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The Toeplitz-Hausdorff and Spectral Inclusion Theorems for Linear Relations

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Abstract

The Toeplitz-Hausdorff Theorem which was established in 1918 asserts that the numerical range of an operator is always convex. We prove this theorem for the numerical range of a linear relation relative to another linear relation. We also show that the closure of this numerical range contains the relative spectrum for these two linear relations. The classical results for a single linear relation can be deduced from the results obtained here.

1 Introduction

The numerical range $\Theta(T)$ of a bounded linear operator T on a Hilbert space $\mathcal{H}$ is the collection of complex numbers of the form $\langle Tx, x\rangle$, where x

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ranges over the unit vectors in $\mathcal{H}$. This concept which is useful in studying linear operators has attracted the attention of many authors in the past few decades (see for example [1, 3, 4, 5, 6, 9, 10], and their references). The concept of the numerical range has various applications in areas such as perturbation theory and generalized eigenvalue problems [1, 3, 5]. The convexity of the numerical range is particularly useful when dealing with perturbation problems involving bounded operators [1, 8].

Convexity is a notion that occurs frequently in the study of the numerical range, especially when looking at applications. Here, we study the convexity of the numerical range of one linear relation with respect to another linear relation and use it to bound the convex hull of the relative spectrum of these linear relations.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and consider the orthogonal sum $\mathcal{H}\oplus\mathcal{H}$ with inner product

$$\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle . \tag{1.1}$$

Any linear subset $\mathcal{T}$ of $\mathcal{H}\oplus\mathcal{H}$ will be called a linear relation. We say that the linear relation $\mathcal{T}$ is closed if it is closed with respect to the topology generated by the inner product in (1.1) via the norm

$$\|(x, y)\|^2 = \|x\|^2 + \|y\|^2.$$

The graph of a linear operator on $\mathcal{H}$ is a linear subset of $\mathcal{H}\oplus\mathcal{H}$, and thus any operator can be seen as a particular example of a linear relation. Note that a linear relation $\mathcal{T}$ is the graph of a linear operator if and only if

$$\{(x, y) \in \mathcal{T} : x = 0\} = \{(0, 0)\}. \tag{1.2}$$

If $\mathcal{T}$ is a linear relation on a Hilbert space $\mathcal{H}$, we denote by $D(\mathcal{T})$, $R(\mathcal{T})$, and $\ker\mathcal{T}$ the sets

$$\begin{aligned} D(\mathcal{T}) &:= \{x \in \mathcal{H} : (x, y) \in \mathcal{T} \text{ for some } y \in \mathcal{H}\}, \\ R(\mathcal{T}) &:= \{y \in \mathcal{H} : (x, y) \in \mathcal{T} \text{ for some } x \in \mathcal{H}\}, \\ \ker\mathcal{T} &:= \{x \in D(\mathcal{T}) : (x, 0) \in \mathcal{T}\}, \end{aligned}$$

called the domain, range, and kernel of $\mathcal{T}$ respectively, which are linear subsets of $\mathcal{H}$. If $x \in D(\mathcal{T})$, then we define $\mathcal{T}x$ to be the set

$$\mathcal{T}x := \{y \in R(\mathcal{T}) : (x, y) \in \mathcal{T}\}.$$

Let $LR(\mathcal{H})$ denote the collection of all linear relations on a Hilbert space $\mathcal{H}$. Then $\mathcal{T} \in LR(\mathcal{H})$ if for every $x, y \in D(\mathcal{T})$ and any $\alpha \in \mathbb{C}$, we have

1. $\mathcal{T}(x+y) = \mathcal{T}x + \mathcal{T}y$,
2. $\mathcal{T}(\alpha x) = \alpha \mathcal{T}x$.

The equalities in (1) and (2) above are understood to be set equalities.

