International Journal of Mathematics and
Computer Science, **17**(2022), no. 1, 33–39

Some results about numerical radius inequalities

Wasim Audeh, Manal Al-Labadi

Department of Mathematics
Faculty of Arts and Sciences
University of Petra
Amman, Jordan

email: waudeh@uop.edu.jo, manal.allabadi@uop.edu.jo

(Received March 17, 2021, Revised June 23, 2021,
Accepted June 25, 2021)

Abstract

Let $X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n$ be operators on a complex separable Hilbert space, $p \geq 2$. Then
$w^p\left(\sum_{i=1}^{n}(X_i + Y_i)\right) \leq 2^{p-1}\left\|\sum_{i=1}^{n}\left(|X_i|^p + |X_i^*|^p + |Y_i|^p + |Y_i^*|^p\right)\right\|.$
This inequality contains several considerable special cases.

1 Introduction

Let $B(H)$ denote the bounded linear operators on the Hilbert space H. For $X \in B(H)$, define

$$w(X) = \sup\left\{|\langle Xx, x\rangle| : x \in H,\ \ \|x\| = 1\right\}.$$

and

$$\|X\| = \sup\left\{\|Xx\| : x \in H,\ \ \|x\| = 1\right\}.$$

Gustafson and Rao [4] proved that

$$\frac{\|.\|}{2} \leq w(.) \leq \|.\|, \tag{1.1}$$

Key words and phrases: Inequality, Numerical radius, Operator, Norm.
AMS (MOS) Subject Classifications: 15A42, 47A63, 47B15.
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

where $w(X)=\frac{\|X\|}{2}$ if $X^2=0$ and $w(X)=\|X\|$ if X is normal.

For $X\in B(H)$, we have $w(X^n)\leq (w(X))^n$ for all natural numbers. For more results on the topic of numerical radius for operators, we refer the reader to [1], [2], [3], [6], [7] and [8]. Kittaneh [8] proved that

$$2w^2(X)\leq \|X^*X+XX^*\|. \tag{1.2}$$

Several new results are provided in this paper.

2 Main Results

The following lemmas are necessary in our paper. Throughout this paper, let $f,g\geq 0$ and continuous on $[0,\infty)$ where $f(t)g(t)=t$ for all nonnegative real numbers.

Lemma 2.1. *Let $m,n\in\mathbb{R}$, $r\geq 2$. Then*

$$|m+n|^r+|m-n|^r\geq 2(|m|^r+|n|^r). \tag{2.3}$$

Lemma 2.2. *Let $m_i,n_i>0$ for $i=1,2,...,n$ and $r\geq 1$. Then*

$$\left(\sum_{i=1}^{n}(m_i+n_i)^r\right)^{1/r}\leq\left(\sum_{i=1}^{n}m_i^r\right)^{1/r}+\left(\sum_{i=1}^{n}n_i^r\right)^{1/r}. \tag{2.4}$$

Lemma 2.3. *Let $m,n\geq 0$, $r\geq 1$. Then*

$$m^r+n^r\leq (m+n)^r\leq 2^{r-1}(m^r+n^r). \tag{2.5}$$

Lemma 2.4. *Let $X\in B(H)$ be self-adjoint operator. Then*

$$|\langle Xx,x\rangle|\leq\langle |X|\,x,x\rangle. \tag{2.6}$$

Lemma 2.5. *Let $X\in B(H)$. Then*

$$|\langle Xx,y\rangle|^2\leq\langle |X|\,x,x\rangle\langle |X^*|\,y,y\rangle \tag{2.7}$$

and more generally,

$$|\langle Xx,y\rangle|^2\leq\left\langle f^2\left(|X|\right)x,x\right\rangle\left\langle g^2\left(|X^*|\right)y,y\right\rangle. \tag{2.8}$$

The following is our main result.

