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A Study on Moments of Generalized Order Statistics from Lomax-Exponential Distribution and Characterization

Yousef F. Alharbi

Department of Mathematics
College of Science
Taibah University
Madinah, Saudi Arabia

email: ymatrafe@taibahu.edu.sa

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Abstract

The main objective of this paper is to study the moment properties of generalized order statistics based on Lomax-exponential distribution. We obtain the explicit expression and recurrence relations between moments of generalized order statistics. Moreover, we deduce results for order statistics and record values and some numerical computations are carried out. Furthermore, we give a characterization of the considered distribution through moment properties.

1 Introduction

The Lomax-exponential distribution (LED) was first proposed in 2015 by Ijaz et al. [22] with some applications on real life problems. It gives more flexibility compared to their initial distributions (Lomax or exponential distributions). This distribution can be used in the analysis of reliability modeling, income and wealth inequality data, business failure life time data, medical, biological data and in other similar fields.

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A modified version of this distribution was proposed in 2018 by Ieran and Kuhe [21]. The survival function for this distribution (SF) is given as

$$\bar{F}(x) = \beta^{\alpha} \left[\beta + \lambda x \right]^{-\alpha}, \quad x > 0; \ \alpha, \beta, \lambda > 0 \tag{1.1}$$

with the corresponding probability density function (p.d.f)

$$f(x) = \alpha \lambda \beta^{\alpha} \left[\beta + \lambda x \right]^{-(\alpha+1)}, \quad x > 0; \ \alpha, \beta, \lambda > 0, \tag{1.2}$$

where $\bar{F}(x) = 1 - F(x)$. For more details about the properties of this distribution, see [22], [21] and [35]. In view of (1.1) and (1.2), we note that

$$\bar{F}(x) = \left(\frac{\beta}{\alpha \lambda} + \frac{1}{\alpha} x \right) f(x). \tag{1.3}$$

1.1 Definition of GOS

The general models of ordered random variables play important roles in the applications of statistical theory. In 1995, Kamps [23] defined the concept of the generalized order statistics (GOS). Moreover, he showed that all well known models of ordered random variables such as order statistics, record values, Pfeifer's records and so on are the sub models of GOS in the distributional and theoretical sense.

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j \geq 0 \ \text{ for } \ 1 \leq i \leq n-1.$$

The random variables $X(1, n, \tilde{m}, k), X(2, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are said to be GOS from an absolutely continuous population with cumulative distribution function (CDF) $F()$ and p.d.f $f()$, if their joint p.d.f is of the form

$$k \Big(\prod_{j=1}^{n-1} \gamma_j \Big) \Big(\prod_{i=1}^{n-1} \big[1 - F(x_i) \big]^{m_i} f(x_i) \Big) \big[1 - F(x_n) \big]^{k-1} f(x_n) \tag{1.4}$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$.

1.2 Particular Cases of GOS

Particular cases of model (1.4) are given as follows:

- The joint density of order statistics is obtained from the model (1.4) by putting $m_1 = m_2 = ... = m_{n-1} = 0$ and $k = 1$, so that $\gamma_r = n - r + 1,\ 1 \leq r \leq n - 1$.

- The joint density based on progressively type-II censored order statistics can be deduced from the model (1.4) by choosing $n = m$, $m_i = R_i$ for $i = 1, 2, ..., m - 1$ and $k = R_m + 1$. Then $\gamma_r = m - r + 1 + \sum_{i=r}^{m} R_i,\ 1 \leq r \leq m$ where R_i is a set of prefixed integer that shows R_i random removal at i^{th} failure from surviving items of an experiment.

- The joint density of upper record values is found from the model (1.4) by letting $m_1 = m_2 = ...m_{n-1} = -1$ and $k = 1$, so that $\gamma_r = 1,\ 1 \leq r \leq n - 1$.

- The joint density of sequential order statistics is extracted from the model (1.4) by supposing $m_i = (n - i + 1)\,\alpha_i - (n - i)\,\alpha_{i+1} - 1$ and $k = \alpha_n,\ \alpha \in R^+,\ i = 1, 2, ..., n-1$. Then $\gamma_r = (n - r + 1)\,\alpha_r,\ 1 \leq r \leq n - 1$.

1.3 Marginal and Joint p.d.f of GOS

There are two cases to be considered:

Case I. $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n-1, i \neq j$.

In view of (1.4), the p.d.f of r^{th} *gos* $X(r, n, \tilde{m}, k)$ is given in [24]

$$f_{X(r,n,\tilde{m},k)}(x) \;=\; C_{r-1} \sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i - 1} f(x), \tag{1.5}$$

where

$$C_{r-1} \;=\; \prod_{i=1}^{r} \gamma_i, \qquad \gamma_i = k + n - i + \sum_{j=1}^{n-1} m_j > 0,$$

and

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \qquad 1 \leq i \leq r \leq n.$$

The joint p.d.f of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \leq r < s \leq n$, is given in [24]

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = C_{s-1} \sum_{j=r+1}^{s} a_j{}^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_j} \times \left[\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right] \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}, \quad x < y, \tag{1.6}$$

where

$$a_j{}^{(r)}(s) = \prod_{\substack{l=r+1 \\ l \neq j}}^{s} \frac{1}{(\gamma_l - \gamma_j)}, \qquad r+1 \leq j \leq s \leq n.$$

Case II : $m_i = m$, $i = 1, 2, \ldots, n-1$.

The p.d.f of r^{th} *gos* $X(r,n,m,k)$ is given in [23]

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)), \tag{1.7}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i \quad , \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & , \quad m \neq -1 \\ \log\left(\dfrac{1}{1-x}\right) & , \quad m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \quad x \in [0,1).$$

The joint p.d.f of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \leq r < s \leq n$, is given in [37]

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!} [\bar{F}(x)]^m g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(x)\, f(y), \quad -\infty \leq x < y \leq \infty. \tag{1.8}$$

1.4 Literature Review

Recurrence relations of moments for GOS for different distributions have been studied by many authors in the literature. For more details, see [1], [5], [6], [9], [10], [11], [12], [14], [15], [16], [24], [25], [30], [31], [32], [33], [36], [37], [38], [39], [41] and references therein.

