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Supereulerian Digraphs with Forbidden Induced Subdigraphs Containing Short Semi-paths

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Abstract

A digraph D is supereulerian if D has a spanning eulerian subdigraph. We investigate forbidden induced subdigraph conditions for a strong digraph to be supereulerian. The subdigraph H is a semi-path in D if its undirected version is a path in $G(D)$. Let SP_k denote the semi-path on k vertices. For $k = 4$, we determine the smallest integer h_k such that if a strong strict digraph D containing a subdigraph H isomorphic to SP_k always satisfies $|A(D[V(H)])| \geq h_k$, then D is supereulerian.

1 Introduction

We consider finite graphs and digraphs. Undefined terms and notations will follow [8] and [6]. We use (u, v) to represent an arc oriented from a vertex u to a vertex v. As in [8], a digraph D is **strict** if D has no loops and if for any pair of distinct vertices $u, v \in V(D)$, there is at most one arc in D oriented from u to v. Throughout this paper, we only consider strict digraphs. We use $D \cong D'$ to mean that the two digraphs D and D' are isomorphic. For

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an integer $n > 0$, we use K_n^* to denote the complete digraph on n vertices. Hence for every pair of distinct vertices $u, v \in V(K_n^*)$, there is exactly one arc (u, v) in $A(K_n^*)$. For a digraph D, the underlying graph of D, denoted by $G(D)$, is obtained from D by erasing the orientations of all arcs of D.

Following [6], for a digraph D with $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) \ : \ x \in X, y \in Y\}.$$

When $Y = V(D) - X$, we define

$$\partial_D^+(X) = (X, V(D) - X)_D \text{ and } \partial_D^-(X) = (V(D) - X, X)_D.$$

For a vertex $v \in V(D)$, $d_D^+(v) = |\partial_D^+(\{v\})|$ and $d_D^-(v) = |\partial_D^-(\{v\})|$ are the **out-degree** and the **in-degree** of v in D, respectively. Finally, we define the following notations: $\delta^+(D) = \min\{d_D^+(v) \ : \ v \in V(D)\}$ and $\delta^-(D) = \min\{d_D^-(v) \ : \ v \in V(D)\}$. Let $N_D^+(v) = \{u \in V(D) - v : (v, u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) - v : (u, v) \in A(D)\}$ denote the **out-neighbourhood** and **in neighbourhood** of v in D, respectively. We call the vertices in $N_D^+(v)$, $N_D^-(v)$ the **out-neighbours**, **in-neighbours** of v. For a digraph D and a subdigraph S of D, an (x, y)-dipath Q is called an (S, S)-dipath if $V(Q) \cap V(S) = \{x, y\}$. When the digraph D is understood from the context, we often omit the subscript D.

Boesch, Suffel, and Tindell [7] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning Eulerian subgraphs. Pulleyblank [17] later in 1979 proved that determining whether a graph is supereulerian is NP-complete. Since then, there have been lots of researches on this topic. For the literature of supereulerian graphs, see Catlin's first survey [9] on the topic and its updates [10] and [16].

It is natural to study supereulerian digraphs. A digraph D is **eulerian** if D is connected and for any $v \in V(D)$, $d_D^+(v) = d_D^-(v)$; and is **supereulerian** if D contains a spanning eulerian subdigraph. The main problem is to determine supereulerian digraphs. Some earlier studies were done by Gutin [12, 13], and recent developments can be found in [1, 2, 3, 5, 14, 15], among others.