For $\mathcal{S}, \mathcal{T} \in LR(\mathcal{H})$ and $\zeta \in \mathbb{C}$, we define $\mathcal{T}+\mathcal{S}$, $\zeta\mathcal{T}$, and $\mathcal{T}^{-1}$ to be the linear relations

$$\begin{aligned}
&\mathcal{T}+\mathcal{S} := \{(x, y+z) : (x,y) \in \mathcal{T},\ (x,z) \in \mathcal{S}\}\,,\\
&\zeta\mathcal{T} := \{(x, \zeta y) : (x,y) \in \mathcal{T}\}\,,\\
&\mathcal{T}^{-1} := \{(y,x) : (x,y) \in \mathcal{T})\}\,.
\end{aligned}$$

Let $\mathcal{T}$ be a linear relation on a Hilbert space $\mathcal{H}$. We say that

(1) $\mathcal{T}$ is bounded if there exists a real number $\alpha > 0$ such that $\|y\| \le \alpha\|x\|$ for all $(x,y) \in \mathcal{T}$,

(2) $\mathcal{T}$ is bounded below if there exists a real number $\beta > 0$ such that $\beta\|x\| \le \|y\|$ for all $(x,y) \in \mathcal{T}$.

We see from (2) that for a linear relation $\mathcal{T}$ which is bounded below, $\mathcal{T}^{-1}$ is a single valued bounded linear operator. For more work on linear relations, see [2, 7].

The following result can be found in [7].

Theorem 1.1. *Let $\mathcal{T}$ be a linear relation on a Hilbert space $\mathcal{H}$ and let $x \in D(\mathcal{T})$. Then $y \in \mathcal{T}x$ if and only if $\mathcal{T}x = \mathcal{T}(0) + y$.*

For linear relations $\mathcal{A}$ and $\mathcal{T}$ on a Hilbert space $\mathcal{H}$, we define the numerical range $\Theta_{\mathcal{A}}(\mathcal{T})$ of $\mathcal{T}$ relative to $\mathcal{A}$ to be the subset of $\mathbb{C}$ given by

$$\Theta_{\mathcal{A}}(\mathcal{T}) = \{\langle y,z\rangle : y \in \mathcal{T}x, z \in \mathcal{A}x \text{ with } \|z\| = 1,\ x \in \mathcal{D}(\mathcal{T}) \cap \mathcal{D}(\mathcal{A})\}\,. \quad (1.3)$$

If $\mathcal{A}$ is the identity relation $\mathcal{I}$ on $\mathcal{H}$, then $\Theta_{\mathcal{I}}(\mathcal{T})$ is simply the classical numerical range $\Theta(\mathcal{T})$ of $\mathcal{T}$ given by

$$\Theta(\mathcal{T}) := \{\langle y,x\rangle : y \in \mathcal{T}x, \|x\| = 1\}.$$

2 Main results

2.1 Convexity of the numerical range

We begin by considering a few results that are useful in establishing the Toeplitz-Hausdorff Theorem for linear relations.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{T}$ be linear relations on a Hilbert space $\mathcal{H}$ with $D(\mathcal{T}) = D(\mathcal{A})$. If $\Theta_{\mathcal{A}}\mathcal{T} \neq \mathbb{C}$, then $\mathcal{T}(0) \perp R(\mathcal{A})$.*

Proof. Assume that $\Theta_{\mathcal{A}}(\mathcal{T}) \neq \mathbb{C}$. Then there exists $\alpha \in \mathbb{C}$ such that $\alpha \notin \Theta_{\mathcal{A}}\mathcal{T}$. Since $\mathcal{T}$ is not single valued, $\mathcal{T}(0) \neq \{0\}$. Hence there exists $h \in \mathcal{T}(0), h \neq 0$. Let $k \in D(\mathcal{A})$, $w \in \mathcal{A}k$ with $\|w\| = 1$, and let $w' \in \mathcal{T}k$. Then $w' + \xi h \in \mathcal{T}k$ for every $\xi \in \mathbb{C}$. Then $\langle w' + \xi h, w\rangle \in \Theta_{\mathcal{A}}\mathcal{T}$, for all $\xi \in \mathbb{C}$ which implies

$$\langle w', w\rangle + \xi\langle h, w\rangle \in \Theta_{\mathcal{A}}(\mathcal{T}) \text{ for all } \xi \in \mathbb{C}.$$

If $\langle h, w\rangle \neq 0$, then $\langle w', w\rangle + \xi\langle h, w\rangle = \alpha$ for some suitable choice of ξ, contradicting the fact that $\alpha \notin \Theta_{\mathcal{A}}\mathcal{T}$; that is, $\langle h, w\rangle = 0$ for all $h \in \mathcal{T}0$ and $w \in R(\mathcal{A})$ with $\|w\| = 1$. Hence $\mathcal{T}(0) \perp R(\mathcal{A})$. □