The characterization of probability distribution have significant applications in natural and applied sciences and plays an essential role in statistical studies. There is several approaches to characterize a probability distribution. For more details about characterization, one may refer to [3], [4], [7], [8], [13], [18], [26], [27], [28], [29], [34], [40] and references therein.

1.5 Gauss' Hypergeometric Function

To find the expectation properties of GOS for Lomax-exponential distribution, we will require to use Gauss' hypergeometric function to find the exact results. It is defined as

$$ {}_2F_1 = \sum_{u=0}^{\infty} \frac{(a)_u \, (b)_u}{(c)_u} \frac{x^u}{u!}, \tag{1.9} $$

where $c \neq 0, -1, -2, -3, \ldots$. This function converges if any of the following conditions is satisfied

$$ \begin{aligned} &(i)\, |x| < 1; \\ &(ii)\, |x| = 1, Re(c - a - b) > 0. \end{aligned} $$

For more details, see [2].

This paper is organized as follows. In the section 2, we study the single moments of GOS for LED, whereas the product moments are discussed in section 3. Moreover, some special cases of single and product moment properties of GOS are demonstrated. Furthermore, the first four moments of order statistics and upper record values are tabulated. In section 4, we obtain characterization results based on the moments properties are obtained. Finally, we conclude our paper in section 5.

2 Single Moments

This section contains the exact expression as well as the relation between moments of GOS based on LED. For convenience with the calculation and discussion, we consider

$$E[X^p(r,n,\tilde{m},k)] = \mu^{(p)}_{r,n,\tilde{m},k}.$$

Theorem 2.1. *Suppose Case I is satisfied. For LED given in* (1.1) *and for* $n \in N, \tilde{m} \in \mathbb{R},\ k > 0,\ 1 \le r \le n,\ p = 1,\ 2,\ ...$

$$\mu^{(p)}_{r,n,\tilde{m},k} = \alpha\, C_{r-1} \left(\frac{\beta}{\lambda}\right)^p \sum_{i=1}^{r} a_i(r)\, B\left(p+1,\, \alpha\,\gamma_i - p\right), \tag{2.10}$$

where $p \le \alpha\,\gamma_i$ *and* $B(a,\, b) = \int_0^1 x^{a-1}\,(1-x)^{b-1}\,dx = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$, *which is the complete beta function.*

Proof. In view of (1.1), (1.2) and (1.5), we obtain

$$\begin{aligned}
\mu^{(p)}_{r,n,\tilde{m},k} &= C_{r-1} \int_0^\infty x^p \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i-1} f(x)\,dx \\
&= C_{r-1} \int_0^\infty x^p \sum_{i=1}^{r} a_i(r) \left[\beta^\alpha \left[\beta + \lambda\,x\right]^{-\alpha}\right]^{\gamma_i-1} \\
&\times \left\{\alpha\,\lambda\,\beta^\alpha \left[\beta + \lambda\,x\right]^{-(\alpha+1)}\right\} dx \\
&= \frac{\alpha\,\lambda}{\beta}\, C_{r-1} \sum_{i=1}^{r} a_i(r) \int_0^\infty x^p \left(1 + \frac{\lambda}{\beta}\,x\right)^{-(\alpha\,\gamma_i+1)} dx.
\end{aligned} \tag{2.11}$$

From [19] $p.$ 315, we have

$$\int_0^\infty x^{\mu-1}\,(1+\beta\,x)^{-v}\,dx = \beta^{-\mu}\,B\left(\mu,\, v - \mu\right). \tag{2.12}$$

Now by applying (2.12) on the right hand side of (2.11) with $\mu = p + 1$ and $v = \alpha\,\gamma_i + 1$, we get

$$\mu^{(p)}_{r,n,\tilde{m},k} = \alpha\, C_{r-1} \left(\frac{\beta}{\lambda}\right)^p \sum_{i=1}^{r} a_i(r)\, B\left(p+1,\, \alpha\,\gamma_i - p\right).$$

The proof of Theorem 2.1 is complete.

Corollary 2.2. *For the condition stated in Theorem 2.1 and for case II, the relation* (2.10) *reduces to single moment of GOS as*

$$\mu_{r,n,m,k}^{(p)} = \frac{C_{r-1}}{(r-1)!}\,\frac{\alpha}{(m+1)^{r-1}}\left(\frac{\beta}{\lambda}\right)^p \sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}B\left(p+1,\,\alpha\,\gamma_{r-j}-p\right) \quad (2.13)$$

where $p \le \alpha\,\gamma_{r-j}$.

Proof. We have

$$g_m^{r-1}\left(F(x)\right) = \left[\frac{1}{m+1}\left\{1-[1-F(x)]^{m+1}\right\}\right]^{r-1}.$$

By expanding the above binomial expression, the p.d.f given in (1.7) can be written as

$$f_{X(r,n,m,k)}(x) = \frac{1}{(m+1)^{r-1}}\,\frac{C_{r-1}}{(r-1)!}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}[\bar{F}(x)]^{\gamma_{r-j}-1}f(x).$$

Now, Corollary 2.2 can be proved along the lines of the proof of Theorem 2.1.

Remark 2.3. *When* $m = 0$ *and* $k = 1$ *in* (2.13)*, the exact single moments of order statistics for LED are*

$$\mu_{r:n}^{(p)} = \alpha\,C_{r:n}\left(\frac{\beta}{\lambda}\right)^p \sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}B\left(p+1,\,\alpha\,(n+j-r+1)-p\right),$$

where $\mu_{r:n}^{(p)} = E(X_{r:n}^p)$ *represents the* p^{th} *moment of* r^{th} *order statistics and* $C_{r:n} = \frac{n!}{(r-1)!\,(n-r)!}$.

For different sample sizes $n = 1, ...5$, the first four moments of order statistics from LED are found using the results in Remark 2.3 for different values of the parameters. The results are presented in tables 1 and 2.
In tables 1 and 2, we note that the well known property of order statistics $E\left(\sum_{i=1}^{n} X_{i:n}^p\right) = n\,E(X)^p$ [17] is satisfied.