Theorem 2.6. *Let $X_1,X_2,...,X_n,Y_1,Y_2,...,Y_n\in B(H)$, $r\geq 2$. Then*

$$w^r\left(\sum_{i=1}^{n}(X_i+Y_i)\right)\leq 2^{r-2}\left\|\sum_{i=1}^{n}\left(|X_i|^r+|X_i^*|^r+|Y_i|^r+|Y_i^*|^r\right)\right\|. \tag{2.9}$$

Proof.

$$\begin{aligned}
\left|\left\langle \sum_{i=1}^{n}(X_i+Y_i)x,x\right\rangle\right|^r &\leq \sum_{i=1}^{n}\left|\langle (X_i+Y_i)x,x\rangle\right|^r \\
&= \sum_{i=1}^{n}\left|\langle X_ix,x\rangle+\langle Y_ix,x\rangle\right|^r \\
&\leq \sum_{i=1}^{n}\left(\left|\langle X_ix,x\rangle\right|+\left|\langle Y_ix,x\rangle\right|\right)^r \\
&\quad \text{(by triangle inequality)} \\
&\leq \left[\left(\sum_{i=1}^{n}\left|\langle X_ix,x\rangle\right|^r\right)^{1/r}+\left(\sum_{i=1}^{n}\left|\langle Y_ix,x\rangle\right|^r\right)^{1/r}\right]^r \\
&\quad \text{(by Minkowski‘s inequality)} \\
&\leq 2^{r-1}\left[\sum_{i=1}^{n}\left|\langle X_ix,x\rangle\right|^r+\sum_{i=1}^{n}\left|\langle Y_ix,x\rangle\right|^r\right] \\
&\quad \text{(by Lemma 2.3)} \\
&\leq 2^{r-1}\left[\sum_{i=1}^{n}\langle |X_i|\,x,x\rangle^{r/2}\langle |X_i^*|\,x,x\rangle^{r/2}+\sum_{i=1}^{n}\langle |Y_i|\,x,x\rangle^{r/2}\langle |Y_i^*|\,x,x\rangle^{r/2}\right] \\
&\quad \text{(by Lemma 2.4 and Lemma 2.5)} \\
&\leq 2^{r-2}\left[\begin{array}{c}\sum_{i=1}^{n}\left(\langle |X_i|\,x,x\rangle^r+\langle |X_i^*|\,x,x\rangle^r\right)+\\ \sum_{i=1}^{n}\left(\langle |Y_i|\,x,x\rangle^r+\langle |Y_i^*|\,x,x\rangle^r\right)\end{array}\right] \\
&\quad \text{(by A-G mean inequality)} \\
&\leq 2^{r-2}\left[\begin{array}{c}\sum_{i=1}^{n}\left(\langle |X_i|^r\,x,x\rangle+\langle |X_i^*|^r\,x,x\rangle\right)+\\ \sum_{i=1}^{n}\left(\langle |Y_i|^r\,x,x\rangle+\langle |Y_i^*|^r\,x,x\rangle\right)\end{array}\right] \\
&= 2^{r-2}\left[\sum_{i=1}^{n}\left(\begin{array}{c}\langle |X_i|^r\,x,x\rangle+\langle |X_i^*|^r\,x,x\rangle+\\ \langle |Y_i|^r\,x,x\rangle+\langle |Y_i^*|^r\,x,x\rangle\end{array}\right)\right] \\
&\leq 2^{r-2}\left[\sum_{i=1}^{n}\langle\left(|X_i|^r+|X_i^*|^r+|Y_i|^r+|Y_i^*|^r\right)x,x\rangle\right].
\end{aligned}$$

It follows that,

$$w^r\left(\sum_{i=1}^{n}(X_i+Y_i)\right) \le 2^{r-2}\left\|\sum_{i=1}^{n}\left(|X_i|^r+|X_i^*|^r+|Y_i|^r+|Y_i^*|^r\right)\right\|.$$

□

Corollary 2.7. *Let $X, Y \in B(H)$, $r \ge 2$. Then*

$$w^r(X+Y) \le 2^{r-2}\left\||X|^r+|X^*|^r+|Y|^r+|Y^*|^r\right\|. \tag{2.10}$$

Proof. Letting $X_i = Y_i = 0$ for $i = 2, 3, ..., n$ in (2.9), we obtain (2.10). □

It should be mentioned that if we let $X = Y$ in (2.10), we obtain

$$w^r(X) \le \frac{1}{2}\left\||X|^r+|X^*|^r\right\|. \tag{2.11}$$

Now, if we let $r = 2$ in inequality (2.11), we retrieve inequality (1.2).

