\{a^i,b^j\}S\{a^i,b^j\}]))$.

Proof. (1) $\Longrightarrow$ (2): Let $a,b,c \in S$ such that $c \leq au$ and $c \leq vb$ for some $u,v \in S$. Let $i,j \in \mathbb{N}$. Put

$$B = (\{a^i,b^j\}S\{a^i,b^j\}].$$

Then B is a bi-ideal of S with $a,b \in \sqrt{B}$. Then $\sqrt{B}$ is a quasi-ideal of S. Therefore

$$c \in (\sqrt{B}S] \cap (S\sqrt{B}] \subseteq \sqrt{B}.$$

Hence $c^n \in B$ for some $n \in \mathbb{N}$.

(2) $\Longrightarrow$ (1): Let B be a bi-ideal of S and let $x \in (\sqrt{B}S] \cap (S\sqrt{B}]$. Then $x \leq au$ and $x \leq vb$ for some $a,b \in \sqrt{B}$ and $u,v \in S$. Now there exist $i,j \in \mathbb{N}$ such that $a^i,b^j \in B$. Then $x^n \in (\{a^i,b^j\}S\{a^i,b^j\}]$. Consider

$$x^n \in (\{a^i,b^j\}S\{a^i,b^j\}] \subseteq (BSB] \subseteq (B] = B.$$

Hence $x \in \sqrt{B}$. From $(\sqrt{B}] = \sqrt{B}$, we deduce that B is a quasi-ideal of S. □

Theorem 2.13. *Let $(S,\cdot,\leq)$ be an ordered semigroup with identity. Then the radical of every quasi-ideal of S is a subsemigroup of S if and only if*

$$\forall a,b \in S\ \forall i,j \in \mathbb{N}\ ((ab)^n \in (\{a^i,b^j\}S] \cap (S\{a^i,b^j\}]).$$

Proof. Assume that the radical of every quasi-ideal of S is a subsemigroup of S. Let $a,b \in S$, and let $i,j \in \mathbb{N}$. Put

$$Q = (\{a^i,b^j\}]S \cap (S\{a^i,b^j\}].$$

Then Q is a quasi-ideal of S with $a,b \in \sqrt{Q}$. By assumption, $\sqrt{Q}$ is a subsemigroup of S, and $ab \in \sqrt{Q}$. Hence

$$(ab)^n \in (\{a^i,b^j\}]S \cap (S\{a^i,b^j\}] \text{ for some } n \in \mathbb{N}.$$

Conversely, assume that

$$\forall a,b \in S\ \forall i,j \in \mathbb{N}\ ((ab)^n \in (\{a^i,b^j\}S] \cap (S\{a^i,b^j\}]).$$

Let Q be a quasi-ideal of S, and let $a,b \in \sqrt{Q}$. Then $a^i \in Q$ and $b^j \in Q$ for some $i,j \in \mathbb{N}$. By assumption, there exists $n \in \mathbb{N}$ such that $(ab)^n \in (\{a^i,b^j\}S] \cap (S\{a^i,b^j\}]$. Consider

$$(ab)^n \in (\{a^i,b^j\}S] \cap (S\{a^i,b^j\}] \subseteq (QS] \cap (QS] \subseteq Q.$$

Then $ab \in \sqrt{Q}$ and so $\sqrt{Q}$ is a subsemigroup of S. □

Theorem 2.14. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. Then the following conditions are equivalent:*

(1) *the radical $\sqrt{(A]}$ is a quasi-ideal of S for every subsemigroup A of S;*

(2) $\forall a, b, c \in S\ (a\tau_r c \wedge b\tau_l c \Longrightarrow \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ (c^n \in (\langle a^i, b^j \rangle]))$.

Proof. (1)$\Longrightarrow$ (2): Let $a, b, c \in S$ such that $a\tau_r c$ and $b\tau_l c$. Then $c \leq au$ and $c \leq vb$ for some $u, v \in S$. Let $i, j \in \mathbb{N}$. Consider

$$A = \langle a^i, b^j \rangle.$$

Then, $\sqrt{A}$ is a quasi-ideal of S. By $c \leq au$ and $c \leq vb$,

$$c \in (\sqrt{A}S] \cap (S\sqrt{A}] \subseteq \sqrt{A}.$$

Hence $c^n \in (\langle a^i, b^j \rangle]$ for some $n \in \mathbb{N}$.

