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Power Menger algebra of terms induced by order-decreasing transformations and superpositions

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Abstract

In this paper, power sets of the set of terms induced by order-decreasing transformations are introduced. The power Menger algebra which consists of power set of the set of terms and superposition operation satisfying the superassociative law is constructed and some of its algebraic properties are studied.

1 Introduction and preliminaries

Generally, an n-ary term of type τ is inductively defined as follows: The first one is each variable $x_i \in X_n$. The second one is the composition between

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n-ary terms of type τ, $t_1, \ldots, t_{n_i}$, under the n_i-ary operation symbol f_i. Here, $X_n = \{x_1, \ldots, x_n\}$, $n \geq 1$ is a finite set whose elements are called variables and the type is the sequence $\tau = (n_i)_{i \in I}$ of the natural numbers which are arities of the symbols f_i. By $W_\tau(X_n)$, we denote the set of all n-ary terms of type τ. More information and recent trends in the study of terms can be found in [1, 14, 17]. For advanced applications of terms in algebra, we refer the reader to [5, 12, 15, 20].

There are several methods to define special classes of terms. Some of them are terms induced by many kinds of transformations, which have been widely studied by many authors; for example, full terms, strongly full terms, generalized full terms, $K^*(n, r)$-full terms; see [4, 18, 21]. In this paper, we focus our attention on the concept of order-decreasing full terms [22] and type $\tau_n = (n_i)_{i \in I}$, where $n_i = n$ for all $i \in I$. The set of all full transformations on a finite set $\{1, \ldots, n\}$ is denoted by T_n. It is clear that the set

$$OD_n = \{\alpha \in T_n \mid \alpha(k) \leq k, \forall k = 1, \ldots, n\}$$

of all order-decreasing transformations is a submonoid of T_n with respect to a composition of mappings. Formally, an *n-ary order-decreasing full term* of type τ_n is inductively defined by:

(i) If f_i is an n-ary operation symbol of type τ_n and $\alpha \in OD_n$, then $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary order-decreasing full term of type τ_n.

(ii) If f_i is an n-ary operation symbol of type τ_n and $t_1, \ldots, t_n$ are n-ary order-decreasing full terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is an n-ary order-decreasing full term of type τ_n.

The set of all order-decreasing full terms of type τ_n will be denoted by $W_{\tau_n}^{OD_n}(X_n)$.

As an example, let us consider the type $\tau_2 = (2, 2)$ with two binary operation symbols f and g. Since $OD_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \right\}$, we obtain

$$f(x_1, x_1), g(x_1, x_2), f(g(x_1, x_1), f(x_1, x_1)), g(f(x_1, x_1), g(x_1, x_2)) \in W_{\tau_2}^{OD_2}(X_2),$$

whereas $f(x_2, x_1) \notin W_{\tau_2}^{OD_2}(X_2)$ since $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \notin OD_2$ and by the definition, $x_1, x_2, g(x_2, f(x_2, x_1)), f(x_1, g(x_1, x_2)), f(f(x_1, x_1), g(x_2, x_1)) \notin W_{\tau_2}^{OD_2}(X_2)$.

The superposition S^n on $W_{\tau_n}^{OD_n}(X_n)$, $n \in \{1, 2, \ldots\}$, can be applied as the mapping

$$S^n : (W_{\tau_n}^{OD_n}(X_n))^{n+1} \to W_{\tau_n}^{OD_n}(X_n)$$

defined by:

(i) $S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n) = f_i(t_{\alpha(1)},\ldots,t_{\alpha(n)})$, where $\alpha \in OD_n$;

(ii) $S^n(f_i(r_1,\ldots,r_n),t_1,\ldots,t_n) = f_i(S^n(r_1,t_1,\ldots,t_n),\ldots,S^n(r_n,t_1,\ldots,t_n))$.

As a consequence, the algebra $\mathcal{MA}_{OD_n}(\tau_n) := (W^{OD_n}_{\tau_n}(X_n), S^n)$ is constructed. It was proved in [22] that $\mathcal{MA}_{OD_n}(\tau_n)$ is a Menger algebra because the superassociativity is satisfied. More research topics regarding Menger algebras can be found in [2, 7, 8, 13, 19].