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{T}$ be linear relations on a Hilbert space $\mathcal{H}$ with $D(\mathcal{T}) = D(\mathcal{A})$ such that $\Theta_{\mathcal{A}}(\mathcal{T}) \neq \mathbb{C}$ and let $u \in D(\mathcal{T}) = D(\mathcal{A})$. Then the equality $\langle u_1, w\rangle = \langle u_2, w\rangle$ holds for all $u_1, u_2 \in \mathcal{T}u$ and $w \in R(\mathcal{A})$.*

Proof. Let $u \in D(\mathcal{T}) = D(\mathcal{A})$ and let $u_1, u_2 \in \mathcal{T}u$. By Theorem 1.1, there exists $g \in \mathcal{T}(0)$ such that $u_1 = g + u_2$. Let $w \in R(\mathcal{A})$. It follows from Theorem 2.1 that

$$\langle u_1, w\rangle = \langle g + u_2, w\rangle = \langle g, w\rangle + \langle u_2, w\rangle = 0 + \langle u_2, w\rangle = \langle u_2, w\rangle.$$

□

Lemma 2.3. *Let $\mathcal{A}$ and $\mathcal{T}$ be linear relations on a Hilbert space $\mathcal{H}$ with $D(\mathcal{T}) = D(\mathcal{A})$ such that $\Theta_{\mathcal{A}}(\mathcal{T}) \neq \mathbb{C}$. For $w_1, w_2 \in \Theta_{\mathcal{A}}(\mathcal{T})$ with $w_1 \neq w_2$, let $u, v \in D(\mathcal{T}) = D(\mathcal{A})$ be such that $w_1 = \langle u_1, u_2\rangle$ and $w_2 = \langle v_1, v_2\rangle$ for $u_1 \in \mathcal{T}u$, $u_2 \in \mathcal{A}u$, $v_1 \in \mathcal{T}v$, and $v_2 \in \mathcal{A}v$ with $\|u_2\| = 1 = \|v_2\|$. If $N(\mathcal{A}) = \{0\}$, then $u_2 + zv_2 \neq 0$ for all $z \in \mathbb{C}$.*

Proof. Assume that $u_2 + zv_2 = 0$ for some $z \in \mathbb{C}$. Then

$$u_2 = -zv_2, \tag{2.4}$$

which means that $|z| = 1$ because $\|u_2\| = \|v_2\| = 1$. Since $u_2 \in \mathcal{A}u$ and $v_2 \in \mathcal{A}v$, it follows that $\mathcal{A}u$ and $\mathcal{A}(-zv)$ have an element in common and therefore $\mathcal{A}u = \mathcal{A}(-zv)$. In other words, $\mathcal{A}(u + zv) = \mathcal{A}(0)$ so that

$$u + zv = 0 \tag{2.5}$$

since $N(\mathcal{A}) = \{0\}$. Equality (2.5) implies that

$$u_1, -zv_1 \in \mathcal{T}(-zv). \tag{2.6}$$

Equalities (2.4), (2.6) and Theorem 2.2 imply that

$$w_1 = \langle u_1, u_2\rangle = \langle u_1, -zv_2\rangle = \langle -zv_1, -zv_2\rangle = |z|^2\langle v_1, v_2\rangle = w_2.$$

This contradicts the fact that $w_1 \neq w_2$. We therefore conclude that $u_2 + zv_2 \neq 0$ for any $z \in \mathbb{C}$. □

We now establish the Toeplitz-Hausdorff theorem for the numerical range $\Theta_{\mathcal{A}}(\mathcal{T})$, where $\mathcal{A}$ and $\mathcal{T}$ are linear relations on a Hilbert space $\mathcal{H}$ with $\mathcal{A}$ injective; that is, $N(\mathcal{A}) = \{0\}$.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{T}$ be linear relations on a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is injective, then $\Theta_{\mathcal{A}}(\mathcal{T})$ is convex.*

