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# Numerical radius inequalities involving accretive-dissipative matrices

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**Abstract**

In this article, we present many new bounds for the numerical radius for $2\times 2$ block matrices involving accretive-dissipative matrices. These new bounds are represented using the spectral norm.

## 1 Introduction

Let $M_n(\mathbb{C})$ be the algebra of all $n\times n$ complex matrices. A matrix $T\in M_n(\mathbb{C})$ is called accretive-dissipative if in its Cartesian decomposition $T=A+iB$, the matrices $A$ and $B$ are positive semidefinite, where $A=\Re(T)=\frac{T+T^*}{2}$ and $B=\Im(T)=\frac{T-T^*}{2i}$. For a matrix $A\in M_n(\mathbb{C})$, the numerical radius is given by

$$w(A)=\max\{|(Ax,x)|: x\in\mathbb{C}^n, \|x\|=1\}.$$

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It is well known that the numerical radius defines a norm on $M_n(\mathbb{C})$ and for every $A \in M_n(\mathbb{C})$ we have

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|, \tag{1.1}$$

where $\|\cdot\|$ is the spectral norm. The inequalities in (1.1) are sharp. The first inequality becomes an equality if $A^2 = 0$, while the second inequality becomes an equality if $A$ is normal. An improvement of the second inequality in (1.1) has been given in [3]. It has been shown that if $A \in M_n(\mathbb{C})$, then

$$w(A) \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{1/2}\right). \tag{1.2}$$

For a comprehensive account for the theory of the numerical range, the reader is referred to [2], [5] and [6].

An important property of the numerical radius norm is its weak unitary invariance, that is, for $A \in M_n(\mathbb{C})$,

$$w(U^*AU) = w(A), \text{ for every unitary } U \in M_n(\mathbb{C}). \tag{1.3}$$

We will make use of the following properties of the spectral norm:

$$w(A) = w(A^*), \text{ for every } A \in M_n(\mathbb{C}) \tag{1.4}$$

$$\|A\| = \|A^*A\|^{1/2} = \|AA^*\|^{1/2} \tag{1.5}$$

$$\|\Re(A)\| \leq w(A) \quad \text{and} \quad \|\Im(A)\| \leq w(A) \tag{1.6}$$

$$\left\|\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right\| = \max\left(\|A\|, \|D\|\right) \tag{1.7}$$

$$\left\|\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right\| = \max\left(\|B\|, \|C\|\right) \tag{1.8}$$

In section two of this paper, we establish new numerical radius inequalities for $2 \times 2$ block matrices involving accretive-dissipative matrices.

# 2 Main results

In order to prove our main results, we need the following lemmas. The first lemma was proved by Shebrawi (see, e.g., [4]).

**Lemma 2.1.** *Let* $A, B \in M_n(\mathbb{C})$. *Then*

$$w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(\|A\| + \|AA^* + BB^*\|^{1/2}\right).$$

The following lemma was proved in [1] by Abu-Omar and Kittaneh.

**Lemma 2.2.** *Let* $A, B \in M_n(\mathbb{C})$ *be positive semidefinite. Then*

$$w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{1}{2}\|A + B\|.$$

In 2002, X. Zhan proved the following important inequality involving positive semidefinite matrices [7]:

**Lemma 2.3.** *Let* $A, B \in M_n(\mathbb{C})$ *be positive semidefinite. Then*

$$\|A^r + B^r\| \leq \|A + B\|^r, \quad \text{for } r \geq 1.$$

Now, we will present our results.

**Theorem 2.4.** *Let* $S, T \in M_n(\mathbb{C})$ *be accretive-dissipative, where* $S = A + iB$ *and* $T = C + iD$ *are the Cartesian decompositions of* $S$ *and* $T$. *Then*

$$w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right) \leq \|A + C\| + \|B + D\|.$$