A collection of terms is called a *tree language* [9]. This research direction is also connected with universal algebra and fundamental computer science [10]. Various research directions on tree languages were considered in [3, 6, 11, 16].

The main purpose of this paper is to present the notion of power sets of the set of order-decreasing full terms. A construction of superposition operation on these power sets is given. We then form the algebra consisting of such power sets and superposition operation and investigate some related properties.

2 Main result

Let $P(W^{OD_n}_{\tau_n}(X_n))$ be the set of all subsets of the set of order-decreasing full terms of type τ_n. Each element of $P(W^{OD_n}_{\tau_n}(X_n))$ is called a *tree language* which always plays a crucial role in both universal algebra and automata theory. For instance, let $\tau_3 = (3,3)$ be a type with two ternary operation symbols f and g. Then the following are some elements of $P(W^{OD_3}_{\tau_3}(X_3))$:

$$\emptyset, \{f(x_1,x_1,x_2)\}, \{g(x_1,x_2,x_2), g(f(x_1,x_1,x_3), g(x_1,x_2,x_2), f(x_1,x_2,x_3))\}.$$

On the other hand, the following two sets do not belong to $P(W^{OD_3}_{\tau_3}(X_3))$

$$\{f(x_2,x_1,x_3)\}, \{f(x_3,x_1,x_3)\}, \{f(x_1,x_3,x_3), f(x_3,x_2,x_1), f(x_2,x_2,x_2)\}.$$

On the set $P(W^{OD_n}_{\tau_n}(X_n))$, we define the superposition operation

$$S^n_{nd} : (P(W^{OD_n}_{\tau_n}(X_n)))^{n+1} \to P(W^{OD_n}_{\tau_n}(X_n))$$

by, for all $T, T_1, \ldots, T_n$ in $P(W^{OD_n}_{\tau_n}(X_n))$,

(i) if at least one of $T, T_1, \ldots, T_n$ is empty, then $S^n_{nd}(T,T_1,\ldots,T_n) = \emptyset$;

(ii) if $T = \{f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})\}$ where $\alpha \in OD_n$, then $S^n_{nd}(T,T_1,\ldots,T_n) = \{f_i(t_{\alpha(1)},\ldots,t_{\alpha(n)}) \mid t_{\alpha(k)} \in T_{\alpha(k)}, 1 \le k \le n\}$;

(iii) if $T = \{f_i(t_1, \ldots, t_n)\}$, then

$$\begin{aligned} & S^n_{nd}(T, T_1, \ldots, T_n) \\ = \; & \{f_i(r_1, \ldots, r_n) \mid r_k \in S^n_{nd}(\{t_k\}, T_1, \ldots, T_n), 1 \leq k \leq n\}; \end{aligned}$$

(iv) if T is an arbitrary subset of $W^{OD_n}_{\tau_n}(X_n)$, then

$$S^n_{nd}(T, T_1, \ldots, T_n) = \bigcup_{t \in T} S^n_{nd}(\{t\}, T_1, \ldots, T_n).$$

Example 2.1. *Let $t_3 = (3, 3)$ be a type with ternary operation symbols f and g. Consider $T = \{f(x_1, x_1, x_2), g(x_1, x_2, x_2), f(x_1, x_2, x_3)\}, T_1 = \{f(x_1, x_1, x_1)\}$, $T_2 = \{f(x_1, x_2, x_1)\}$ and $T_3 = \{g(x_1, x_2, x_3)\}$ in $P(W^{OD_3}_{\tau_3}(X_3))$. We have*