Proof. If $\Theta_{\mathcal{A}}(\mathcal{T}) = \mathbb{C}$, then there is nothing to prove. Assume therefore that $\Theta_{\mathcal{A}}(\mathcal{T}) \neq \mathbb{C}$ and let $w_1, w_2 \in \Theta_{\mathcal{A}}(\mathcal{T})$ with $w_1 \neq w_2$. Then there exist $u, v \in D(\mathcal{A}) \cap D(\mathcal{T})$ such that $w_1 = \langle v_1, \widehat{v}_1\rangle$ and $w_2 = \langle v_2, \widehat{v}_2\rangle$ with $v_1 \in \mathcal{T}u$, $\widehat{v}_1 \in \mathcal{A}u$, $v_2 \in \mathcal{T}v$, and $\widehat{v}_2 \in \mathcal{A}v$. Lemma (2.3) implies that $\widehat{v}_1 + z\widehat{v}_2 \neq 0$ for any $z \in \mathbb{C}$. Let t be a real number such that $0 < t < 1$. We look for a $z \in \mathbb{C}$ such that the equation

$$\langle v_1 + zv_2, \widehat{v}_1 + z\widehat{v}_2\rangle = (tw_1 + (1-t)w_2)\|\widehat{v}_1 + z\widehat{v}_2\|^2 \tag{2.7}$$

is satisfied. Equation (2.7) means that $\langle \widetilde{v}_1, \widetilde{v}_2\rangle = tw_1 + (1-t)w_2$ is an element of $\Theta_{\mathcal{A}}(\mathcal{T})$ for

$$\widetilde{v}_1 = \frac{v_1 + zv_2}{\|\widehat{v}_1 + z\widehat{v}_2\|} \quad \text{and} \quad \widetilde{v}_2 = \frac{\widehat{v}_1 + z\widehat{v}_2}{\|\widehat{v}_1 + z\widehat{v}_2\|}.$$

Since

$$\begin{aligned}\langle v_1 + zv_2, \widehat{v}_1 + z\widehat{v}_2\rangle &= \langle v_1, \widehat{v}_1\rangle + \langle v_1, z\widehat{v}_2\rangle + \langle zv_2, \widehat{v}_1\rangle + \langle zv_2, z\widehat{v}_2\rangle \\ &= w_1 + \overline{z}\langle v_1, \widehat{v}_2\rangle + z\langle v_2, \widehat{v}_1\rangle + |z|^2 w_2 \end{aligned}\tag{2.8}$$

and

$$\begin{aligned}(tw_1 + (1-t)w_2)\|\widehat{v}_1 + z\widehat{v}_2\|^2 &= (t(w_1 - w_2) + w_2)\langle \widehat{v}_1 + z\widehat{v}_2, \widehat{v}_1 + z\widehat{v}_2\rangle \\ &= (t(w_1 - w_2) + w_2)\left[\langle \widehat{v}_1, \widehat{v}_1\rangle + \langle \widehat{v}_1, z\widehat{v}_2\rangle + \langle z\widehat{v}_2, \widehat{v}_1\rangle + \langle z\widehat{v}_2, z\widehat{v}_2\rangle\right] \\ &= (t(w_1 - w_2) + w_2)\left[1 + \overline{z}\langle \widehat{v}_1, \widehat{v}_2\rangle + z\langle \widehat{v}_2, \widehat{v}_1\rangle + |z|^2\right],\end{aligned}\tag{2.9}$$

substitution of equations (2.8) and (2.9) into (2.7) yields the equation

$$\begin{aligned}&t(w_1 - w_2)|z|^2 + z(\langle \widehat{v}_2, \widehat{v}_1\rangle(t(w_1 - w_2) + w_2) - \langle v_2, \widehat{v}_1\rangle) \\ &+ \overline{z}(\langle \widehat{v}_1, \widehat{v}_2\rangle(t(w_1 - w_2) + w_2) - \langle v_1, \widehat{v}_2\rangle) + (t-1)(w_1 - w_2) = 0.\end{aligned}$$

This equation can be rewritten in the form

$$p|z|^2 + q.z + r.\overline{z} + s = 0, \tag{2.10}$$

where

$$\begin{aligned} p &= t(w_1 - w_2), \\ q &= \langle \widehat{v}_2, \widehat{v}_1 \rangle \left(t(w_1 - w_2) + w_2 \right) - \langle v_2, \widehat{v}_1 \rangle), \\ r &= \langle \widehat{v}_1, \widehat{v}_2 \rangle \left(t(w_1 - w_2) + w_2 \right) - \langle v_1, \widehat{v}_2 \rangle), \text{ and} \\ s &= (t-1)(w_1 - w_2). \end{aligned}$$

