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On Some Exponential Diophantine Equations

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Abstract

In this paper, we consider Exponential Diophantine equations of the form $p^X + (p + \lambda + 1)^Y = Z^2$ over the set of all positive integers, where $p > 3$, $p + \lambda + 1$ and λ are primes with λ being a non Sophie Germain prime that is congruent to -1 modulo 4. We show that such equations have no solutions in positive integers X, Y and Z.

1 Introduction

Exponential Diophantine equations of the form $p^X + q^Y = Z^2$ have been studied by many researchers [9], [10], [13], [15], [1], [2], [5], [6], [8], [11], [12], [3], [4], [16], [7], [14], [17]. In particular, Gayo and Bacani [10] studied the Diophantine equation $p^X + q^Y = Z^2$ in relation to Mersenne primes and gave a list of some unsolvable cases for this equation. Mina and Bacani [13] showed that the equation has no positive integer solutions if p and $q = p+4$ are cousin primes, and X, Y are of the same parity. In 2018, Burshtein [6] established that this equation has no solutions in positive integers X, Y, Z whenever p and $q = p + 4$ are primes. For primes $p > 3$ and $q = p + 8$, Fernando [8] showed that this equation has no solution (X, Y, Z) in the set of positive integers. In 2021, Dockan and Pakapongpun [9] proved that this equation

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has no solution in positive integers X, Y, Z when $p > 3$ and $q = p + 20$ are primes. In this paper, we prove that the Exponential Diophantine equation $p^X + (p+\lambda+1)^Y = Z^2$ has no positive integer solutions for X, Y and Z when $p > 3$, $p+\lambda+1$ and λ are primes with λ being a non Sophie Germain prime satisfying the condition $\lambda \equiv -1(\text{mod} 4)$.

Among the special cases of our work, we introduce the following theorems:

Theorem 1.1. *([8]) Let $k \geq 2$ be an integer. If $p > 3$ and $p+8$ are primes, then the diophantine equation $p^X + (p+8)^Y = W^{2k}$ has no solutions in positive integers X, Y and W.*

Theorem 1.2. *([9]) The Diophantine equation $p^X + (p+20)^Y = Z^2$, where $p > 3$, $p+20$ are primes, has no solutions in positive integers X, Y and Z.*

Definition 1.3. *A prime p is said to be a Sophie Germain prime if both p and $2p+1$ are prime. A prime p is non Sophie Germain prime if $2p+1$ is not prime.*

The first few non Sophie Germain prime are:

$$7, 13, 17, 19, 31, 37, 43, 47, 59, 61, 67, 71, 73, 79, 97, 101, 103, 107, 109, 127.$$

2 Main results

In this section, we give the main results of our work. Before we give our theorem and the related corollaries, we introduce the following lemma:

Lemma 1. *Let X, Y, Z be positive integers and let p, q be primes bigger than 3 such that $p^X + q^Y = Z^2$. Then $Z^2 \equiv 0(\text{mod} 4)$.*

Proof. Suppose that X, Y, Z are positive integers and $p, q > 3$ are primes. Then p^X, q^Y are odd primes. So $p^X + q^Y$ is an even integer. Therefore, Z^2 is even and hence Z must be even. Consequently, 4 is a factor of Z^2; that is, $Z^2 \equiv 0(\text{mod} 4)$.

Theorem 2.1 (Main Theorem). *Let λ be a non Sophie Germain prime such that $\lambda \equiv -1(\bmod 4)$. Then the Exponential Diophantine Equation $p^X + (p+\lambda+1)^Y = Z^2$, where $p > 3$ and $p+\lambda+1$ are primes, has no solutions in positive integers X, Y and Z.*

Proof. Assume that λ is a non Sophie Germain prime such that $\lambda \equiv -1(\text{mod}4)$. For primes $p > 3$ and $p+\lambda+1$, we have p^X and $(p+\lambda+1)^Y$ are odd positive integers. Using Lemma 1, we get $p^X + (p+\lambda+1)^Y = Z^2 \equiv 0(\text{mod}4)$.
Here, there are two cases for the prime p.