$$\begin{aligned} S^3_{nd}(\{f(x_1, x_1, x_2)\}, T_1, T_2, T_3) &= \{f(t_1, t_1, t_2) \mid t_1 \in T_1, t_2 \in T_2\} \\ &= \{f(f(x_1, x_1, x_1), f(x_1, x_1, x_1), f(x_1, x_2, x_1))\}, \\ S^3_{nd}(\{g(x_1, x_2, x_2)\}, T_1, T_2, T_3) &= \{g(t_1, t_2, t_2) \mid t_1 \in T_1, t_2 \in T_2\} \\ &= \{g(f(x_1, x_1, x_1), f(x_1, x_2, x_1), f(x_1, x_2, x_1))\}, \\ S^3_{nd}(\{f(x_1, x_2, x_3), T_1, T_2, T_3\}) &= \{f(t_1, t_2, t_3) \mid t_1 \in T_1, t_2 \in T_2, t_3 \in T_3\} \\ &= \{f(f(x_1, x_1, x_1), f(x_1, x_2, x_1), g(x_1, x_2, x_3))\}. \end{aligned}$$

It follows that

$$\begin{aligned} & S^3_{nd}(T, T_1, T_2, T_3) \\ = \; & S^3_{nd}(\{f(x_1, x_1, x_2)\}, T_1, T_2, T_3) \cup S^3_{nd}(\{g(x_1, x_2, x_2)\}, T_1, T_2, T_3) \\ & \cup S^3_{nd}(\{f(x_1, x_2, x_3), T_1, T_2, T_3\}) \\ = \; & \{f(f(x_1, x_1, x_1), f(x_1, x_1, x_1), f(x_1, x_2, x_1))\} \cup \{g(f(x_1, x_1, x_1), f(x_1, x_2, x_1), \\ & f(x_1, x_2, x_1))\} \cup \{f(f(x_1, x_1, x_1), f(x_1, x_2, x_1), g(x_1, x_2, x_3))\} \\ = \; & \{f(f(x_1, x_1, x_1), f(x_1, x_1, x_1), f(x_1, x_2, x_1)), g(f(x_1, x_1, x_1), f(x_1, x_2, x_1), \\ & f(x_1, x_2, x_1)), f(f(x_1, x_1, x_1), f(x_1, x_2, x_1), g(x_1, x_2, x_3))\}. \end{aligned}$$

Now we can form the algebra $(P(W^{OD_n}_{\tau_n}(X_n)), S^n_{nd})$ and present some of its interesting properties.

For an order-decreasing full term t of type τ_n, we need the concept of an order-decreasing full term t_β induced by the original order-decreasing term t, where $\beta \in OD_n$. This can be defined inductively as follows:

(1) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ for $i \in I, \alpha \in OD_n$, then

$$t_\beta = f_i(x_{\beta(\alpha(1))}, \ldots, x_{\beta(\alpha(n))}).$$

(2) If $t = f_i(t_1, \ldots, t_n)$, then $t_\beta = f_i((t_1)_\beta, \ldots, (t_n)_\beta)$.

Clearly, t_β is an element in $W_{\tau_n}^{OD_n}(X_n)$. For any nonempty subset T of $W_{\tau_n}^{OD_n}(X_n)$, we define

$$T_\beta := \{t_\beta \mid t \in T \subseteq W_{\tau_n}^{OD_n}(X_n)\}.$$

If $T = \emptyset$, then we define $T_\beta := \emptyset$.

Example 2.2. *Let $t_3 = (3,3)$ be a type with ternary operation symbols f and g. Consider $T = \{f(x_1, x_1, x_3), f(x_1, x_2, x_2), g(x_1, x_2, x_3)\}$ in $P(W_{\tau_3}^{OD_3}(X_3))$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ in OD_3. Then*

$$\begin{aligned} T_\beta &= \{f(x_1, x_1, x_3)_\beta, f(x_1, x_2, x_2)_\beta, g(x_1, x_2, x_3)_\beta\} \\ &= \{f(x_{\beta(1)}, x_{\beta(1)}, x_{\beta(3)}), f(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(2)}), g(x_{\beta(1)}, x_{\beta(2)}, x_{\beta(3)})\} \\ &= \{f(x_1, x_1, x_1), f(x_1, x_2, x_2), g(x_1, x_2, x_1)\}. \end{aligned}$$

We then prove the following propositions.