*Proof.*

$$\begin{aligned}
w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right) &= w\left(\begin{bmatrix} A+iB & 0 \\ C+iD & 0 \end{bmatrix}\right) \\
&= w\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + i\begin{bmatrix} B & 0 \\ D & 0 \end{bmatrix}\right) \\
&\leq w\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} B & 0 \\ D & 0 \end{bmatrix}\right) \\
&= w\left(\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}\right) && \text{(by (1.4))} \\
&\leq \left\|\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}\right\| + \left\|\begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}\right\| && \text{(by (1.1))} \\
&= \left\|\begin{bmatrix} A^2+C^2 & 0 \\ 0 & 0 \end{bmatrix}\right\|^{1/2} + \left\|\begin{bmatrix} B^2+D^2 & 0 \\ 0 & 0 \end{bmatrix}\right\|^{1/2} && \text{(by (1.5))} \\
&= \|A^2+C^2\|^{1/2} + \|B^2+D^2\|^{1/2}.
\end{aligned}$$

Finally, applying Lemma 2.3 with $r = 2$, we get the desired result. □

Now the following corollaries are consequences of Theorem 2.4.

**Corollary 2.5.** $w\left(\begin{bmatrix} T^* & 0 \\ T & 0 \end{bmatrix}\right) \leq \sqrt{2}\left(\|C\| + \|D\|\right).$

*Proof.*

$$\begin{aligned}
w\left(\begin{bmatrix} T^* & 0 \\ T & 0 \end{bmatrix}\right) &= w\left(\begin{bmatrix} C-iD & 0 \\ C+iD & 0 \end{bmatrix}\right) \\
&\leq w\left(\begin{bmatrix} C & 0 \\ C & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & -D \\ 0 & D \end{bmatrix}\right) \\
&\leq \left\|\begin{bmatrix} C & 0 \\ C & 0 \end{bmatrix}\right\| + \left\|\begin{bmatrix} 0 & -D \\ 0 & D \end{bmatrix}\right\| \\
&= \left\|\begin{bmatrix} C^2+C^2 & 0 \\ 0 & 0 \end{bmatrix}\right\|^{1/2} + \left\|\begin{bmatrix} 0 & 0 \\ 0 & D^2+D^2 \end{bmatrix}\right\|^{1/2} && \text{(by (1.5))} \\
&= \sqrt{2}\left(\|C^2\|^{1/2} + \|D^2\|^{1/2}\right) \\
&= \sqrt{2}\left(\|C\| + \|D\|\right).
\end{aligned}$$

□

**Corollary 2.6.** $w\left(\begin{bmatrix} 0 & S \\ 0 & T \end{bmatrix}\right) \le \|A+C\| + \|B+D\|.$

*Proof.* Let $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Then clearly, $U$ is unitary. Hence, by applying (1.3), it follows that

$$\begin{aligned} w\left(\begin{bmatrix} 0 & S \\ 0 & T \end{bmatrix}\right) &= w\left(U^* \begin{bmatrix} 0 & S \\ 0 & T \end{bmatrix} U\right) \\ &= w\left(\begin{bmatrix} T & 0 \\ S & 0 \end{bmatrix}\right) \\ &\le \|A+C\| + \|B+D\|. \end{aligned}$$

□

Applying Theorem 2.4 and Corollary 2.6, we get

**Corollary 2.7.** $w\left(\begin{bmatrix} S & T \\ T & S \end{bmatrix}\right) \le 2\left(\|A+C\| + \|B+D\|\right).$

**Theorem 2.8.**

$$\begin{aligned} w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right) \le\ & \frac{1}{2}\left(\|A+C\| + \|A\|^{1/2}\|A+C\|^{1/2}\right) \\ &+ \frac{1}{2}\left(\|B+D\| + \|B\|^{1/2}\|B+D\|^{1/2}\right). \end{aligned}$$

*Proof.*

$$\begin{aligned} w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right) &= w\left(\begin{bmatrix} A+iB & 0 \\ C+iD & 0 \end{bmatrix}\right) \\ &= w\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + i\begin{bmatrix} B & 0 \\ D & 0 \end{bmatrix}\right) \\ &\le w\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} B & 0 \\ D & 0 \end{bmatrix}\right) \\ &= w\left(\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}\right). \end{aligned}$$

Now applying inequality (1.2), we get

$$w\left(\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}\right) \le \frac{1}{2}\left(\left\|\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}\right\| + \left\|\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}^2\right\|^{1/2}\right).$$