Theorem 2.8. *Let $X, Y \in B(H)$, $r \ge 2$. Then*

$$w^r(X+Y) \le 2^{r-3}\left\||X+Y|^r+|(X+Y)^*|^r+|X-Y|^r+|(X-Y)^*|^r\right\|. \tag{2.12}$$

Proof.

$$\begin{aligned}
|\langle (X+Y)x,x\rangle|^r &= |\langle Xx,x\rangle+\langle Yx,x\rangle|^r \\
&\le 2^{r-1}\left(|\langle Xx,x\rangle|^r+|\langle Yx,x\rangle|^r\right) \\
&\le 2^{r-2}\,|\langle (X+Y)x,x\rangle|^r+|\langle (X-Y)x,x\rangle|^r \\
&\quad \text{(by using Lemma 2.1).} \\
&\le 2^{r-2}\left[\begin{array}{c}\langle |X+Y|\,x,x\rangle^{r/2}\,\langle |(X+Y)^*|\,x,x\rangle^{r/2}+ \\ \langle |X-Y|\,x,x\rangle^{r/2}\,\langle |(X-Y)^*|\,x,x\rangle^{r/2}\end{array}\right] \\
&\quad \text{(by Lemma 2.5)} \\
&\le 2^{r-2}\left[\begin{array}{c}\left\langle |X+Y|^{r/2}\,x,x\right\rangle\left\langle |(X+Y)^*|^{r/2}\,x,x\right\rangle+ \\ \left\langle |X-Y|^{r/2}\,x,x\right\rangle\left\langle |(X-Y)^*|^{r/2}\,x,x\right\rangle\end{array}\right] \\
&\le 2^{r-3}\left[\begin{array}{c}\langle |X+Y|^r\,x,x\rangle+\left\langle |(X+Y)^*|^r\,x,x\right\rangle+ \\ \langle |X-Y|^r\,x,x\rangle+\left\langle |(X-Y)^*|^r\,x,x\right\rangle\end{array}\right] \\
&= 2^{r-3}\left\langle\left(\begin{array}{c}|X+Y|^r+|(X+Y)^*|^r+ \\ |X-Y|^r+|(X-Y)^*|^r\end{array}\right)x,x\right\rangle.
\end{aligned}$$

It follows that

$$w^r(X+Y) \le 2^{r-3} \left\| |X+Y|^r + |(X+Y)^*|^r + |X-Y|^r + |(X-Y)^*|^r \right\|.$$

□

Corollary 2.9. *Let $X \in B(H)$, $r \ge 2$. Then*

$$w^r(X) \le 2^{r-3} \left\| |X|^r + |X^*|^r \right\|. \tag{2.13}$$

Proof. Replacing Y with X in (2.12), we get (2.13). □

Corollary 2.10. *Let $X \in B(H)$, $r \ge 2$. Then*

$$w^2(X) \le \frac{1}{2} \left\| X^*X + XX^* \right\|.$$

Proof. This inequality follows directly from inequality (2.13) by replacing r with 2. □

Theorem 2.11. *Let $X, Y \in B(H)$ be self adjoint operators, $r \ge 2$. Then*

$$w^r(X+Y) \le 2^{r-2} \left\| |X+Y|^r + |X-Y|^r \right\|. \tag{2.14}$$

Proof.

$$\begin{aligned}
|\langle (X+Y)x, x\rangle|^r &= |\langle Xx, x\rangle + \langle Yx, x\rangle|^r \\
&\le 2^{r-1} \left[|\langle Xx, x\rangle|^r + |\langle Yx, x\rangle|^r \right] \\
&\le 2^{r-2} \left[|\langle (X+Y)x, x\rangle|^r + |\langle (X-Y)x, x\rangle|^r \right] \\
&\quad \text{(by Lemma 2.1)} \\
&\le 2^{r-2} \left[\langle |X+Y|\, x, x\rangle^r + \langle |X-Y|\, x, x\rangle^r \right] \\
&\quad \text{(by Lemma 2.4)} \\
&\le 2^{r-2} \left[\langle |X+Y|^r\, x, x\rangle + \langle |X-Y|^r\, x, x\rangle \right] \\
&\le 2^{r-2} \langle (|X+Y|^r + |X-Y|^r)\, x, x\rangle.
\end{aligned}$$