Forbidden induced subgraph conditions have been a widely investigated topic. Given a graph K, a graph G is said to be K**-free** if for each subgraph H of G, if $H \cong K$, then $|E(G[V(H)])| \geq |E(H)| + 1$. Sufficient $K_{1,3}$-free conditions for hamiltonicity have been intensively studied, as seen in [11]. For a vertex w of G, let

$$M_G^i(w) = G[\{x \in V(G) : 1 \leq d_G(w, x) \leq i\}].$$

For $w \in V(G)$, let $N_2(w)$ be the subgraph induced by the set of edges uv, such that either u or v is adjacent to w. A vertex w of G is N^i-locally connected (N_2-locally connected, respectively) if $M_G^i(w)$ ($N_2(w)$, respectively) is connected. If every vertex of G is N^i-locally connected (N_2-locally connected, respectively), then G is **N^i-locally connected (N_2-locally connected, respectively)**. Recently, Saito and Xiong proved the following.

Theorem 1.1. *(Saito and Xiong, [18]) Let H be a connected graph of order at least three, P_k be an undirected path on k vertices, and G be a connected, N^3-locally connected graph. Each of the following holds.*
(i) Every 2-edge connected H-free graph is supereulerian if, and only if H is $K_{1,2}$.
(ii) Every N^2-locally connected H-free graph is supereulerian if and only if H is either $K_{1,2}$ or $K_{1,3}$.
(iii) If G is P_5-free, then G is supereulerian, if G is P_6-free, then G is supereulerian or the Petersen graph.

These motivates the current study on forbidden induced subdigraph sufficient conditions for supereulerian digraphs. For a subdigraph H in D with $V(H) = k$, we say H is a semi-path in D if $G(H)$ is a path in $G(D)$. Throughout the rest of the paper, for an integer $k \geq 3$, SP_k denotes the semi-path on k vertices. A subdigraph H of a digraph D is an SP_k-subdigraph if H is isomorphic to SP_k.

Below we give some examples that are semi-paths in D and some examples that are not semi-paths.

Example.1.

Let $\{u, v, x, y\}$ be distinct vertices in $V(D)$.

(1) The subdigraph consisted of $\{(u, v), (u, y), (x, v)\}$ is a semi-path where $k = 4$.

(2) The subdigraph consisted of $\{(u, v), (v, x), (x, y)\}$ is a semi-path where $k = 4$.

(3) The subdigraph consisted of $\{(u, v), (v, x), (u, x)\}$ is not a semi-path (the undirected version is a cycle).

(4) The subdigraph consisted of $\{(u, v), (u, x), (y, u)\}$ is not a semi-path (the undirected version is not a path).

Definition 1.2. *For integers $h \geq k > 2$, we define $F(SP_k, h)$ to be the family of all strict digraphs such that $D \in F(SP_k, h)$ if and only if D is*

strong and satisfies both of the following.
(i) D contains at least one semi-path SP_k with $|A(D[V(SP_k)])| = h$, and
(ii) for any semi-path SP_k in D, $|A(D[V(SP_k)])| \geq h$.

If $D \in F(SP_k, h)$, then we also call D a $F(SP_k, h)$-digraph. Thus it is of interest to determine the smallest h_k such that every strong strict digraph in $F(SP_k, h_k)$ is supereulerian.

The main purpose of this research is to investigate the behavior of digraphs in $F(SP_k, h)$, $k = 4$, and to determine the value of h_k. We show that $h_4 = 7$. In this paper we deal with semi-paths while in ([5]) they investigated dipaths case. Our result is presented in the following section.

2 Supereulerian digraphs in $F(SP_4, h)$

In this section, we investigate the supereulerianicity of digraphs in $F(SP_4, h)$ with $6 \leq h \leq 7$, and determine the smallest value of h_4 such that every digraph in $F(SP_4, h_4)$ is supereulerian. We need a necessary condition for a digraph to be supereulerian. Let D be a digraph and $U \subset V(D)$. Let $t_0(U)$ be the smallest integer t such that $D[U]$ has a collection of arc disjoint ditrails $T_1, T_2, \cdots, T_t$ with $U = \cup_{i=1}^{t} V(T_i)$. For any subset $A \subseteq V(D) - U$, define $B =: V(D) - U - A$, and

$$h(U, A) =: min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + min\{|(U, B)_D|, |(B, U)_D|\} - t_0(U).$$

Then we have the following proposition.