Proposition 2.3. *Let n be a fixed positive integer. If $T, T_1, \ldots, T_n \in P(W_{\tau_n}^{OD_n}(X_n))$ and $\alpha \in OD_n$, then*

$$S_{nd}^n(T_\alpha, T_1, \ldots, T_n) = S_{nd}^n(T, T_{\alpha(1)}, \ldots, T_{\alpha(n)}).$$

Proof. If $T = \emptyset$, the proof is clear. Let $T = \{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\}$ where $\beta \in OD_n$. Then $T_\alpha = \{f_i(x_{\alpha(\beta(1))}, \ldots, x_{\alpha(\beta(n))})\}$. Thus, we have

$$\begin{aligned} S_{nd}^n(T_\alpha, T_1, \ldots, T_n) &= S_{nd}^n(\{f_i(x_{\alpha(\beta(1))}, \ldots, x_{\alpha(\beta(n))})\}, T_1, \ldots, T_n) \\ &= \{f_i(t_{\alpha(\beta(1))}, \ldots, t_{\alpha(\beta(n))}) \mid t_{\alpha(\beta(k))} \in T_{\alpha(\beta(k))}, 1 \le k \le n\} \\ &= S_{nd}^n(\{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\}, T_{\alpha(1)}, \ldots, T_{\alpha(n)}) \\ &= S_{nd}^n(T, T_{\alpha(1)}, \ldots, T_{\alpha(n)}). \end{aligned}$$

Let $T = \{f_i(r_1, \ldots, r_n)\}$ and assume that, for all $1 \le k \le n$,

$$S_{nd}^n(\{r_k\}_\alpha, T_1, \ldots, T_n) = S_{nd}^n(\{r_k\}, T_{\alpha(1)}, \ldots, T_{\alpha(n)}).$$

Then $T_\alpha = \{f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)})\}$ and so

$$\begin{aligned} & S_{nd}^n(T_\alpha, T_1, \ldots, T_n) \\ = \ & S_{nd}^n(\{f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)})\}, T_1, \ldots, T_n) \\ = \ & \{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in S_{nd}^n(\{r_{\alpha(k)}\}, T_1, \ldots, T_n), 1 \le k \le n\} \\ = \ & \{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in S_{nd}^n(\{r_k\}_\alpha, T_1, \ldots, T_n), 1 \le k \le n\} \\ = \ & \{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in S_{nd}^n(\{r_k\}, T_{\alpha(1)}, \ldots, T_{\alpha(n)})\} \\ = \ & S_{nd}^n(\{f_i(r_1, \ldots, r_n)\}, T_{\alpha(1)}, \ldots, T_{\alpha(n)}) \\ = \ & S_{nd}^n(T, T_{\alpha(1)}, \ldots, T_{\alpha(n)}). \end{aligned}$$

If T is an arbitrary subset of $W_{\tau_n}^{OD_n}(X_n)$, then we obtain
$S^n_{nd}(T_\alpha, T_1, \ldots, T_n) = \bigcup_{t\in T} S^n_{nd}(\{t_\alpha\}, T_1, \ldots, T_n) = \bigcup_{t\in T} S^n_{nd}(\{t\}_\alpha, T_1, \ldots, T_n) = \bigcup_{t\in T} S^n_{nd}(\{t\}, T_{\alpha(1)}, \ldots, T_{\alpha(n)}) = S^n_{nd}(T, T_{\alpha(1)}, \ldots, T_{\alpha(n)})$.

□

Proposition 2.4. *Let n be a fixed positive integer. If $T, T_1, \ldots, T_n \in P(W_{\tau_n}^{OD_n}(X_n))$ and $\alpha \in OD_n$, then*

$$S^n_{nd}(T_\alpha, T_1, \ldots, T_n) = (S^n_{nd}(T, T_1, \ldots, T_n))_\alpha.$$

