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The General Formula for the Ehrhart Polynomial of Polytopes with Applications

Fatema A. Sadiq, Shatha A. Salman, Raghad I. Sabri

Mathematics and Computer Applications
Faculty of Applied Science
University of Technology
Baghdad, Iraq

email: 10122@uotechnology.edu.iq, 100178@uotechnology.edu.iq,
100247@uotechnolg.edu.iq

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Abstract

Recently, Polytopes have shown wide applications in a lot of situations. For example, a cyclic polytope is very important in different areas of science like solutions to extremum problems (the Upper Bound Conjecture). Polytopes serve as bases for diverse constructions (from triangulations to bimatrix games). In addition, we give the general form for the product of simplex polytopes and an algorithm for these computations.

1 Introduction

Ehrhart polynomials of integral polytopes is an active area of research at the intersection of discrete geometry, geometry of numbers, enumerative combinatorics, and combinatorial commutative algebra. An integral polytope is a convex polytope whose vertices are all integral points; other polytopes can be described either as the solution of the system of liner inequality or as convex hull of vertices. Ehrhart [1] showed that for a d-dimensional integral polytope, $L(P,t)$ is a polynomial in t of degree$d$. We still do not know much about

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the coefficients of Ehrhart polynomials for general polytopes except that the leading, second and last coefficients of $L(P, t)$ are the volume of P, one half of the volume of the boundary of P and 1, respectively. Of special importance are cyclic polytopes because they have the largest number of faces of every dimension among all polytopes of a fixed dimension and number of vertices. Cyclic polytope possess important properties [2] and are found in many areas of mathematics, such as higher category theory and algebraic K-theory [3, 4, 5], as well as game theory [6 7]. The Ehrhart polynomial of a cyclic polytope is equal to its volume. The Ehrhart polynomial of its lower envelope it was given by [8] as conjecture. Liu [9] proved the conjecture about the coefficients of the Erhart polynomial of a cyclic polytope. In [10], a theorem for computing the coefficients of the Ehrhart polynomials is given, the cross product are used to find Ehrhart polynomial in $d-$dimension.

2 Basic Concepts

This section deals with some basic concepts that we consider in this paper.

Definition 2.1. [11] A polytope in R^d is the convex hull of finitely many points or, equivalently, an intersection of finitely many closed half-spaces which is bounded. These are called the V-representation and the H-representation of the polytope respectively.

Definition 2.2. [12] *A* $d-$dimensional simplex is a $d-$polytope with exactly $d + 1$ vertices. Equivalently, it is the convex hull of a set of affinely independent points in R^d.

Definition 2.3. [13] The moment curve in R^d is defined by $M : R \to R^d$, where $M(\nu) = (\nu, \nu^2,...,\nu^d)^T \in R^d$.

Definition 2.4. [13] The cyclic C_d (n) polytope of dimension d with n vertices is the convex hull C_d $(v_1, v_2, ..., v_n)$=conv $\{M(v_1), M(v_2), ..., M(v_n)\}$ of distinct points $M(v_i)$ where $n > d$ with $\{v_1 < v_2 < ... < v_n\}$ on the moment curve.

Theorem 2.5. [9] Let $C_d(V)$ be an integral cyclic polytope where $V = \{v_1,v_2,...,v_n\}$ $L(C_d(V), t) = \sum_{j=0}^{d} vol(tC_j(V)) = \sum_{j=0}^{d} vol_j(C_j(V))\, t^j$ ....*(2.1)* where vol_j $(tC_j(V)$ is the volume of $tC_j(V)$ in a j-dimensional space and vol_0 $(tC_0(V)) = 1$.

Theorem 2.6. [9] For any integral set V with $n = V = d+1$, (meaning that $C_d(n)$ is simplex), $vol(tC_d(V))$ $=t^d/d!\prod_{1\leq i<k\leq d+1}(v_k - v_i)$ ....*(2.2)*

Theorem 2.7. [10] Let $C_d(V)$ be an integral cyclic polytope, where $V = \{v_1, v_2, ..., v_n\}$ for $v_i = 1, 2, \ldots, n$ and $i = 1, 2, ..., n$, the following are satisfied:
(i) if C_d (n) is not simplex (one can decomposed it into simplices), then we can use equation (2.2) to compute its volume.
(ii) In a one dimensional space, the volume of cyclic polytope tC_1 (V) in the interval $[v_1, v_n]$ is equal to $v_n - v_1$.