Dividing through equation (2.10) by p, we get

$$|z|^2 + \frac{q}{p}.z + \frac{r}{p}.\overline{z} + \frac{s}{p} = 0. \tag{2.11}$$

Let $z = x + iy$, $\left(\dfrac{q}{p}\right) = \alpha + i\beta$, and $\left(\dfrac{r}{p}\right) = \delta + i\gamma$, where x, y, α, β, δ, and γ are some real constants. Substitution of $\left(\dfrac{q}{p}\right)$, $\left(\dfrac{r}{p}\right)$, and $\left(\dfrac{s}{p}\right)$ into equation (2.11)and equating real and imaginary parts to zero gives rise to the simultaneous equations

$$x^2 + y^2 + mx + ny - \frac{(1-t)}{t} = 0 \tag{2.12}$$

and

$$lx + ky = 0, \tag{2.13}$$

where k, l, m, and n are some fixed real constants.

Equation (2.12) represents a circle with the origin as one of the points in its interior (since the constant term in this equation is negative). If at least one of the constants k and l is nonzero, then equation (2.13) represents a straight line through the origin and therefore intersects the circle (2.12) in two distinct points. We can, therefore, always find (at least) two distinct complex numbers $z = z_1$ and $z = z_2$ which satisfy equation (2.7). Hence $\Theta_{\mathcal{A}}(\mathcal{T})$ is convex. □

If $\mathcal{A}$ is the identity relation $\mathcal{I}$ on $\mathcal{H}$, then $\Theta_{\mathcal{A}}(\mathcal{T})$ is the classical numerical range $\Theta(\mathcal{T})$. Therefore, it follows that the classical numerical range $\Theta(\mathcal{T})$ of a linear relation $\mathcal{T}$ is convex.

2.2 The Spectral Inclusion

Let $\mathcal{T}$ be a linear relation on a Hilbert $\mathcal{H}$ and $\mathcal{I}$ the identity linear relation on $\mathcal{H}$; that is, $\mathcal{I}$ is the identity operator on $\mathcal{H}$.

(1) The set $\rho(\mathcal{T}) = \{\lambda \in \mathbb{C} : (\mathcal{T} - \mathcal{I}\lambda)^{-1} \in \mathcal{B}(\mathcal{H})\}$, where $\mathcal{B}(\mathcal{H})$ is the class of bounded linear operators defined on the whole space $\mathcal{H}$ is called the resolvent set of $\mathcal{T}$

(2) The spectrum of $\mathcal{T}$ is the set $\sigma(\mathcal{T}) = \mathbb{C} \setminus \rho(\mathcal{T})$.

A scalar λ such that $N(\mathcal{T} - \lambda\mathcal{I}) \neq \{0\}$ is called an eigenvalue of $\mathcal{T}$. In other words, a scalar λ is called an eigenvalue of $\mathcal{T}$ if there exists a non zero vector $x \in \mathcal{H}$ such that $(x, \lambda x) \in G(\mathcal{T})$. If λ is an eigenvalue of $\mathcal{T}$, then the nonzero subspace $N(\mathcal{T} - \lambda\mathcal{I})$ is called the eigenspace corresponding to eigenvalue λ. Clearly, if λ is an eigenvalue of $\mathcal{T}$, then $\lambda \in \sigma(\mathcal{T})$.

If the identity linear relation $\mathcal{I}$ in the definition of the resolvent set above is replaced by an arbitrary linear relation $\mathcal{A}$, then we get what we call the resolvent set of $\mathcal{T}$ relative to $\mathcal{A}$, given by

$$\rho_{\mathcal{A}}(\mathcal{T}) = \{\lambda \in \mathbb{C} : (\mathcal{T} - \lambda\mathcal{A})^{-1} \in \mathcal{B}(\mathcal{H})\}.$$

As before, we define the spectrum $\sigma_{\mathcal{A}}(\mathcal{T})$ of $\mathcal{T}$ relative to $\mathcal{T}$ to be the set $\mathbb{C} \setminus \rho_{\mathcal{A}}(\mathcal{T})$.