Table 1: First Four Moments of Order Statistics for LED ($\lambda = 1, \beta = 0.75$, $\alpha = 4.5$, 6.5).

n	r	$\lambda = 1, \beta = 0.75$							
		$\alpha = 4.5$				$\alpha = 6.5$			
		p=1	p=2	p=3	p=4	p=1	p=2	p=3	p=4
1	1	0.2143	0.1288	0.1929	1.1571	0.1364	0.0455	0.0292	0.035
2	1	0.0938	0.02	0.0075	0.0045	0.0625	0.0085	0.0019	0.0006
	2	0.3348	0.2371	0.3782	2.3098	0.2102	0.0824	0.0565	0.0695
3	1	0.06	0.0078	0.0017	0.0005	0.0405	0.0035	0.0005	9.1713
	2	0.1613	0.4462	0.0192	0.0125	0.1064	0.0186	0.0048	0.0017
	3	0.4216	0.3333	0.5576	3.4584	0.2621	0.1143	0.0824	0.1034
4	1	0.0441	0.0041	0.0006	0.0001	0.03	0.0019	0.0002	2.5012
	2	0.1076	0.0189	0.0048	0.0017	0.0722	0.0083	0.0013	0.0003
	3	0.2149	0.0703	0.0336	0.0233	0.1407	0.029	0.0083	0.0032
	4	0.4905	0.4209	0.7323	4.6034	0.3026	0.1427	0.1071	0.1368
5	1	0.0349	0.0026	0.0003	4.7759	0.0238	0.0012	8.931	9.4011
	2	0.081	0.0105	0.0019	0.0004	0.0548	0.0047	0.0006	8.7457
	3	0.1475	0.0315	0.0092	0.0036	0.0938	0.0136	0.0025	0.0006
	4	0.2597	0.0962	0.0499	0.0364	0.1690	0.0392	0.0121	0.0049
	5	0.5482	0.0502	0.9029	5.7452	0.3360	0.1686	0.1308	0.1697

Remark 2.4. *For $m \to -1$, the exact moments of k^{th} upper record values from LED are*

$$\mu^{(p)}_{U^{(k)}(n)} = \left(\frac{\beta}{\lambda}\right)^p \sum_{u=0}^{p} (-1)^{u+p} \binom{p}{u} \left(\frac{\alpha\, k}{\alpha\, k - u}\right)^n,$$

where $\mu^{(p)}_{U^{(k)}(n)}(x)$ denotes the p^{th} moment of sequence of k^{th} upper record values.

Proof. From (1.7), when $m \to -1$, we get

$$\mu^{(p)}_{U^{(k)}(n)} = \frac{k^n}{\Gamma n} \int_0^\infty x^p \,\left[-\ln\left(1 - F(x)\right)\right]^{r-1} \left[\bar{F}(x)\right]^{k-1} f(x)\, dx.$$

Table 2: First Four Moments of Order Statistics for LED($\lambda = 2, \beta = 2.5$, $\alpha =$ 4.5, 6.5).

n	r	$\lambda = 2, \beta = 2.5$							
		$\alpha = 4.5$				$\alpha = 6.5$			
		p=1	p=2	p=3	p=4	p=1	p=2	p=3	p=4
1	1	0.3571	0.3571	0.8929	8.9286	0.2273	0.1263	0.1353	0.2706
2	1	0.1562	0.0558	0.0349	0.0349	0.1042	0.0237	0.0089	0.0049
	2	0.558	0.6585	1.7508	17.8223	0.3504	0.2289	0.2617	0.5362
3	1	0.1	0.0217	0.0078	0.0041	0.0676	0.0097	0.0022	0.0007
	2	0.2686	0.1239	0.0891	0.0965	0.1774	0.0517	0.0222	0.0134
	3	0.7027	0.9258	2.5817	26.6852	0.4369	0.3174	0.3814	0.7976
4	1	0.0735	0.0115	0.0029	0.001	0.05	0.0052	0.0008	0.0002
	2	0.1794	0.0525	0.0224	0.0133	0.1203	0.0230	0.0062	0.0023
	3	0.3581	0.1954	0.1558	0.1797	0.2345	0.0805	0.0383	0.0245
	4	0.8175	1.1692	3.3903	35.5203	0.5044	0.3964	0.4958	1.0553
5	1	0.0581	0.0071	0.0014	0.0004	0.0397	0.0033	0.0004	7.2539
	2	0.1351	0.0291	0.0089	0.0037	0.0913	0.013	0.0026	0.0007
	3	0.2459	0.0876	0.0427	0.0277	0.1638	0.038	0.0117	0.0046
	4	0.4329	0.2672	0.2311	0.2810	0.2816	0.1088	0.056	0.0378
	5	0.9137	1.3947	4.1802	44.3302	0.5601	0.4683	0.6057	1.3097

In view of (1.1), (1.2) and (1.3), we obtain

$$\begin{aligned} \mu^{(p)}_{U^{(k)}(n)} &= \frac{k^n}{\Gamma n} \int_0^\infty x^p \left[-\ln\left(1 + \frac{\lambda}{\beta}\, x\right)^{-\alpha} \right]^{r-1} \left(1 + \frac{\lambda}{\beta}\, x\right)^{-\alpha\, k} \\ &\times \frac{\alpha\, \lambda}{\beta} \left(1 + \frac{\lambda}{\beta}\, x\right)^{-1} dx. \end{aligned} \tag{2.14}$$

Using the transformation $-\ln\left(1+\frac{\lambda}{\beta}x\right)^{-\alpha}=t$ and $\frac{\alpha\lambda}{\beta}\left(1+\frac{\lambda}{\beta}x\right)^{-1}dx=dt$ in (2.14), we get

$$\begin{aligned}\mu^{(p)}_{U^{(k)}(n)} &= \frac{k^n}{\Gamma n}\left(\frac{\beta}{\lambda}\right)^p\sum_{u=0}^{p}(-1)^{u+p}\binom{p}{u}\int_0^\infty e^{-\left(k-\frac{u}{\alpha}\right)t}\,t^{n-1}\,dt\\ &= \frac{k^n}{\Gamma n}\left(\frac{\beta}{\lambda}\right)^p\sum_{u=0}^{p}(-1)^{u+p}\binom{p}{u}\frac{\Gamma n}{\left(k-\frac{u}{\alpha}\right)^n}\\ &= \left(\frac{\beta}{\lambda}\right)^p\sum_{u=0}^{p}(-1)^{u+p}\binom{p}{u}\left(\frac{\alpha k}{\alpha k-u}\right)^n.\end{aligned}$$

Thus, the proof is complete.