Proof. If T is empty, the theorem is clear. Let $T = \{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\}$ where $\beta \in OD_n$. Then

$$\begin{aligned}
S^n_{nd}(T_\alpha, T_1, \ldots, T_n) &= S^n_{nd}((\{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\})_\alpha, T_1, \ldots, T_n)\\
&= S^n_{nd}(\{f_i(x_{\alpha(\beta(1))}, \ldots, x_{\alpha(\beta(n))})\}, T_1, \ldots, T_n)\\
&= \{f_i(t_{\alpha(\beta(1))}, \ldots, t_{\alpha(\beta(n))}) \mid t_{\alpha(\beta(k))} \in T_{\alpha(\beta(k))}, 1 \le k \le n\}\\
&= \{f_i(t_{\beta(1)}, \ldots, t_{\beta(n)})_\alpha \mid t_{\beta(k)} \in T_{\beta(k)}, 1 \le k \le n\}\\
&= (\{f_i(t_{\beta(1)}, \ldots, t_{\beta(n)}) \mid t_{\beta(k)} \in T_{\beta(k)}, 1 \le k \le n\})_\alpha\\
&= (S^n_{nd}(\{f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})\}, T_1, \ldots, T_n))_\alpha\\
&= (S^n_{nd}(T, T_1, \ldots, T_n))_\alpha.
\end{aligned}$$

Let $T = \{f_i(r_1, \ldots, r_n)\}$ and assume that, for all $1 \le k \le n$,

$$S^n_{nd}(\{r_k\}_\alpha, T_1, \ldots, T_n) = (S^n_{nd}(\{r_k\}, T_1, \ldots, T_n))_\alpha.$$

Then

$$\begin{aligned}
&S^n_{nd}(T_\alpha, T_1, \ldots, T_n)\\
=\ &S^n_{nd}((\{f_i(r_1, \ldots, r_n)\})_\alpha, T_1, \ldots, T_n)\\
=\ &S^n_{nd}(\{f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)})\}, T_1, \ldots, T_n)\\
=\ &\{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in S^n_{nd}(\{r_{\alpha(k)}\}, T_1, \ldots, T_n), 1 \le k \le n\}\\
=\ &\{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in S^n_{nd}(\{r_k\}_\alpha, T_1, \ldots, T_n), 1 \le k \le n\}\\
=\ &\{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in (S^n_{nd}(\{r_k\}, T_1, \ldots, T_n))_\alpha, 1 \le k \le n\}\\
=\ &\{f_i(s_1, \ldots, s_n)_\alpha \mid (s_k)_\alpha \in (S^n_{nd}(\{r_k\}, T_1, \ldots, T_n))_\alpha, 1 \le k \le n\}\\
=\ &(\{f_i(s_1, \ldots, s_n) \mid s_k \in S^n_{nd}(\{r_k\}, T_1, \ldots, T_n)\})_\alpha, 1 \le k \le n\\
=\ &(S^n_{nd}(\{f_i(r_1, \ldots, r_n)\}, T_1, \ldots, T_n))_\alpha\\
=\ &(S^n_{nd}(T, T_1, \ldots, T_n))_\alpha.
\end{aligned}$$

If T is an arbitrary subset of $W_{\tau_n}^{OD_n}(X_n)$, then
$S^n_{nd}(T_\alpha, T_1, \ldots, T_n) = \bigcup_{t\in T} S^n_{nd}(\{t_\alpha\}, T_1, \ldots, T_n) = \bigcup_{t\in T} S^n_{nd}(\{t\}_\alpha, T_1, \ldots, T_n)$

$= \bigcup_{t\in T}(S^n_{nd}(\{t\}, T_1, \ldots, T_n))_\alpha = (\bigcup_{t\in T} S^n_{nd}(\{t\}, T_1, \ldots, T_n))_\alpha =$ $(S^n_{nd}(T, T_1, \ldots, T_n))_\alpha$. The proof is complete. □

By Propositions 2.3 and 2.4, we have the following corollary.