Definition 2.8. [14] For two polytopes $P \subseteq R^{d_p}$ and $Q \subseteq R^{dq}$ of dimension d_P and d_q, the product of P and Q is $P \times Q = \{(p, q)$, where $p \subseteq P, q \subseteq Q\} \subseteq R^{dp+dp}$.

Theorem 2.9. [14] If P is a d_P-dimensional integral polytope in R^{d_P} and Q is d_q-dimensional integral polytope in R^{d_q}, then the Ehrhart polynomial of $P \times Q = L_P(t)L_Q(t)$.

3 The Ehrhart polynomial of cyclic with different number of vertices

The Ehrhart polynomial for cyclic polytope are computed combinatorically using the general formula given in table 1.

The description of the table is as follows:
The coefficients a_1 equal to the number of vertices minus 1.
The coefficients a_2 equal to the number of vertices multiply by i, where i is taken as $1, 2$, and after two steps, r is added to the index, where $r = 2, 3, 4, 5, 6, ...$
This gives the general formula for the cyclic polytope with different number of vertices.
$L(C_2(1, 2, 3), t) = t^2 + 2t + 1.$
$L(C_2(1, 2, 3, 4), t) = 4t^2 + 3t + 1.$
$L(C_2(1, 2, 3, 4, 5), t) = 10t^2 + 4t + 1.$
$L(C_2(1, 2, 3, 4, 5, 6), t) = 20t^2 + 5t + 1.$
$L(C_2(1, 2, 3, 4, 5, 6, 7), t) = 35t^2 + 6t + 1.$
$L(C_2(1, 2, 3, 4, 5, 6, 7, 8), t) = 56t^2 + 7t + 1.$
$L(C_2(1, 2, 3, 4, 5, 6, 7, 8, 9), t) = 84t^2 + 8t + 1.$
$L(C_2(1, 2, 3, 4, 5, 6, 7, 8, 9, 10), t) = 120t^2 + 9t + 1..$

Table 1. The General Coefficients Formula of $C_d(n)$

Number of vertices n	$n*i$ and $n*(i+r)$	a_2	a_1 =n-1
4	1	4	3
5	2	10	4
6	$3+2$	20	5
7	5	35	6
8	7	56	7
9	9+3	84	8
10	12	120	9
11	15	165	10
12	18+4	220	11
13	22	286	12
14	26	364	13
15	30+5	470	14
16	35	576	15
17	40	680	16
18	45+6	816	17
19	51	969	18
20	57	1149	19
21	63+7	1330	20
22	70	1540	21
23	77	2541	23

3.1 General formula for the product of two cyclic polytopes

Let $C_2(n)$ be a cyclic polytope and $\times$ denotes the product of two cyclic polytopes. Let $L(C_2(n),t)$=a_2 t^2+a_1 $t+a_0$ be the Ehrhart polynomial of a cyclic polytope with n vertices. Then
$L_{C_2\times C_2}$ $(t) = L(C_2(n),t)$ $.L(C_2(n),t)$.
=b_4 t^4+b_3 t^3+b_2 t^2+b_1t+b_0,
where
$b_4 = a_2^2, b_3 = 2(a_2a_1), b_2 = 2(a_2 + a_1^2), b_1 = 2(n-1), b_0 = 1.$
This is given in Table 2. For each n vertices, we obtain the Ehrhart polynomial for the product of two cyclic polytopes.

Table 2. The coefficients of the Ehrhart polynomial..

n	b_4	b_3	b_2	b_1
3	1	4	6	4
4	16	24	17	6
5	100	80	36	8
6	400	200	65	10
7	1225	420	106	12
8	3136	784	161	14
9	7056	1344	232	16
10	14400	2160	321	18

Now an algorithm for all of computation is given as
Input n: number of vertices, $n > 3$.
Output :$a_2, a_1, b_4, b_3, b_2, b_1$.
Step 1 : $k = 4, i = 1, j = 1, c = 2$.
Step 2 : a_1=$k - 1, b_1 = 2(k - 1)$.
Step 3: if $(c = 0)$.
$j = j + 1, a_2 = (k \times i) + j, i = i + j, k = k + 1, c = 2$.
otherwise
step 4: $a_2 = k \times i, i = i + j, k = k + 1, c = c - 1$
step 5: if $(k >= n)$.
Otherwise go to 3
Step 6: end
Step 7: b_4=a_2^2, b_3=$2(a_2 a_1)$, b_2=$(2a_2 + a_1^2)$