In table 3, we present the first four moments of upper record values using the results in Remark 2.4 for different values of parameter α and for sample sizes $n=1,...5$.

Table 3: First Four Moments of Upper Record Values for LED

n	$\lambda=1,\beta=0.75$							
	$\alpha=4.5$				$\alpha=6.5$			
	p=1	p=2	p=3	p=4	p=1	p=2	p=3	p=4
1	0.2143	0.1286	0.1929	0.1571	0.1364	0.0455	0.0292	0.0351
2	0.4898	0.5253	1.3665	18.6135	0.2975	0.1648	0.1602	0.2834
3	0.844	1.452	6.2775	205.1865	0.488	0.4007	0.5552	1.4032
4	1.2995	3.3932	23.9224	1990.213	0.7131	0.8165	1.556	5.515
5	1.885	7.2388	82.625	18407.67	0.9791	1.5058	3.8577	18.9695
	$\alpha=8.5$				$\alpha=10$			
1	0.1	0.0231	0.0094	0.0063	0.0833	0.0156	0.005	0.0025
2	0.2133	0.0794	0.0471	0.0433	0.1759	0.0525	0.0241	0.0162
3	0.3418	0.1827	0.1475	0.1801	0.2788	0.1179	0.0723	0.0632
4	0.4873	0.3514	0.3717	0.5879	0.3931	0.2211	0.1743	0.1924
5	0.6523	0.6101	0.824	1.6599	0.5201	0.3739	0.3692	0.5053

Theorem 2.5. *Under the similar conditions stated in Theorem 2.1, the recurrence relation moment of GOS for LED is given as*

$$\mu^{(p)}_{r,n,\tilde{m},k}-\mu^{(p)}_{r-1,n,\tilde{m},k}=p\,C_{r-2}\left(\frac{\beta}{\lambda}\right)^p\sum_{i=1}^{r}a_i(r)\,B\left(p,\,\alpha\gamma_i-p\right)\qquad(2.15)$$

and

$$\mu^{(p)}_{r,n,\tilde{m},k} - \mu^{(p)}_{r-1,n,\tilde{m},k} = \frac{p}{\alpha\,\gamma_r}\left[\mu^{(p)}_{r,n,\tilde{m},k} + \frac{\beta}{\lambda}\,\mu^{(p-1)}_{r,n,\tilde{m},k}\right] \tag{2.16}$$

where $p \leq \alpha\,\gamma_i$.

Proof. From [9], we have

$$\begin{aligned} &E\left[\xi\{X(r,n,m,k)\}\right] - E\left[\xi\{X(r-1,n,m,k)\}\right] \\ &\qquad = C_{r-2}\int_{-\infty}^{\infty}\xi'(x)\sum_{i=1}^{r}a_i\,(r)\,\left[\bar{F}(x)\right]^{\gamma_i}\,dx. \end{aligned} \tag{2.17}$$

By putting $\xi(x) = x^p$ in (2.17), we get

$$\mu^{(p)}_{r,n,\tilde{m},k} - \mu^{(p)}_{r-1,n,\tilde{m},k} = p\,C_{r-2}\int_0^{\infty}x^{p-1}\sum_{i=1}^{r}a_i(r)\,\left[\bar{F}(x)\right]^{\gamma_i}\,dx. \tag{2.18}$$

Now, using (1.1) in (2.18), we obtain

$$\begin{aligned} \mu^{p}_{r,n,\tilde{m},k} - \mu^{p}_{r-1,n,\tilde{m},k} &= p\,C_{r-2}\sum_{i=1}^{r}a_i(r)\int_0^{\infty}x^{p-1}\left[\beta^{\alpha}\left[\beta + \lambda\,x\right]^{-\alpha}\right]^{\gamma_i}\,dx \\ &= p\,C_{r-2}\sum_{i=1}^{r}a_i(r)\int_0^{\infty}x^{p-1}\left(1 + \frac{\lambda}{\beta}\,x\right)^{-\alpha\,\gamma_i}\,dx. \end{aligned} \tag{2.19}$$

The expression (2.15) can be obtained by applying (2.12) in (2.19).
The expression in (2.16) can be easily proved using (1.3) in (2.18).

Corollary 2.6. *For the conditions stated in Theorem 2.1, the recurrence relations of single moment of GOS for Case II is given as*

$$\begin{aligned} \mu^{(p)}_{r,n,m,k} - \mu^{(p)}_{r-1,n,m,k} &= \frac{p\,C_{r-2}}{(r-1)!\,(m+1)^{r-1}}\left(\frac{\beta}{\lambda}\right)^{p}\sum_{j=0}^{r-1}(-1)^j \\ &\quad\times\binom{r-1}{j}B\left(p,\,\alpha\,\gamma_{r-j} - p\right),\quad m \neq -1, \end{aligned} \tag{2.20}$$

where $p \leq \alpha\,\gamma_{r-j}$. *Also,*

$$\mu^{(p)}_{r,n,m,k} - \mu^{(p)}_{r-1,n,m,k} = \frac{p}{\alpha\,\gamma_r}\left[\mu^{(p)}_{r,n,m,k} + \frac{\beta}{\lambda}\,\mu^{(p-1)}_{r,n,m,k}\right]. \tag{2.21}$$

Proof. The expression (2.20) can be proved along the lines of Corollary 2.2. Moreover, when $\gamma_i \neq \gamma_j$ but $m_i = m,\ i = 1, 2 ..., n-1$, the p.d.f given in (1.5) and (1.7) are equivalent (See Khan et al. (2006)). Thus, (2.21) can be obtained by replacing $\tilde{m}$ by m in (2.16).