Corollary 2.5. *Let n be a fixed positive integer. If $T, T_1, \ldots, T_n \in P(W^{OD_n}_{\tau_n}(X_n))$ and $\alpha \in OD_n$, then*

$$S^n_{nd}(T, T_{\alpha(1)}, \ldots, T_{\alpha(n)}) = (S^n_{nd}(T, T_1, \ldots, T_n))_\alpha.$$

Theorem 2.6. *Let n be a fixed positive integer. If $T, T_1, \ldots, T_n, S_1, \ldots, S_n \in P(W^{OD_n}_{\tau_n}(X_n))$, then S^n_{nd} satisfies the superassociative law:*

$$S^n_{nd}(S^n_{nd}(T, S_1, \ldots, S_n), T_1, \ldots, T_n)$$
$$= S^n_{nd}(T, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n)).$$

Proof. If $T = \emptyset$, the proof is obvious. Without loss of generality, we may assume that all of $T_1, \ldots, T_n, S_1, \ldots, S_n$ are nonempty. Let $T = \{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}$, where $\alpha \in OD_n$. Then

$$\begin{aligned}
& S^n_{nd}(S^n_{nd}(T, S_1, \ldots, S_n), T_1, \ldots, T_n) \\
= \;& S^n_{nd}(S^n_{nd}(\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}, S_1, \ldots, S_n), T_1, \ldots, T_n) \\
= \;& S^n_{nd}(\{f_i(s_{\alpha(1)}, \ldots, s_{\alpha(n)}) \mid s_{\alpha(k)} \in S_{\alpha(k)}, 1 \le k \le n\}, T_1, \ldots, T_n) \\
= \;& \{f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)}) \mid r_{\alpha(k)} \in S^n_{nd}(\{s_{\alpha(k)} \mid s_{\alpha(k)} \in S_{\alpha(k)}, 1 \le k \le n\}, \\
& T_1, \ldots, T_n)\} \\
= \;& \{f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)}) \mid r_{\alpha(k)} \in S^n_{nd}(S_{\alpha(k)}, T_1, \ldots, T_n), 1 \le k \le n\} \\
= \;& S^n_{nd}(\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n)) \\
= \;& S^n_{nd}(T, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n)).
\end{aligned}$$

Now let $T = \{f_i(r_1, \ldots, r_n)\}$, where $r_1, \ldots, r_n \in W^{OD_n}_{\tau_n}(X_n)$ and assume that, for all $1 \le k \le n$, the theorem is satisfied. Then we have

$$\begin{aligned}
& S^n_{nd}(S^n_{nd}(T, S_1, \ldots, S_n), T_1, \ldots, T_n) \\
= \;& S^n_{nd}(S^n_{nd}(\{f_i(r_1, \ldots, r_n)\}, S_1, \ldots, S_n), T_1, \ldots, T_n) \\
= \;& S^n_{nd}(\{f_i(s_1, \ldots, s_n) \mid s_k \in S^n_{nd}(\{r_k\}, S_1, \ldots, S_n), 1 \le k \le n\}, T_1, \ldots, T_n) \\
= \;& \{f_i(p_1, \ldots, p_n) \mid p_k \in S^n_{nd}(\{s_k \mid s_k \in S^n_{nd}(\{r_k\}, S_1, \ldots, S_n)\}, T_1, \ldots, T_n), \\
& 1 \le k \le n\} \\
= \;& \{f_i(p_1, \ldots, p_n) \mid p_k \in S^n_{nd}(S^n_{nd}(\{r_k\}, S_1, \ldots, S_n), T_1, \ldots, T_n), 1 \le k \le n\} \\
= \;& \{f_i(p_1, \ldots, p_n) \mid p_k \in S^n_{nd}(\{r_k\}, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, \\
& S^n_{nd}(S_n, T_1, \ldots, T_n))\} \\
= \;& S^n_{nd}(\{f_i(r_1, \ldots, r_n)\}, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n)) \\
= \;& S^n_{nd}(T, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n)).
\end{aligned}$$

If T is an arbitrary subset of $W_{\tau_n}^{OD_n}(X_n)$, then we obtain

$$\begin{aligned}
& S^n_{nd}(S^n_{nd}(T, S_1, \ldots, S_n), T_1, \ldots, T_n) \\
= \; & S^n_{nd}(\bigcup_{t \in T} S^n_{nd}(\{t\}, S_1, \ldots, S_n), T_1, \ldots, T_n) \\
= \; & \bigcup_{t \in T} S^n_{nd}(S^n_{nd}(\{t\}, S_1, \ldots, S_n), T_1, \ldots, T_n) \\
= \; & \bigcup_{t \in T} S^n_{nd}(\{t\}, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n)) \\
= \; & S^n_{nd}(T, S^n_{nd}(S_1, T_1, \ldots, T_n), \ldots, S^n_{nd}(S_n, T_1, \ldots, T_n)).
\end{aligned}$$