Proposition 2.1. *(Hong, Lai and Liu, Proposition 2.1 of [14]) If D has a spanning eulerian subdigraph, then for any $U \subset V(D)$, we have $h(U, A) \geq 0$.*

Digraphs in $F(SP_4, h)$ with $h = 6$ are not necessarily supereulerian, as can be seen in the example below.

Example.2.

Let $\alpha, \beta, k > 0$ be integers with $\alpha, \beta \geq k+1$, and let A and B be two disjoint set of vertices with $|A| = \alpha$ and $|B| = \beta$. Let $\ell \geq \alpha\beta + 1$ be an integer, and U be a set of vertices disjoint from $A \cup B$ with $|U| = \ell$. We construct a digraph $D = D(\alpha, \beta, k, \ell)$ such that $V(D) = A \cup B \cup U$ and the arcs of D are given as required in (D1) and (D2) below. (See Figure 4.1. Page 31 in [4]).
(D1) $D[A \cup B] \cong K^*_{\alpha+\beta}$ is a complete digraph.
(D2) For every vertex $u \in U$, and for every $v \in A$, $(u, v) \in A(D)$ and for every $w \in B$, $(w, u) \in A(D)$. Thus for any $u \in U$, we have $N_D^+(u) = A$ and $N_D^-(u) = B$. No two vertices in U are adjacent.

Direct computation yields

$$h(U,A) = |\partial_D^+(A)| + |(U,B)_D| - t_0(U) = \alpha\beta - |U| < 0,$$

and so by Proposition 2.1, any $D = D(\alpha, \beta, k, \ell)$ is nonsupereulerian. By Definition 1.2, $D \in F(SP_4, 6)$. Considering $SP_4' = \{(v_1, u_1), (v_1, v_2), (v_2, u_2)\}$ we find $|A(D[V(SP_4')])| = 6$ where $\{v_1, v_2\} \in A$ and $\{u_1, u_2\} \in U$. Thus Example 2 indicates that $F(SP_4, 6)$ contains infinitely many nonsupereulerian digraphs.

Theorem 2.2. *Each of the following holds.*
(i) Every digraph D in $F(SP_4, 7)$ is supereulerian.
(ii) $h_4 = 7$.

Proof.
As (ii) follows from (i) and Example 2, it suffices to prove (i). Assume that $D \in F(SP_4, 7)$. By contradiction, assume that D is a nonsupereulerian digraph.

Since D is strong, D must have an eulerian subdigraph. Let S be an eulerian subdigraph of D such that among all eulerian subdigraphs of D

$$|V(S)| \text{ is maximized.} \tag{2.1}$$

If $|V(S)| = |V(D)|$, then S is a spanning eulerian subdigraph of D and we are done. Assume by contradiction that $|V(D)| > |V(S)| > 1$. Hence $V(D) - V(S) \neq \emptyset$. Since D is strong, there exists an (S, S)-dipath Q on at least three vertices. Let Q be chosen so that:

$$\text{the length of the shortest dipath } P \text{ in } S \text{ between the endpoints of } Q \text{ is minimized.} \tag{2.2}$$

Assume that $V(Q) \cap V(S) = \{z, r\}$, where z, r are the first and the last vertex of Q and P.
If $P = (z, r)$, then $S - (z, r) + Q$ is an eulerian subdigraph with at least one more vertex than S, contrary to (2.1), moreover, by the maximality of S, z cannot equal r.
Therefore, $|V(P)| \geq 3$ and $|V(Q)| \geq 3$. Let $P = zy_1, y_2, ..., y_d r$ and $Q = zu_1...u_k r$.
There exists at least a vertex $y_c \in \{y_1, y_2, ..., y_d\}$ such that

$$\{(y_c, z), (r, y_c)\} \cap A(S) = \phi, \tag{2.3}$$

otherwise $S - P + Q$ is a closed ditrail greater than S, this because $S - P + Q$ remains connected, and for each vertex in $S - P + Q$ the in-degrees and out-degrees are equal. In the rest of the proof we will consider y_c as the first vertex in $\{y_1, y_2, ..., y_d\}$ that satisfies (2.3).