Applying (1.5) and Lemma 2.3, we get

$$\left\| \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} A^2 + C^2 & 0 \\ 0 & 0 \end{bmatrix} \right\|^{1/2} = \|A^2 + C^2\|^{1/2} \leq \|A + C\|,$$

and

$$\begin{aligned} \left\| \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}^2 \right\|^{1/2} &= \left\| \begin{bmatrix} A^2 & AC \\ 0 & 0 \end{bmatrix} \right\|^{1/2} \\ &= \left\| \begin{bmatrix} A^2 & AC \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^2 & 0 \\ CA & 0 \end{bmatrix} \right\|^{1/4} \\ &= \left\| \begin{bmatrix} A^4 + AC^2A & 0 \\ 0 & 0 \end{bmatrix} \right\|^{1/4} \\ &= \|A^4 + AC^2A\|^{1/4} = \|A\left(A^2 + C^2\right)A\|^{1/4} \\ &\leq \|A\|^{1/4}\|A^2 + C^2\|^{1/4}\|A\|^{1/4} \\ &\leq \|A\|^{1/2}\|A + C\|^{1/2}. \end{aligned}$$

Consequently, we get

$$w\left(\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(\|A + C\| + \|A\|^{1/2}\|A + C\|^{1/2}\right).$$

Similarly,

$$w\left(\begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(\|B + D\| + \|B\|^{1/2}\|B + D\|^{1/2}\right).$$

The result now follows from the previous two inequalities. □

For $A, B, C, D \in M_n(\mathbb{C})$ let

$$\alpha = \sqrt{\max\left(\|A\|^2, \|C\|^2\right) + \|AC\|},$$

and

$$\beta = \sqrt{\max\left(\|B\|^2, \|D\|^2\right) + \|BD\|}.$$

Then we have the following theorem.

**Theorem 2.9.** *If $S$ and $T$ are accretive-dissipative, then*

$$w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right) \le \alpha + \beta.$$

*Proof.*

$$\begin{aligned} w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right) &= w\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + i\begin{bmatrix} B & 0 \\ D & 0 \end{bmatrix}\right) \\ &\le w\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} B & 0 \\ D & 0 \end{bmatrix}\right) \\ &\le \left\|\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\right\| + \left\|\begin{bmatrix} B & 0 \\ D & 0 \end{bmatrix}\right\| && \text{(by (1.1))} \\ &\le \left\|\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}\right\|^{1/2} + \left\|\begin{bmatrix} B & 0 \\ D & 0 \end{bmatrix}\begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}\right\|^{1/2} && \text{(by (1.5))} \\ &= \left\|\begin{bmatrix} A^2 & AC \\ CA & C^2 \end{bmatrix}\right\|^{1/2} + \left\|\begin{bmatrix} B^2 & BD \\ DB & D^2 \end{bmatrix}\right\|^{1/2}. \end{aligned}$$

Applying (1.7) and (1.8), we get

$$\begin{aligned} \left\|\begin{bmatrix} A^2 & AC \\ CA & C^2 \end{bmatrix}\right\| &= \left\|\begin{bmatrix} A^2 & 0 \\ 0 & C^2 \end{bmatrix} + \begin{bmatrix} 0 & AC \\ CA & 0 \end{bmatrix}\right\| \\ &\le \left\|\begin{bmatrix} A^2 & 0 \\ 0 & C^2 \end{bmatrix}\right\| + \left\|\begin{bmatrix} 0 & AC \\ CA & 0 \end{bmatrix}\right\| \\ &= \max\left(\|A^2\|, \|C^2\|\right) + \max\left(\|AC\|, \|CA\|\right) \\ &= \max\left(\|A\|^2, \|C\|^2\right) + \|AC\|. \end{aligned}$$

Thus,

$$\left\|\begin{bmatrix} A^2 & AC \\ CA & C^2 \end{bmatrix}\right\|^{1/2} \le \sqrt{\max\left(\|A\|^2, \|C\|^2\right) + \|AC\|}.$$

Similarly,

$$\left\|\begin{bmatrix} B^2 & BD \\ DB & D^2 \end{bmatrix}\right\|^{1/2} \le \sqrt{\max\left(\|B\|^2, \|D\|^2\right) + \|BD\|}.$$