Case 1. $p \equiv 1(\text{mod}4)$. In this case, $p^X \equiv 1(\text{mod}4)$ and $(p+\lambda+1)^Y \equiv 1(\text{mod}4)$. So, $Z^2 \equiv 2(\text{mod}4)$ which contradicts the fact that $Z^2 \equiv 0(\text{mod}4)$.

Case 2. $p \equiv -1(\text{mod}4)$. For this case, we have four subcases for X and Y.

Subcase 2-1. Both X and Y are even positive integers. Then $X = 2\mu$, $Y = 2\nu$ for some integers $\mu, \nu \geq 1$.
It follows that $p^X \equiv 1(\text{mod}4)$ and $(p+\lambda+1)^Y \equiv 1(\text{mod}4)$. Consequently, $Z^2 \equiv 2(\text{mod}4)$ which is contradiction since $Z^2 \equiv 0(\text{mod}4)$.

Subcase 2-2. Both X and Y are odd positive integers. Then $X = 2\mu+1$, $Y = 2\nu+1$ for some integers $\mu, \nu \geq 0$.
For this case, we have $p^X \equiv -1(\text{mod}4)$ and $(p+\lambda+1)^Y \equiv -1(\text{mod}4)$. Therefore, $Z^2 \equiv 2(\text{mod}4)$ which is impossible since $Z^2 \equiv 0(\text{mod}4)$.

Subcase 2-3. $X = 2\mu+1$, $Y = 2\nu$ for some integers $\mu \geq 0, \nu \geq 1$.
This case implies that

$$p^{2\mu+1} + (p+\lambda+1)^{2\nu} = Z^2. \tag{2.1}$$

If we assume that $q = p+\lambda+1$, then Equation 2.1 becomes $p^{2\mu+1} + q^{2\nu} = Z^2$. Therefore,

$$p^{2\mu+1} = Z^2 - q^{2\nu} = (Z - q^\nu)(Z + q^\nu). \tag{2.2}$$

From Equation 2.2, we get $Z - q^\nu = p^{\mu_1}$ and $Z + q^\nu = p^{\mu_2}$ for some non-negative integers $\mu_1 < \mu_2$ and $\mu_1 + \mu_2 = 2\mu+1$. It follows that

$$p^{\mu_1}(p^{\mu_2-\mu_1} - 1) = p^{\mu_2} - p^{\mu_1} = 2q^\nu. \tag{2.3}$$

Equation 2.3 implies that p^{μ_1} divides the even integer $2q^\nu$. This is true only if $\mu_1 = 0$. Consequently, we have $p^{\mu_2} - 1 = 2q^\nu$ and

$\mu_2 = 2\mu + 1$. It follows that

$$p^{2\mu+1} - 1 = (p-1)\sum_{k=0}^{2\mu} p^k = 2q^\nu. \quad (2.4)$$

Equation 2.4 implies that $2q^\nu$ has $p-1$ as an even divisor. So, $p - 1 = 2q^m$ for some non-negative integer $m < \nu$.
First, let us assume that $m = 0$. In this case, $p = 3$ which contradicts the fact that $p > 3$. Secondly, if $m \neq 0$, then $p - 1 = 2q^m$ which is impossible since

$$p - 1 < p < q < 2q^m.$$

Subcase 2-4. $X = 2\mu$, $Y = 2\nu + 1$ for some integers $\mu \geq 1, \nu \geq 0$.
For this case, we also assume that $q = p + \lambda + 1$. So, $p^{2\mu} + q^{2\nu+1} = Z^2$. Consequently,

$$q^{2\nu+1} = Z^2 - p^{2\mu} = (Z - p^\mu)(Z + p^\mu). \quad (2.5)$$

By the Unique Factorization Theorem and Equation 2.5, there exists some non-negative integers $\eta_3 < \eta_4$ with $\eta_3 + \eta_4 = 2\nu + 1$ such that $Z - p^\mu = q^{\eta_3}$ and $Z + p^\mu = q^{\eta_4}$. It follows that

$$q^{\eta_3}(q^{\eta_4 - \eta_3} - 1) = q^{\eta_4} - q^{\eta_3} = 2p^\mu. \quad (2.6)$$