4 The product of two polytopes with Ehrhart polynomial

In this section, two polytopes are taken with two simplicies of dimension two. We find the general formula for the product of them. This means that the Ehrhart polynomial in four dimensions can be found. The formula is given as follows:
For m=2
If the selected polytopes is the simplex with vertices $\{0, e_1, e_1 + e_2\}$, then the Ehrhart Polynomial is given as:
$L_P = 1/2t^2 + 3/2t + 1$, therefore the product is

$L_P \times L_P$=$1/4t^4$+$6/4t^3$+$13/4t^2$+$3t+1$ for P^2 , in same procedure the following are obtained as:
For $m = 3, 4, 5, 6, 7$ and 8
ν=$\{0, 2e_1, 2(e_1 + e_2)\}$, $L_P = 2t^2 + 3t + 1$, $L_P \times L_P$=$4t^4 + 12t^3 + 13t^2 + 6t + 1$
ν=$\{0, 3e_1$,$3(e_1$+$e_2)\}$, $L_P = 9/2t^2$+$9/2t$+1, $L_P \times L_P$=$81/4t^4$+$162/4t^3$+$117/4$ $t^2 + 18/2t + 1$
ν=$\{0, 4e_1, 4(e_1+e_2)\}$, $L_P = 8t^2+6t+1$, $L_P \times L_P$=$64t^4+96t^3+52t^2+12t+1$
ν=$\{0, 5e_1, 5(e_1+e_2)\}$, $L_P = 25/2t^2+15/2t+1$, $L_P \times L_P$=$625/4t^4+750/4t^3+$ $325/4t^2 + 15t + 1$
ν=$\{0, 6e_1, 6(e_1+e_2)\}$, $L_P = 18t^2+9t+1$, $L_P \times L_P = 324t^4+324t^3+117t^2+$ $18t + 1$
ν=$\{0, 7e_1, 7(e_1 + e_2)\}$, $L_P = 49/2t^2 + 21/2t + 1$, $L_P \times L_P$= $2401/4t^4 +$ $2058/4t^3 + 637/4t^2 + 21t + 1$.
Now, we obtain the general formula for the product:
i) for even interval m, the Ehrhart coefficients
$d_4 = y^4/4, d_3 = 3/2y^3, d_2 = 13/4y^2, y = 1, 3, 5, 7, \ldots$ and $d_1 = 3k, k = 1, 2, 3, 4$
ii) for odd interval m, the Ehrhart coefficients
$d_4 = 4x^4, d_3 = 12x^3, d_2 = 13x^2, x = 1, 2, 3, 4, \ldots \text{and} d_1 = 3k, k = 1, 2, 3, 4$
The results are given in table 3.

Table 3. The coefficients of the Ehrhart polynomial

m	d_4	d_3	d_2	d_1
2	1/4	6/4	13/4	3
3	4	12	13	6
4	81/4	162/4	117/4	9
5	64	96	52	12
6	625/4	750/4	3025/4	15
7	324	324	117	18
8	2401/4	1029/4	637/4	21
9	1024	768	208	24
.	.	.	.	.
.	.	.	.	.

Now, an algorithm for all of computation is given as:
Input n
Output :d_4, d_3, d_2, d_1
Step 1: $y = -1, k = 0, x = 0, m = 2, k = k + 1$
Step 2: if $(m \mod 2) = 0$
Step 3: $y = y + 2, d_4 = y^4/4, d_3 = 3/2y^3, d_2 = 13/4y^2, d_1 = 3k$ otherwise
Step 4: $x = x + 1, d_4 = 4x^4, d_3 = 12x^3, d_2 = 13x^2, d_1 = 3k$
Step 5: End

5 Conclusion

There is a lot of search that deals with the Ehrhart polynomial. However, the general formula in $d-$dimension poses some difficulty. In our work, we gave the general formula for two cyclic polytope and we have taken the simplex polytope with algorithms.

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