Now, the result follows from the last two inequalities. □

**Corollary 2.10.**

$$w\left(\begin{bmatrix} S & 0 \\ -T & 0 \end{bmatrix}\right) \leq \alpha + \beta.$$

*Proof.* Let $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Then, clearly, $U$ is unitary, The result follows since

$$\begin{aligned} w\left(\begin{bmatrix} S & 0 \\ -T & 0 \end{bmatrix}\right) &= w\left(U^* \begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix} U\right) \\ &= w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right). \end{aligned}$$

□

**Theorem 2.11.**

$$w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right) \leq \left(\|A\| + \|B\| + \frac{\|C\| + \|D\|}{2}\right).$$

*Proof.*

$$\begin{aligned} w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right) &= w\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + i \begin{bmatrix} B & 0 \\ D & 0 \end{bmatrix}\right) \\ &\leq w\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} B & 0 \\ D & 0 \end{bmatrix}\right) \\ &\leq w\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}\right) \\ &+ w\left(\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix}\right) \\ &= \|A\| + \frac{\|C\|}{2} + \|B\| + \frac{\|D\|}{2}. \end{aligned}$$

Which follows since $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$ are normal. □

**Theorem 2.12.** $w\left(\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}\right) \leq \|S + T\|.$

*Proof.* Let $S = A + iB$ and $T = C + iD$ be the Cartesian decompositions of $S$ and $T$. Then we have

$$
\begin{aligned}
w\left(\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}\right) &= w\left(\begin{bmatrix} 0 & A \\ C & 0 \end{bmatrix} + i\begin{bmatrix} 0 & B \\ D & 0 \end{bmatrix}\right) \\
&\le w\left(\begin{bmatrix} 0 & A \\ C & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & B \\ D & 0 \end{bmatrix}\right) \\
&\le \frac{1}{2}\left(\|A+C\| + \|B+D\|\right) \qquad \text{(by Lemma 2.2)} \\
&= \frac{1}{2}\left(\|\Re(S+T)\| + \|\Im(S+T)\|\right) \\
&\le \frac{1}{2}\left(w(S+T) + w(S+T)\right) \qquad \text{(by (1.6))} \\
&= w(S+T) \\
&\le \|S+T\|. \qquad \text{(by (1.1))}
\end{aligned}
$$

□

**Theorem 2.13.** $w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right) \le \frac{1}{2}\left(\|A\| + \|A+C\| + \|B\| + \|B+D\|\right).$

*Proof.* Let $S = A + iB$ and $T = C + iD$ be the Cartesian decompositions of $S$ and $T$. Then, by applying Lemma 2.1 and Lemma 2.3, we get

$$
\begin{aligned}
w\left(\begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}\right) &= w\left(\begin{bmatrix} S^* & T^* \\ 0 & 0 \end{bmatrix}\right) \\
&= w\left(\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix} - i\begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}\right) \\
&\le w\left(\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}\right) \\
&\le \frac{1}{2}\left(\|A\| + \|AA^* + CC^*\|^{1/2} + \|B\| + \|BB^* + DD^*\|^{1/2}\right) \\
&\le \frac{1}{2}\left(\|A\| + \|A+C\| + \|B\| + \|B+D\|\right).
\end{aligned}
$$

□

# References


[1] Amer Abu-Omar, Numerical radius inequalities for $n \times n$ operator matrices, Linear Algebra and its Applications, **468,** (2015), 18–26.

[2] Fuad Kittaneh, Notes on Some Inequalities for Hilbert Space Operators, **24,** no. 2, Publications of the Research Institute for Mathematical Sciences, (1988), 283–293.

[3] Fuad Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, Studia Mathematica, **158,** no. 1, (2003), 11–17.

[4] Khalid Shebrawi, Numerical radius inequalities for certain $2 \times 2$ operator matrices II, Linear Algebra and its Applications, **523,** (2017), 1–12.

[5] Karl E. Gustafson, Duggirala K. M. Rao, Numerical range, 1997.

[6] P. R. Halmos, A Hilbert space problem book, Second Edition, Graduate Texts in Mathematics, Springer-Verlag, New York, 1982.

[7] Xingzhi Zhan, Matrix Inequalities, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, (2002).

[8] Rajendra Bhatia, Positive definite matrices, Princeton University Press, 2007.