We consider three cases:

Case 1. $|V(Q)| \geq 4$:

In this case, $SP_4' = \{(u_1, u_2), (z, u_1), (z, y_1)\}$ is an SP_4 in D. By (2.1) and (2.2), we conclude that $\{(u_1, z), (u_2, z), (y_1, u_1), (u_1, y_1), (y_1, u_2), (u_2, y_1)\} \cap A(D) = \emptyset$. It follows that $|A(D[V(SP')])| < 7$, contrary to the assumption that $D \in F(SP_4, 7)$.

Case 2. $|V(Q)| = 3$ and $|V(P)| = 3$:

In this case, $SP_4'' = \{(u_1, r), (z, u_1), (z, y_1)\}$ is an SP_4 in D. By (2.1) and (2.2), we conclude that $\{(r, u_1), (u_1, z), (u_1, y_1), (y_1, u_1)\} \cap A(D) = \emptyset$. By the assumption that $D \in F(SP_4, 7)$ and (2.3) we have $|\{(y_1, z), (r, y_1)\} \cap (A(D) - A(S))| \geq 1$. If $(y_1, z) \in (A(D) - A(S))$, then $S - (y_1, r) + \{(y_1, z), (z, u_1), (u_1, r)\}$ is a greater closed ditrail which violates (2.1).
If $(r, y_1) \in (A(D) - A(S))$, then $S - (z, y_1) + \{(z, u_1), (u_1, r), (r, y_1)\}$ is a greater closed ditrail which violates (2.1).

Case 3. $|V(Q)| = 3$ and $|V(P)| \geq 4$:

(i) $y_c = y_1$:

Let $SP_4^{(3)} = \{(u_1, r), (z, u_1), (z, y_1)\}$.
By (2.1) and (2.2), we conclude that $\{(r, u_1), (u_1, z), (u_1, y_1), (y_1, u_1)\} \cap A(D) = \emptyset$. By the assumption that $D \in F(SP_4, 7)$ we have $|\{(y_1, r), (r, y_1)\} \cap A(D)| \geq 1$. Then there exists an arc a in $A(D)$ such that $a \in \{(y_1, r), (r, y_1)\} \cap A(D)$.
Let $SP_4^{(4)} = \{(y_1, y_2), a, (u_1, r)\}$.
By (2.1) and (2.2), we conclude that $\{(r, u_1), (y_2, u_1), (u_1, y_2), (u_1, y_1), (y_1, u_1)\} \cap A(D) = \emptyset$.
By the assumption that $D \in F(SP_4, 7)$ we have $|\{(y_1, r), (r, y_1)\} \cap (A(D))| = 2$. By (2.3) we have $(r, y_1) \in A(D) - A(S)$.
Thus $S - (z, y_1) + \{(z, u_1), (u_1, r), (r, y_1)\}$ is a greater closed ditrail which violates (2.1).

(ii) $y_c = y_2$:

By (i) $|\{(y_1, r), (r, y_1)\} \cap (A(D))| = 2$.
Let $SP_4^{(5)} = \{(y_1, y_2), (y_1, r), (u_1, r)\}$.

By (2.1) and (2.2), we conclude that $\{(r,u_1),(y_2,u_1),(u_1,y_2),(u_1,y_1),(y_1,u_1)\}\cap A(D)=\emptyset$.
By the assumption that $D\in F(SP_4,7)$ we have $|\{(y_2,r),(r,y_2)\}\cap(A(D))|=2$. By (2.3) we have $(r,y_2)\in A(D)-A(S)$.
Thus $S-\{(z,y_1),(y_1,y_2)\}+\{(z,u_1),(u_1,r),(r,y_2)\}$ is a greater closed ditrail which violates (2.1).