Remark 2.7. *By putting $m_i = 0$; $i = 1, 2, \ldots, n-1$ and $k = 1$, we obtain the recurrence relation for single moments of order statistics as*

$$\mu_{r:n}^{(p)} - \mu_{r-1:n}^{(p)} = \frac{p}{(n-r-1)\,\alpha}\left[\mu_{r:n}^{(p)} + \frac{\beta}{\lambda}\,\mu_{r:n}^{(p-1)}\right].$$

Remark 2.8. *Let $m_i \to -1$; $i = 1, 2, \ldots, n-1$. Then the recurrence relation of single moments of k^{th} upper record values is*

$$\mu_{U^{(k)}(n)}^{(p)} - \mu_{U^{(k)}(n-1)}^{(p)} = \frac{p}{k\,\alpha}\left[\mu_{U^{(k)}(n)}^{(p)} + \frac{\beta}{\lambda}\,\mu_{U^{(k)}(n)}^{(p-1)}\right].$$

3 Product Moments

For LED, the product moments of GOS are considered in this section. For the sake of convenience throughout the calculations and discussion, we assume

$$E[X^p(r,n,\tilde{m},k)X^q(s,n,\tilde{m},k)] = \mu_{r,s,n,\tilde{m},k}^{(p,q)}.$$

Theorem 3.1. *Let Case I be satisfied. For LED given in* (1.1) *and* $n \in N, \tilde{m} \in \mathbb{R},\ K > 0,\ 1 \leq r < s \leq n,\ p, q = 1, 2, ...,$

$$\begin{aligned}\mu_{r,s,n,\tilde{m},k}^{(p,q)} &= \alpha^2\, C_{s-1} \left(\frac{\beta}{\lambda}\right)^{p+q} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s) \sum_{u=0}^{\infty} (-1)^u \frac{1}{q-\alpha\gamma_j} \\ &\times \frac{(\alpha\gamma_j+1)_u\,(\alpha\gamma_j-q)_u}{(\alpha\gamma_j-q+1)_u}\, B\left(p+q-\alpha\,\gamma_j-u+1,\ \alpha\,\gamma_i+u-p-q\right) \quad (3.22)\end{aligned}$$

where $q \leq \alpha$ *and* $p+q < \alpha\gamma_i + u$.

Proof. We know that

$$\mu^{(p,q)}_{r,s,n,\tilde{m},k} = C_{s-1} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s) \int_0^\infty \int_x^\infty x^p\, y^q \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_j} [\bar{F}(x)]^{\gamma_i} \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}\, dy\, dx$$

$$= C_{s-1} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s) \int_0^\infty \int_x^\infty x^p\, y^q\, [\bar{F}(y)]^{\gamma_j-1} [\bar{F}(x)]^{\gamma_i-\gamma_j-1} f(x)\, f(y)\, dy\, dx$$

$$= C_{s-1} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s) \int_0^\infty x^p\, [\bar{F}(x)]^{\gamma_i-\gamma_j-1} f(x)$$

$$\times \left\{ \int_x^\infty y^q\, [\bar{F}(y)]^{\gamma_j-1} f(y)\, dy \right\} dx. \tag{3.23}$$

Now, we consider

$$\begin{aligned} I(x) &= \int_x^\infty y^q\, [\bar{F}(y)]^{\gamma_j-1} f(y)\, dy \\ &= \int_x^\infty y^q \left[\beta^\alpha [\beta + \lambda y]^{-\alpha}\right]^{\gamma_j-1} \left\{\alpha \lambda \beta^\alpha [\beta + \lambda y]^{-(\alpha+1)}\right\} dy \\ &= \alpha \lambda \beta^{\alpha\gamma_j} \int_x^\infty y^q\, [\beta + \lambda y]^{-(\alpha\gamma_j+1)}\, dy \\ &= \frac{\alpha \lambda}{\beta} \int_x^\infty y^q \left(1 + \frac{\lambda}{\beta} y\right)^{-(\alpha\gamma_j+1)} dy. \end{aligned} \tag{3.24}$$

From [19] $p.\,315$, we have

$$\int_u^\infty x^{\mu-1} (1 + \beta x)^{-v}\, dx = \frac{u^{\mu-v}}{\beta^v (\mu - v)}\; {}_2F_1\left(v, v-\mu, v-\mu+1; \frac{-1}{\beta u}\right). \tag{3.25}$$

Applying (3.25) and (1.9) in the last term on the right hand side of (3.24), we obtain

$$\begin{aligned} I(x) &= \alpha \lambda^{-\alpha\gamma_j} \beta^{\alpha\gamma_j} \frac{x^{q-\alpha\gamma_j}}{q - \alpha\gamma_j}\; {}_2F_1\left(\alpha\gamma_j + 1,\, \alpha\gamma_j - q,\, \alpha\gamma_j - q + 1; \frac{-\beta}{\lambda x}\right) \\ &= \alpha \sum_{u=0}^\infty (-1)^u \left(\frac{\lambda}{\beta}\right)^{-(\alpha\gamma_j+u)} \frac{x^{q-\alpha\gamma_j-u}}{q - \alpha\gamma_j} \frac{(\alpha\gamma_j+1)_u\, (\alpha\gamma_j - q)_u}{(\alpha\gamma_j - q + 1)_u}. \end{aligned} \tag{3.26}$$

Now, using (3.26) in (3.23), we obtain

$$\begin{aligned}\mu^{(p,q)}_{r,s,n,\tilde{m},k} &= \alpha\, C_{s-1} \sum_{i=1}^{r} a_i\,(r) \sum_{j=r+1}^{s} a_j^{(r)}\,(s)\\ &\times \sum_{u=0}^{\infty}(-1)^u \left(\frac{\lambda}{\beta}\right)^{-(\alpha\,\gamma_j+u)} \frac{1}{q-\alpha\,\gamma_j}\,\frac{(\alpha\gamma_j+1)_u\,(\alpha\gamma_j-q)_u}{(\alpha\gamma_j-q+1)_u}\\ &\times \int_0^{\infty} x^{p+q-\alpha\,\gamma_j-u}\,[\bar{F}(x)]^{\gamma_i-\gamma_j-1}\,f(x)\,dx. \qquad (3.27)\end{aligned}$$

Again we consider

$$\begin{aligned}J(x) &= \int_0^{\infty} x^{p+q-\alpha\,\gamma_j-u}\,[\bar{F}(x)]^{\gamma_i-\gamma_j-1}\,f(x)\,dx\\ &= \int_0^{\infty} x^{p+q-\alpha\,\gamma_j-u}\,\left[\beta^{\alpha}\,[\beta+\lambda\,y]^{-\alpha}\right]^{\gamma_i-\gamma_j-1}\left\{\alpha\,\lambda\,\beta^{\alpha}\,[\beta+\lambda\,x]^{-(\alpha+1)}\right\}dx\\ &= \frac{\alpha\,\lambda}{\beta}\int_0^{\infty} x^{p+q-\alpha\,\gamma_j-u}\left[1+\frac{\lambda}{\beta}\,x\right]^{-(\alpha(\gamma_i-\gamma_j)+1)} dx. \qquad (3.28)\end{aligned}$$

Using (2.12) in the last term on the right hand side of (3.28), we get

$$J(x) = \alpha\,\left(\frac{\beta}{\lambda}\right)^{p+q-\alpha\,\gamma_j-u} B\left(p+q-\alpha\,\gamma_j-u+1,\,\alpha\,\gamma_i+u-p-q\right). \qquad (3.29)$$

Using (3.29) in (3.27), we get (3.22). Thus, the proof of Theorem 3.1 is complete.