This finishes the proof. □

Using Theorem 2.6, the algebra $(P(W_{\tau_n}^{OD_n}(X_n)), S^n_{nd})$ is an example of a Menger algebra and is called the *power Menger algebra of terms induced by order-decreasing transformations.*

Finally, as we mentioned in Section 1, we study the relationship between the Menger algebra of order-decreasing full terms and the power Menger algebra of terms induced by order-decreasing transformations. To do this, we need the following technical lemma.

Lemma 2.7. *Let $t, t_1, \ldots, t_n \in W_{\tau_n}^{OD_n}(X_n)$. Then the equation*

$$\{S^n(t, t_1, \ldots, t_n)\} = S^n_{nd}(\{t\}, \{t_1\}, \ldots, \{t_n\}),$$

holds.

Proof. We give a proof by induction on the complexity of the order-decreasing full term t. We begin with the case when $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ and $\alpha \in OD_n$. Then $\{S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n)\} = \{f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)})\}$ $= S^n_{nd}(\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})\}, \{t_1\}, \ldots, \{t_n\}) = S^n_{nd}(\{t\}, \{t_1\}, \ldots, \{t_n\})$. Now assume that $t = f_i(s_1, \ldots, s_n)$ and

$$\{S^n(s_k, t_1, \ldots, t_n)\} = \widehat{S}^n(\{s_k\}, \{t_1\}, \ldots, \{t_n\}),$$

for every $k = 1, \ldots, n$. Then

$$\begin{aligned}
& \{S^n(t, t_1, \ldots, t_n)\} \\
= \; & \{S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n)\} \\
= \; & \{f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n))\} \\
= \; & \{f_i(r_1, \ldots, r_n) \mid r_k \in \{S^n(s_k, t_1, \ldots, t_n)\}, 1 \leq k \leq n\} \\
= \; & \{f_i(r_1, \ldots, r_n) \mid r_k \in \widehat{S}^n(\{s_k\}, \{t_1\}, \ldots, \{t_n\}), 1 \leq k \leq n\} \\
= \; & \widehat{S}^n(\{f_i(s_1, \ldots, s_n)\}, \{t_1\}, \ldots, \{t_n\}) \\
= \; & \widehat{S}^n(\{t\}, \{t_1\}, \ldots, \{t_n\}).
\end{aligned}$$

□

We can now prove the following theorem.

Theorem 2.8. *The algebra* $(W_{\tau_n}^{OD_n}(X_n), S^n)$ *can be embedded in to the algebra* $(P(W_{\tau_n}^{OD_n}(X_n)), S_{nd}^n)$.

Proof. Let t be an order-decreasing full term of type τ_n. Define a mapping $\rho : W_{\tau_n}^{OD_n}(X_n) \to P(W_{\tau_n}^{OD_n}(X_n))$ by $\rho(t) = \{t\}$. Obviously, ρ is an injective mapping. We now prove that this mapping preserves the operations of those two algebras. To do this, let $t, t_1, \ldots, t_n$ be arbitrary elements in $W_{\tau_n}^{OD_n}(X_n)$. Then $\rho(S^n(t, t_1, \ldots, t_n)) = \{S^n(t, t_1, \ldots, t_n)\} = S_{nd}^n(\{t\}, \{t_1\}, \ldots, \{t_n\}) = S_{nd}^n(\rho(t), \rho(t_1), \ldots, \rho(t_n))$ by Lemma 2.7. This shows that the mapping ρ is monomorphism from the algebra $(W_{\tau_n}^{OD_n}(X_n), S^n)$ to $(P(W_{\tau_n}^{OD_n}(X_n)), S_{nd}^n)$. □

3 Conclusions

It is commonly known that tree languages have a closed connection with theoretical computer science, especially the theory of automata. Sets of terms can be regarded as tree languages in the sense of algebraic machine. In this paper, we studied a specific kind of tree languages generated by order-decreasing full terms. Under the superposition of tree languages, several properties were studied and the Power Menger algebra consisting of power set of of the set of order-decreasing full terms and superposition satisfying the superassociative law was constructed. As a result, we found that there was a monomorphism from the Menger algebra of order-decreasing full terms to the power Menger algebra of them.