From Equation 2.6, q^{η_3} becomes a divisor of $2p^\mu$ which is possible only if $\eta_3 = 0$. Thus, $q^{\eta_4} - 1 = q^{2\nu+1} - 1 = 2p^\mu$. Here, there are two possibilities:

(i) If $\nu = 0$, then $q - 1 = 2p^\mu$ or $p + \lambda = 2p^\mu$. It follows that $p(2p^{\mu-1} - 1) = \lambda$, where p, λ are primes. Consequently, $p = \lambda$ since p divides λ. Moreover, we get $q = p + \lambda + 1 = 2\lambda + 1$ which contradicts the fact that q is a prime but $2\lambda + 1$ is not.

(ii) If $\nu > 0$, then

$$q^{2\nu+1} - 1 = (q-1)\sum_{i=0}^{2\nu} q^i = 2p^\mu. \quad (2.7)$$

Equation 2.7 implies that $q - 1$ is an even divisor of $2p^\mu$. More precisely, we obtain $q - 1 = 2p^{\mu_0}$ for some non-negative integer μ_0

less than μ. There are two possibilities for μ_0 to consider in the previous equality.

(ii-1) If $\mu_0 = 0$, then $q = 3$ which contradicts the fact that $q > 3$.

(ii-2) If $\mu_0 > 0$, then $p+\lambda = q-1 = 2p^{\mu_0}$. Hence, $\lambda = p(2p^{\mu_0-1}-1)$. It follows that p is a divisor of λ. Thus $p = \lambda$ since both of p and λ are primes. Consequently, $q = p + \lambda + 1 = 2\lambda + 1$ which is impossible since $2\lambda + 1$ is not prime.

The first result that related to our main theorem (Theorem 2.1), is the following corollary.

Corollary 2.2. *The Exponential Diophantine Equation* $11^X + 31^Y = Z^2$ *has no solutions in positive integers* X, Y *and* Z*.*

Proof. By applying the Theorem 2.1 for $p = 11$ and $\lambda = 19$, the result comes.

The second result that related to our main theorem (Theorem 2.1), is the following corollary.

Corollary 2.3. *For primes* $p > 3$, $p+32$*, the Exponential Diophantine Equation* $p^X + (p + 32)^Y = Z^2$ *has no solutions in positive integers* X, Y *and* Z*.*

Proof. Again, we can apply Theorem 2.1 to get the result. Here, p is any prime bigger than 3 and $\lambda = 31$.

In Table 1 which illustrates some particular cases of the exponential Diophantine Equation $p^X + q^Y = Z^2$ wherein no solutions exist in positive integers X, Y an Z.

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Table 1: List of Some Unsolvable cases of $p^X + q^Y = Z^2$

Conditions on p, q, X, Y and Z	Reference
$p = 3, q = 128$	[10]
$p = 7, q = 8$	[10]
$p = 31, q = 8$	[10]
$p = 127, q = 32$	[10]
$p = 8191, q = 128$	[10]
$p, q = p + 4k$ are primes, $k \geq 2$	[13]
X, Y are of the same parity	
$p > 3, q = p + 4$ are primes	[6]
$p > 3, q = p + 8$ are primes	[8]
$Z = W^k; k \geq 2$	(Special case of Theorem 2.1)
$p = 4, q = 7$	[17]
$p = 4, q = 11$	[17]
$p > 3, q = p + 20$ are primes	[9]
	(Special case of Theorem 2.1)
$p = 31, q = 41$	[11]
$p = 61, q = 71$	[11]
$p = 379, q = 397$	[2]
$p = 61, q = 67$	[12]
$p = 67, q = 73$	[12]
$p = 7, q = 31$	[16]

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