Corollary 3.2. *For the conditions stated in Theorem 3.1 and for case II, the relation for product moment of GOS for LED is*

$$\begin{aligned}\mu^{(p,q)}_{r,s,n,m,k} &= \frac{\alpha^2\,C_{s-1}}{(r-1)!\,(s-r-1)!}\,\frac{1}{(m+1)^{s-2}}\sum_{j=0}^{r-1}\sum_{h=0}^{s-r-1}\sum_{u=0}^{\infty}(-1)^{j+h+u}\binom{r-1}{j}\\ &\times \binom{s-r-1}{h}\left(\frac{\beta}{\lambda}\right)^{p+q}\frac{1}{q-\alpha\gamma_{s-h}}\,\frac{(\alpha\gamma_{s-h}+1)_u\,(\alpha\gamma_{s-h}-q)_u}{(\alpha\gamma_{s-h}-q+1)_u}\\ &\times B\left(p+q-\alpha\,\gamma_{s-h}-u+1,\,\alpha\,(m+1)(s+j-r-h)-p-q+\alpha\,\gamma_{s-h}+u\right) \qquad (3.30)\end{aligned}$$

where $q \le \alpha$ *and* $p+q-\alpha\gamma_{s-h} - u < \alpha\,(m+1)(s+j-r-h)$.

Proof. By expanding $g_m^{r-1}(F(x))$ and $[h_m(F(y)) - h_m(F(x))]^{s-r-1}$ binomially in (1.8) after noting that

$$h_m(y) - h_m(x) = \frac{1}{m+1}\left[(1-x)^{m+1} - (1-y)^{m+1}\right]$$

and $g_m(x) = h_m(x) - h_m(0)$. Then, the p.d.f given in (1.8) can be expanded as

$$f_{X(r,n,m,k),X(s,n,m,k)} = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!\,(m+1)^{s-2}} \sum_{j=0}^{r-1} \sum_{h=0}^{s-r-1} (-1)^{j+h} \binom{r-1}{j}$$
$$\times \binom{s-r-1}{h} [\bar{F}(x)]^{(s-r+j-h)(m+1)-1} [\bar{F}(y)]^{\gamma_{s-j}-1} f(x)\, f(y).$$

Now, Corollary can be proved along the lines of proof of Theorem 3.1.

Theorem 3.3. *For LED and under the conditions stated in Theorem 3.1, the recurrence relations of product moments of GOS are*

$$\begin{aligned} \mu^{(p,q)}_{r,s,n,\tilde{m},k} - \mu^{(p,q)}_{r,s-1,n,\tilde{m},k} &= q\,\alpha\,C_{s-2} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s) \sum_{u=0}^{\infty} (-1)^u \left(\frac{\beta}{\lambda}\right)^{p+q} \\ &\times \frac{1}{q-\alpha\gamma_j} \frac{(\alpha\gamma_j)_u\,(\alpha\gamma_j - q)_u}{(\alpha\gamma_j - q + 1)_u} \\ &\times B\,(p+q-\alpha\,\gamma_j - u + 1,\, \alpha\,\gamma_i + u - p - q) \qquad (3.31) \end{aligned}$$

where $q \le \alpha$ *and* $p + q < \alpha\gamma_i + u$. *In addition,*

$$\mu^{(p,q)}_{r,s,n,\tilde{m},k} - \mu^{(p,q)}_{r,s-1,n,\tilde{m},k} = \frac{q}{\alpha\,\gamma_s}\left[\mu^{(p,q)}_{r,s,n,\tilde{m},k} + \frac{\beta}{\lambda}\,\mu^{(p,q-1)}_{r,s,n,\tilde{m},k}\right]. \qquad (3.32)$$

Proof. From [9], we have

$$\begin{aligned} &E\,[\xi\{X(r,n,\tilde{m},k), X(s,n,\tilde{m},k)\}] - E\,[\xi\{X(r,n,\tilde{m},k), X(s-1,n,\tilde{m},k)\}] \\ &= C_{s-2} \int_{-\infty}^{\infty} \int_{x}^{\infty} \frac{\partial}{\partial y}\xi\,(x,y) \left[\sum_{j=r+1}^{s} a_j^{(r)}(s) \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_j}\right] \\ &\times \left[\sum_{i=1}^{r} a_i\,(r)\,[\bar{F}(x)]^{\gamma_i}\right] \frac{f(x)}{\bar{F}(x)}\,dy\,dx. \qquad (3.33) \end{aligned}$$

Let $\xi(x,y) = x^p\,y^q$. Then

$$\mu^{(p,q)}_{r,s,n,\tilde{m},k} - \mu^{(p,q)}_{r,s-1,n,\tilde{m},k}$$
$$= q\,C_{s-2} \int_{0}^{\infty} \int_{x}^{\infty} x^p\,y^{q-1} \left[\sum_{j=r+1}^{s} a_j^{(r)}(s) \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_j}\right] \left[\sum_{i=1}^{r} a_i\,(r)\,[\bar{F}(x)]^{\gamma_i}\right] \frac{f(x)}{\bar{F}(x)}\,dy\,dx$$

$$= q\, C_{s-2} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_0^\infty x^p\, [\bar{F}(x)]^{\gamma_i-\gamma_j-1} f(x) \left\{ \int_x^\infty y^{q-1} [\bar{F}(y)]^{\gamma_j}\, dy \right\} dx.$$

Now, proceeding along the lines of proof of Theorem 3.1, the theorem can be proved using (3.25), (1.9) and (2.12).
The relations (3.32) can be easily proved using (1.3) in (3.33).