One of the many applications of the numerical range of an operator T is to bound its spectrum, that is, $\sigma(T) \subset \overline{\Theta(T)}$ where $\sigma(T)$ denote the spectrum of T and $\Theta(T)$ is its numerical range. This is the famous Spectral Inclusion Theorem ([4, Thoerem 1.2-1]). In this section, we prove the spectral inclusion theorem for linear relations. The proof is given in terms of the relative spectrum and the relative numerical range of $\mathcal{T}$ with respect to $\mathcal{A}$ defined above.

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{T}$ be everywhere defined linear relations on a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is bounded below with $R(\mathcal{A}) = \mathcal{H}$, then $\sigma_{\mathcal{A}}(\mathcal{T}) \subset \overline{\Theta_{\mathcal{A}}(\mathcal{T})}$.*

Proof. Assume that there exists a $\lambda \in \sigma_{\mathcal{A}}(\mathcal{T})$ such that $\lambda \notin \overline{\Theta_{\mathcal{A}}(\mathcal{T})}$ and let $d\left(\lambda, \overline{\Theta_{\mathcal{A}}(\mathcal{T})}\right)$ denote the distance between λ and $\overline{\Theta_{\mathcal{A}}(\mathcal{T})}$; that is,

$$d\left(\lambda, \overline{\Theta_{\mathcal{A}}(\mathcal{T})}\right) := \inf\{|\mu - \lambda| : \mu \in \overline{\Theta_{\mathcal{A}}(\mathcal{T})}\}.$$

Let $x \in \mathcal{H}$. Since $R(\mathcal{A}) = \mathcal{H}$, there exists $z \in \mathcal{H}$ such that $x \in \mathcal{A}z$. If $y \in \mathcal{T}z$, then the definition of $d\left(\lambda, \overline{\Theta_{\mathcal{A}}(\mathcal{T})}\right)$ above implies that

$$|\langle (y - \lambda x), x\rangle| = \|x\|^2 \left| \frac{\langle y, x\rangle}{\|x\|^2} - \lambda \right| \geq d\left(\lambda, \overline{\Theta_{\mathcal{A}}(\mathcal{T})}\right) \|x\|^2, \tag{2.14}$$

where we have used the fact that $\dfrac{y}{\|x\|} \in \mathcal{T}\left(\dfrac{z}{\|x\|}\right)$ and $\dfrac{x}{\|x\|} \in \mathcal{A}\left(\dfrac{z}{\|x\|}\right)$ since $y \in \mathcal{T}z$ and $x \in \mathcal{A}z$ and both $\mathcal{A}$ and $\mathcal{T}$ are linear.

Application of the Cauchy Schwarz inequality to (2.14) yields

$$\|y - \lambda x\| \geq d\left(\lambda, \overline{\Theta_{\mathcal{A}}(\mathcal{T})}\right) \|x\| \geq \beta \|z\| \tag{2.15}$$

for all $z \in \mathcal{H}$, $y \in \mathcal{T}z$ and $x \in \mathcal{A}z$. The last inequality in (2.15) follows from the fact that $\mathcal{A}$ is bounded below and that $d\left(\lambda, \overline{\Theta_{\mathcal{A}}(\mathcal{T})}\right) > 0$ since $\lambda \notin \overline{\Theta_{\mathcal{A}}(\mathcal{T})}$.

Using (2.15), we conclude that $(\mathcal{T} - \lambda\mathcal{A})^{-1}$ is single valued and bounded. In addition, from (2.14), We conclude that any vector orthogonal to the range $R(\mathcal{T} - \lambda\mathcal{A})$ of $\mathcal{T} - \lambda\mathcal{A}$ must vanish, meaning that $R(\mathcal{T} - \lambda\mathcal{A})$ is dense in $\mathcal{H}$. Thus $\lambda \in \rho_{\mathcal{A}}(\mathcal{T})$, contradicting the fact that $\lambda \in \sigma_{\mathcal{A}}(\mathcal{T})$. This contradiction implies that $\lambda \in \overline{\Theta_{\mathcal{A}}(\mathcal{T})}$ and that $\sigma_{\mathcal{A}}(\mathcal{T}) \subset \overline{\Theta_{\mathcal{A}}(\mathcal{T})}$. □

Note that if we replace the linear relation $\mathcal{A}$ in Theorem 2.5 with the identity relation $\mathcal{I}$, we get the classical inclusion $\sigma(\mathcal{T}) \subset \overline{\Theta(\mathcal{T})}$.

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