(2)$\Longrightarrow$ (1): Let A be a subsemigroup of S and let

$$x \in (\sqrt{(A]}S] \cap (S\sqrt{(A]}].$$

Then $x \leq au$ and $x \leq vb$ for some $a, b \in \sqrt{(A]}$ and $u, v \in S$. Thus $a^i, b^j \in (A]$ for some $i, j \in \mathbb{N}$. Then $x^n \in (\langle a^i, b^j \rangle] \subseteq (A]$ and so $x \in \sqrt{(A]}$. Since $(\sqrt{(A]}] = \sqrt{(A]}$, $\sqrt{(A]}$ is a quasi-ideal of S. □

Theorem 2.15. *Let $(S, \cdot, \leq)$ be an ordered semigroup with identity. Then the following conditions are equivalent:*

(1) *the radical of every quasi-ideal of S is a quasi-ideal of S;*

(2) $\forall a, b \in S\ (\sqrt{(\{a, b\}S] \cap (S\{a, b\}]}$ *is a quasi-ideal of* S);

(3) $\forall a, b, c \in S\ (a\tau_r c \wedge b\tau_l c \Longrightarrow \exists n \in \mathbb{N}\ (c^n \in (\{a^2, b^2\}S] \cap (S\{a^2, b^2\}])$;

(4) $\forall a, b, c \in S\ (a\tau_r c \wedge b\tau_l c \Longrightarrow \forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ (c^n \in (\{a^k, b^k\}S] \cap (S\{a^k, b^k\}])$.

Proof. (1) $\Longrightarrow$ (2): Since $(\{a, b\}S] \cap (S\{a, b\}]$ is a quasi-ideal of S, $\sqrt{(\{a, b\}S] \cap (S\{a, b\}]}$ is a quasi-ideal of S.

(2) $\Longrightarrow$ (3): Let $a, b, c \in S$ such that $c \leq au$ and $c \leq va$ for some $u, v \in S$. Put

$$Q = (\{a^2, b^2\}S] \cap (S\{a^2, b^2\}].$$

Then $\sqrt{Q}$ is a quasi-ideal of S. Obviously, $a, b \in \sqrt{Q}$. We have

$$c \in (\sqrt{Q}S] \cap (S\sqrt{Q}] \subseteq \sqrt{Q}.$$

This implies $c^n \in Q$ for some $n \in \mathbb{N}$.

(3)$\Longrightarrow$ (4): Let $a, b, c \in S$ be such that $c \leq au$ and $c \leq va$ for some $u, v \in S$. Then

$$c^n \in (\{a^2, b^2\}S] \cap (S\{a^2, b^2\}]$$

for some $n \in \mathbb{N}$. We have

$$(\{a^2, b^2\}S] \cap (S\{a^2, b^2\}] \subseteq (\{a, b\}S] \cap (S\{a, b\}].$$

Then $c^n \in (\{a, b\}S] \cap (S\{a, b\}]$.

Now, assume that there is $m \in \mathbb{N}$ such that

$$c^m \in (\{a^k, b^k\}S] \cap (S\{a^k, b^k\}]$$

for some $k \in \mathbb{N}$. Then

$$(c^m)^l \in (\{a^{2k}, b^{2k}\}S] \cap (S\{a^{2k}, b^{2k}\}]$$

for some $l \in \mathbb{N}$. Consider

$$\begin{aligned} & (\{a^{2k}, b^{2k}\}S] \cap (S\{a^{2k}, b^{2k}\}] \\ = \; & (\{a^{k+1}a^{k-1}, b^{k+1}b^{k-1}\}S] \cap (S\{a^{k-1}a^{k+1}, b^{k-1}b^{k+1}\}] \\ \subseteq \; & (\{a^{k+1}, b^{k+1}\}S] \cap (S\{a^{k+1}, b^{k+1}\}]. \end{aligned}$$

Hence $(c^m)^l \in (\{a^{k+1}, b^{k+1}\}S] \cap (S\{a^{k+1}, b^{k+1}\}]$.

(4) $\Longrightarrow$ (1): Let Q is a quasi-ideal of S. Let $x \in (\sqrt{Q}S] \cap (S\sqrt{Q}]$. Then $b \leq au$ and $x \leq va$ for some $a, b \in \sqrt{Q}$ and $u, v \in S$. Then $a^k, b^l \in Q$ for some $k, l \in \mathbb{N}$. Then $x^n \in (\{a^{k+l}, b^{k+l}\}S] \cap (S\{a^{k+l}, b^{k+k}\}]$. Consider

$$x^n \in (\{a^k, b^k\}S] \cap (S\{a^k, b^k\}] \subseteq (QS] \cap (SQ] \subseteq Q.$$

Consequently, $x^n \in Q$ and so $x \in \sqrt{Q}$. Since $(\sqrt{Q}] = \sqrt{Q}$, (1) follows. □

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