Corollary 3.4. *Suppose Case II is true and under the conditions stated in Theorem 3.1, the recurrence relations of the product moments are*

$$\begin{aligned} &\mu^{(p,q)}_{r,s,n,m,k} - \mu^{(p,q)}_{r,s-1,n,m,k} \\ =\;& \frac{q\,\alpha\, C_{s-2}}{(r-1)!\,(s-r-1)!} \frac{1}{(m+1)^{s-2}} \sum_{j=0}^{r-1} \sum_{h=0}^{s-r-1} \sum_{u=0}^{\infty} (-1)^{j+h+u} \binom{r-1}{j} \binom{s-r-1}{h} \\ \times\;& \left(\frac{\beta}{\lambda}\right)^{p+q} \frac{1}{q-\alpha\gamma_{s-h}} \frac{(\alpha\gamma_{s-h})_u\, (\alpha\gamma_{s-h}-q)_u}{(\alpha\gamma_{s-h}-q+1)_u} \\ \times\;& B\,(p+q-\alpha\,\gamma_{s-h}-u+1,\; \alpha\,(m+1)(s+j-r-h)-p-q+\alpha\,\gamma_{s-h}+u) \end{aligned} \tag{3.34}$$

where $q \le \alpha$ *and* $p+q-\alpha\gamma_{s-h} - u < \alpha\,(m+1)(s+j-r-h)$. *Further, we have*

$$\mu^{(p,q)}_{r,s,n,m,k} - \mu^{(p,q)}_{r,s-1,n,m,k} = \frac{q}{\alpha\,\gamma_s} \left[\mu^{(p,q)}_{r,s,n,m,k} + \frac{\beta}{\lambda}\, \mu^{(p,q-1)}_{r,s,n,m,k} \right]. \tag{3.35}$$

Remark 3.5. *If* $m_i = 0$; $i = 1, 2, \ldots, n-1$ *and* $k = 1$, *then the relation for product moment of order statistics for LED is given by*

$$\mu^{(p,q)}_{r,s:n} - \mu^{(p,q)}_{r,s-1:n} = \frac{q}{\alpha\,(n-s+1)} \left[\mu^{(p,q)}_{r,s:n} + \frac{\beta}{\lambda}\, \mu^{(p,q-1)}_{r,s:n} \right]$$

where $\mu^{(p,q)}_{r,s:n} = E[X^p_{r:n} . X^q_{s:n}]$.

Remark 3.6. *Let* $m_i \to -1$; $i = 1, 2, \ldots, n-1$. *Then the product moment of* k^{th} *upper record values is written as*

$$\mu^{(p,q)}_{U^{(k)}(n,m)} - \mu^{(p,q)}_{U^{(k)}(n,m-1)} = \frac{q}{\alpha\, k} \left[\mu^{(p,q)}_{U^{(k)}(n,m)} + \frac{\beta}{\lambda}\, \mu^{(p,q-1)}_{U^{(k)}(n,m)} \right],$$

where $E\big[X^p_{U^{(k)}(n)} X^q_{U^{(k)}(m)}\big] = \mu^{(p,q)}_{U^{(k)}(n,m)}$.

4 Characterization

In this section, the characterization results of LED are considered through recurrence relations of the moments of GOS. Moreover, the characterization results through conditional expectation are studied.

Theorem 4.1. *Fix a positive integer* k *and let* p *be a non-negative integer. A necessary and sufficient condition for a random variable* X *to be distributed with p.d.f given in* (1.2) *is that*

$$\left(1-\frac{p}{\alpha\,\gamma_r}\right)\mu^{p}_{r,n,\tilde{m},k}=\mu^{p}_{r-1,n,\tilde{m},k}+\frac{p\,\beta}{\alpha\,\lambda\,\gamma_r}\,\mu^{p-1}_{r,n,\tilde{m},k}. \tag{4.36}$$

Proof. The necessary part follows from (2.16). On the other hand, suppose the relation in (4.36) is satisfied. Now, from [9], for $\xi(x)=x^p$, we obtain

$$p\,C_{r-2}\int_{-\infty}^{\infty}x^{p-1}\sum_{i=1}^{r}a_i\,(r)\,\left[\,\bar{F}(x)\,\right]^{\gamma_i}\,dx$$
$$=\frac{p\,C_{r-1}}{\alpha\,\gamma_r}\int_{-\infty}^{\infty}x^{p-1}\left(\frac{\beta}{\lambda}+x\right)\sum_{i=1}^{r}a_i\,(r)\,\left[\,\bar{F}(x)\,\right]^{\gamma_i-1}f(x)\,dx.$$

This implies

$$\frac{p\,C_{r-1}}{\alpha\,\gamma_r}\int_{-\infty}^{\infty}x^{p-1}\sum_{i=1}^{r}a_i\,(r)\,\left[\,\bar{F}(x)\,\right]^{\gamma_i-1}\left\{\alpha\,\bar{F}(x)-\left(\frac{\beta}{\lambda}+x\right)\,f(x)\right\}dx=0. \tag{4.37}$$

Applying the extension of *Müntz-Szász* theorem, given in [20], to (4.37), we get

$$\frac{\bar{F}(x)}{f(x)}=\frac{1}{\alpha}\left(\frac{\beta}{\lambda}+x\right).$$

Thus, $f(x)$ is the p.d.f given in (1.2). Hence, Theorem 4.1 holds.