References

[1] K. Denecke, Partial clones, Asian-Eur. J. Math., **13,** (2020), No. 8, Article ID: 2050161, 19 pages.

[2] K. Denecke, P. Glubudom, Green's quasiorder on Menger algebras of terms, Thai J. Math., **4,** (2006), 49–61.

[3] K. Denecke, P. Glubudom, J. Koppitz, Power clones and non-deterministic hypersubstitutions, Asian-Eur. J. Math., **1,** (2008), 115–128.

[4] K. Denecke, P. Jampachon, Clones of full terms, Algebra Discrete Math., **4,** (2004), 1–11.

[5] K. Denecke, J. Koppitz, S. Shtrakov, Multi-hypersubstitutions and colored solid varieties, Int. J. Algebra Comput., **16,** (2006), 797–815.

[6] K. Denecke, N. Sarasit, Semigroups of tree languages, Asian-Eur. J. Math., **1,** (2008), 489–507.

[7] W. A. Dudek, V. S. Trokhimenko, Menger algebras of associative and self-distributive n-ary operations, Quasigroups Relat. Syst., **26,** (2018), 45–52.

[8] W. A. Dudek, V. S. Trokhimenko, Menger algebras of idempotent n-ary operations, Stud. Sci. Math. Hung., **55,** (2019), 260–269.

[9] F. Gecseg, M. Steinby, Tree languages, in: Handbook of Formal Languages, Vol. 3, Chapter 1, Tree Languages, pp. 1-68, Springer-Verlag 1997.

[10] J. Hopcroft, J. Ullmann, Introduction to Automata Theory, Languages, and Computation, Addison-Wesley Longman, 1990.

[11] P. Kitpratyakul, B. Pibaljommee, Semigroups of linear tree languages, Asian-Eur. J. Math., **11,** (2018), no. 6, Article ID: 1850091, 19 pages.

[12] J. Koppitz, All *Reg*-solid varieties of commutative semigroups, Semigroup Forum, **78,** (2009), 148–156.

[13] T. Kumduang, S. Leeratanavalee, Menger hyperalgebras and their representations, Commun. Algebra, **49,** (2021), no. 4, 1513–1533. DOI: 10.1080/00927872.2020.1839089

[14] T. Kumduang, S. Leeratanavalee, Semigroups of terms, tree languages, Menger algebra of n-ary functions and their embedding theorems, symmetry, **13,** (2021), no. 4, 558, https://doi.org/10.3390/sym13040558

[15] S. Leeratanavalee, Outermost-strongly solid variety of commutative semigroups, Thai J. Math., **14,** (2016), no. 2, 305–313.

[16] S. Lekkoksung, Monoid of nd-full hypersubstitutions, Discuss. Math. Gen. Algebra Appl., **39,** (2019), 165–179.

[17] D. Phusanga, J. Koppitz, The semigroup of linear terms, Asian-Eur. J. Math., **13**, (2020), no. 1, Article ID: 20500059, 9 pages.

[18] S. Phuapong, S. Leeratanavalee, The algebra of generalized full terms, Int. J. Open Probl. Comput. Sci. Math., **4,** (2011), 54–65.

[19] B. M. Schein, V. S.Trokhimenko, Algebras of multiplace functions, Semigroup Forum, **17,** (1979), 1–64.

[20] S. Shtrakov, J. Koppitz, Stable varieties of semigroups and groupoids, Algebra Univers., **75,** (2016), 85–106.

[21] K. Wattanatripop, T. Changphas, The clone of $K^*(n, r)$-full terms, Discuss. Math. Gen. Algebra Appl., **39,** (2019), 277–288.

[22] K. Wattanatripop, T. Changphas, The Menger algebra of terms induced by order-decreasing transformations, Commun. Algebra, (2021), DOI: 10.1080/00927872.2021.1888385