(iii) $y_c=y_j$, where $3\le j\le d$:

Repeating the procedure in (ii) till we get $|\{(y_{j-1},r),(r,y_{j-1})\}\cap(A(D))|=2$.
Let $SP_4^{(6)}=\{(y_{j-1},y_j),(y_{j-1},r),(u_1,r)\}$.
By (2.1) and (2.2), we conclude that $\{(r,u_1),(y_j,u_1),(u_1,y_j),(u_1,y_{j-1}),(y_{j-1},u_1)\}\cap A(D)=\emptyset$.
By the assumption that $D\in F(SP_4,7)$ we have $|\{(y_j,r),(r,y_j)\}\cap(A(D))|=2$. By (2.3) we have $(r,y_j)\in A(D)-A(S)$.
Thus $S-\{(z,y_1),(y_1,y_2),...,(y_{j-1},y_j)\}+\{(z,u_1),(u_1,r),(r,y_j)\}$ is a greater closed ditrail which violates (2.1).
This completes our proof.

References

[1] M. J. Algefari, H.-J. Lai, Supereulerian digraphs with large arc-strong connectivity, J. Graph Theory, **81,** no. 4, (2016), 393–402.

[2] M. J. Algefari, K. A. Alsatami, H.-J. Lai, J. Liu, Supereulerian digraphs with given local structures, Information Processing Letters, **116,** no. 5, (2016), 321–326.

[3] M. J. Algefari, H.-J. Lai, J. Xu, Locally dense supereulerian digraphs, Discrete Applied Mathematics, **238,** (2018), 24–31.

[4] M. J. Algefari, On Supereulerian Digraphs, (2016), https://doi.org/10.33915/etd.5061.

[5] M. J. Algefari, H.-J. Lai, J. Liu, X. Zhang, Supereulerian digraphs with forbidden induced subdigraphs containing short dipaths, Ars Combinatoria, **147,** (2019), 289–302.

[6] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, 2nd Edition, Springer-Verlag, London, 2009.

[7] F. T. Boesch, C. Suffel, R. Tindell, The spanning subgraphs of eulerian graphs, J. Graph Theory, **1,** (1977) 79–84.

[8] J. A. Bondy, U. S. R. Murty, Graph Theory. Springer, New York, 2008.

[9] P. A. Catlin, Supereulerian graphs: a survey, J. Graph Theory, 16, (1992), 177–196.

[10] Z. H. Chen, H.-J. Lai, Reduction techniques for super-Eulerian graphs and related topics-a survey, *Combinatorics and graph theory*, **1,**, (1995), 53–69.

[11] R. Faudree, E. Flandrin, Z. Ryjáček, Claw-free graphs, – A survey, Discrete Math., **164,** (1997), 87–147.

[12] G. Gutin, Cycles and paths in directed graphs. Ph.D Thesis, School of Mathematics, Tel Aviv University, 1993.

[13] G. Gutin, Connected $(g; f)$-factors and supereulerian digraphs. Ars Combin., **54,** (2000), 311–317.

[14] Y. Hong, H.-J. Lai and Q. Liu, Supereulerian digraphs, Discrete Math., 330, (2014) 87-95.

[15] Y. Hong, Q. Liu and H.-J. Lai, Ore-type degree condition of supereulerian digraphs, Discrete Math., 339, (2016) 2042-2050.

[16] H.-J. Lai, Y. Shao and H. Yan, An update on supereulerian graphs, WSEAS Transactions on Mathematics 12, (2013) 926-940.

[17] W. R. Pulleyblank, A note on graphs spanned by Eulerian graphs, J. Graph Theory, 3, (1979) 309-310.

[18] A. Saito and L. Xiong, Forbidden subgraphs and weak locally connected graphs. Preprint.