Theorem 4.2. *Fix a positive integer* k *and let* $p,\,q$ *are non-negative integers. A necessary and sufficient condition for a random variables* X *to be distributed with p.d.f given in* (1.2) *is that*

$$\left(1-\frac{p}{\alpha\,\gamma_s}\right)\mu^{(p,q)}_{r,s,n,\tilde{m},k}=\mu^{(p,q)}_{r,s-1,n,\tilde{m},k}+\frac{p\,\beta}{\alpha\,\lambda\,\gamma_s}\,\mu^{(p,q-1)}_{r,s,n,\tilde{m},k}. \tag{4.38}$$

Proof. The necessary part follows from (3.32). To prove sufficiency part, supposed the relation in (4.38) is satisfied.
Now, using Athar and Islam (2004) for $\xi(x, y) = x^p\, y^q$, we get

$$\begin{aligned} & q\, C_{s-2} \int_{-\infty}^{\infty} \int_{x}^{\infty} x^p\, y^{q-1} \left[\sum_{j=r+1}^{s} a_j^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_j} \right] \\ \times & \left[\sum_{i=1}^{r} a_i(r)\, [\bar{F}(x)]^{\gamma_i} \right] \frac{f(x)}{\bar{F}(x)}\, dy\, dx. \\ = & \frac{q\, C_{s-1}}{\alpha\, \gamma_s} \int_{-\infty}^{\infty} \int_{x}^{\infty} x^p\, y^{q-1} \left(\frac{\beta}{\lambda} + y \right) \left[\sum_{j=r+1}^{s} a_j^{(r)}(s) \left[\frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_j} \right] \\ \times & \left[\sum_{i=1}^{r} a_i(r)\, [\bar{F}(x)]^{\gamma_i} \right] \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}\, dy\, dx. \end{aligned}$$

This implies

$$\begin{aligned} \frac{q\, C_{s-1}}{\alpha\, \gamma_s} \int_{-\infty}^{\infty} \int_{x}^{\infty} x^p\, y^{q-1} \left[\sum_{i=1}^{r} a_i(r)\, [\bar{F}(x)]^{\gamma_i} \right] \left[\sum_{j=r+1}^{s} a_j^{(r)}(s) \left[\frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_j} \right] \frac{f(x)}{\bar{F}(x)} \\ \times \left\{ \alpha - \left(\frac{\beta}{\lambda} + y \right) \frac{f(y)}{\bar{F}(y)} \right\} dy\, dx = 0. \quad (4.39) \end{aligned}$$

From [20], we apply the extension of *Müntz-Szász* to (4.39). So, we get

$$\frac{\bar{F}(y)}{f(y)} = \frac{1}{\alpha} \left(\frac{\beta}{\lambda} + y \right).$$

Thus, $f(y)$ is a p.d.f given in (1.2). So, the proof of Theorem 4.2 is complete.

Theorem 4.3. *Let* $X(r, n, m, k)$, $r = 1, 2, ..., n$ *be* r^{th} *GOS based on continuous df F() and E(X) exists. Then, for two consecutive values* r *and* $r+1$, *such that* $1 \le r < r+1 \le n$,

$$E\left[X(r+1, n, m, k) | X(r, n, m, k) = x\right] = \frac{\alpha\, \gamma_{r+1}}{\alpha\, \gamma_{r+1} - 1}\, x + \frac{\beta}{\lambda\, (\alpha\, \gamma_{r+1} - 1)} \quad (4.40)$$

if and only if

$$\bar{F}(x) = \beta^{\alpha} \left[\beta + \lambda\, x \right]^{-\alpha} = \left[1 + \frac{\lambda}{\beta}\, x \right]^{-\alpha}, \; x > 0; \; \alpha, \beta, \lambda > 0. \quad (4.41)$$

Proof. Khan and Alzaid [27] have proved that

$$E\left[h\left(X(r+1,\,n,\,m,\,k)\right|X(r,\,n,\,m,\,k)=x)\right]\;=\;a^{*}\,h(x)\;+\;b^{*} \qquad (4.42)$$

if and only if

$$\bar{F}(x)\;=\;[\,a\,h(x)\;+\;b\,]^{c} \qquad (4.43)$$

where $a^{*}=\prod_{j=r+1}^{s}\left(\frac{c\,\gamma_j}{1+c\,\gamma_j}\right)$ and $b^{*}\;=\;-\,\frac{b}{a}\,(1-a^{*})$.

By comparing (4.41) and (4.43), it is not difficult to see that

$$a\;=\;\frac{\lambda}{\beta},\;b\;=\;1,\;c\;=\;-\,\alpha\text{ and }h(x)\;=\;x.$$

Therefore, it is easy to prove the theorem in view of (4.42).

Corollary 4.4. *For the r^{th} order statistics $X_{r:n},\ r=1,\,2,\,...,n$ and under the conditions stated in Theorem 4.3*

$$\begin{aligned}E\left(X_{r+1:n}|X_{r:n}=x\right)\;&=\;\frac{\alpha\;(n\;-\;r)}{\alpha\,(n\;-\;r)-1}\,x\;+\;\frac{\beta}{\lambda\,(\alpha\,(n\;-\;r)-1)}\\&=\;\frac{\alpha\,\lambda\,(n\;-\;r)\,x\;+\;\beta}{\lambda\;(\alpha\,(n\;-\;r)\;-\;1)}\end{aligned}$$

and consequently

$$E\left(X_{n:n}|X_{n-1:n}=x\right)\;=\;\frac{\alpha}{\alpha\;-\;1}\,x\;+\;\frac{\beta}{\lambda\,(\alpha\;-\;1)}\;=\;\frac{\alpha\,\lambda\,x\;+\;\beta}{\lambda\,(\alpha\;-\;1)}$$

if and only if

$$\bar{F}(x)\;=\;\beta^{\alpha}\;[\,\beta\;+\;\lambda\,x\,]^{-\,\alpha},\;\;x>0;\;\;\alpha,\,\beta,\,\lambda\,>0.$$

Remark 4.5. *We note that a similar characterization result can be also found for adjacent records as*

$$E\left[X_{U(n)}|X_{U(n-1)}=x\right]\;=\;E\left[X|X\geq x\right]\;=\;\frac{\alpha}{\alpha\;-\;1}\,x\;+\;\frac{\beta}{\lambda\,(\alpha\;-\;1)}.$$

5 Conclusions

The Lomax-exponential distribution proposed by Ijaz et al. [22] gives enough flexibility to model real-life problems. The main purpose was to study moments of GOS for Lomax-exponential distribution. We derived the exact expressions for single and product moments and the recurrence relations for GOS from Lomax exponential distribution. Numerical results of the first four moments of order statistics and upper record values were computed. Since the characterization of probability distribution plays significant role in statistical studies, the given characterization results are useful for researchers in applied and natural sciences. The results of all sub models of GOS can be deduced from our results in this study.

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