It follows that

$$w^r(X+Y) \le 2^{r-2} \left\| |X+Y|^r + |X-Y|^r \right\|.$$

□

Corollary 2.12. *Let X be self adjoint operator in $B(H)$, $r \ge 2$. Then*

$$w^r(X) \le 2^{r-2} \left\| |X|^r \right\|, \tag{2.15}$$

and

$$w^2(X) \le \left\| |X|^2 \right\| = \left\| X^2 \right\|. \tag{2.16}$$

Proof. Replacing Y with X in inequality (2.14), we get inequality (2.15). Letting $r = 2$ in inequality (2.15), we get inequality (2.16). □

Theorem 2.13. *Let X, Y be self adjoint operators in $B(H)$. Then*

$$w(XY) \le \frac{1}{2} \left\| |XY| + |(XY)^*| \right\|. \tag{2.17}$$

Proof.

$$\begin{aligned} |\langle XYx, x\rangle| &\le \langle |XY|\, x, x\rangle^{1/2} \langle |(XY)^*|\, x, x\rangle^{1/2} \\ &\quad \text{(by Lemma 2.5)} \\ &\le \frac{1}{2} \left[\langle |XY|\, x, x\rangle + \langle |(XY)^*|\, x, x\rangle\right] \\ &= \frac{1}{2} \langle (|XY| + |(XY)^*|)\, x, x\rangle. \end{aligned}$$

It follows that

$$w(XY) \le \frac{1}{2} \left\| |XY| + |(XY)^*| \right\|.$$

□

Theorem 2.14. *Let $X \in B(H)$, $r \ge 1$. Then*

$$w^r(Y^*X) \le \frac{1}{2} \left\| f^{2r} |X| + \left(Y^* g^2 (|X^*|)\, Y\right)^r \right\|. \tag{2.18}$$

Proof.

$$\begin{aligned} |\langle Y^*Xx, x\rangle|^r &= |\langle Xx, Yx\rangle|^r \\ &\le \left\langle f^2(|X|)\, x, x\right\rangle^{r/2} \left\langle g^2(|X^*|)\, Yx, Yx\right\rangle^{r/2} \\ &\quad \text{(by Lemma 2.5)} \\ &= \left\langle f^2(|X|)\, x, x\right\rangle^{r/2} \left\langle Y^* g^2(|X^*|)\, Yx, x\right\rangle^{r/2} \\ &\le \left\langle f^{2r}(|X|)\, x, x\right\rangle^{1/2} \left\langle \left(Y^* g^2(|X^*|)\, Y\right)^r x, x\right\rangle^{1/2} \\ &\le \frac{1}{2} \left[\left\langle f^{2r}(|X|)\, x, x\right\rangle + \left\langle \left(Y^* g^2(|X^*|)\, Y\right)^r x, x\right\rangle\right] \\ &= \frac{1}{2} \left\langle \left[f^{2r}(|X|) + \left(Y^* g^2(|X^*|)\, Y\right)^r\right] x, x\right\rangle. \end{aligned}$$

Now we get

$$w^r(Y^*X) \leq \frac{1}{2} \left\| f^{2r} |X| + \left(Y^* g^2 (|X^*|)\, Y\right)^r \right\|.$$

□

Remark 2.15. *Inequality (1.2) is a special case of inequality (2.18) which follows by letting $f(t) = g(t) = t^{1/2}$, $r = 2$, $Y = I$.*

References

[1] A. Abu-Omar, F. Kittaneh, Numerical radius inequalities for $n \times n$ operator matrices, Linear Algebra Appl., **468,** (2015), 18–26.

[2] A. Abu-Omar, F. Kittaneh, A generalization of the numerical radius, Linear Algebra Appl., **569,** (2019), 323–334.

[3] R. Bhatia, Matrix Analysis, Springer, New York, 1997.

[4] K. E. Gustafson, D. K. M. Rao, Numerical range, the field of values of linear operators and matrices, Springer, New York, 1997.

[5] P. R. Halmos, , A Hilbert space problem book, 2nd ed., Springer, New York, 1982.

[6] O. Hirzallah, F. Kittaneh, K. Shebrawi, Numerical radius inequalities for commutators of Hilbert space operators, Numer. Funct. Anal. Optim., **32,** (2011), 739–749.

[7] J. C. Hou, H. K. Du, Norm inequalities of positive operator matrices, **22,** (1995), 281–294.

[8] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, Studia math., **168,** (2005), 73–80.