International Journal of Mathematics and Computer Science, **17**(2022), no. 1, 377–394

On the locating chromatic number of trees

Yusuf Hafidh, Edy Tri Baskoro

Combinatorial Mathematics Research Group
Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung
Jalan Ganesa 10 Bandung, Indonesia

emails: yusufhafidh@math.itb.ac.id, ebaskoro@math.itb.ac.id

(Received July 3, 2021, Accepted September 4, 2021)

Abstract

A palm $S_n(a_1, a_2, \cdots, a_n)$ is simply a subdivision of a star S_n on every edge in $a_i - 1$ times (for the i^{th}-edge). In this paper, we derive some 'better' upper bound of the locating chromatic number of a tree by using the locating coloring of its composing palms. We also determine the locating chromatic number of a palm itself. We prove the complexity of the locating chromatic number of a regular palm tree $S_n(k)$ and an olive $S_n(1, 2, \cdots, n)$, namely $\chi_L(S_n(k)) = \Theta(n^{1/k})$; $\chi_L(S_n(3)) = (1 + o(1))\sqrt[3]{4n}$; and $\chi_L(O_n) = \left\lceil \log_3\left(\frac{n}{4}\right)\right\rceil + 3$.

1 Introduction

Let $G = (V, E)$ be a simple connected graph. For any $u \in V$ and $S \subseteq V$, the *distance* from vertex u to S is defined by $d(u, S) = \min\{d(u, v) \mid v \in S\}$. A set $S \subseteq V$ *resolves* two vertices u and v if $d(u, S) \neq d(v, S)$. Let $c : V \to \{1, 2, \cdots, k\}$ be a k-coloring of G and $c^{-1}(i) = \{v \in V \mid c(v) = i\}$. A coloring c is called a *locating k-coloring* (or simply a *locating coloring*) if for every two vertices, there exists a color class $c^{-1}(i)$ that resolves them. The *locating chromatic number* of G, denoted by $\chi_L(G)$, is the smallest integer k such that G has a locating k-coloring. The *color code* of a vertex v with

Key words and phrases: Locating chromatic number, tree, graph
AMS Subject Classifications: 05C15, 05C12.
Corresponding author: E. T. Baskoro
ISSN 1814-0432, 2022, http://ijmcs.future-in-tech.net

respect to c is given by $r_c(v) = (d(v, c^{-1}(1)), d(v, c^{-1}(2)), \cdots, d(v, c^{-1}(k)))$. The locating chromatic number is also called the metric chromatic number [12].

Several algorithms have been proposed to compute some upper bound for the locating chromatic number of trees [2, 6, 11, 14]. In almost all of these algorithms [6, 11, 14], all the vertices which are not in an end-path (a path joining a leaf to its nearest branch) are colored by using only two colors. In particular, one of the graphs with the locating chromatic number far from all the known upper bounds is an olive O_n. From the algorithms in [11] and [14], we have $\chi_L(O_n) \leq n+1$; and from [6], we have $\chi_L(O_n) \leq \lceil \sqrt{n} \rceil + 1$. The exact value for $\chi_L(O_n)$ is $\left\lceil \log_3 \left(\frac{n}{4}\right) \right\rceil + 3$ as stated in Corollary 4.3.

The known algorithms for computing the locating chromatic number of a tree T yield values still far from the exact one since they do not employ an optimal locating coloring in the composing palms of T. The coloring algorithms in [11] and [14] use $n+1$ colors in a palm with n leaves and the coloring algorithm in [6] uses $\lceil \sqrt{n} \rceil + 1$ colors. This situation implies that a 'better' coloring algorithm on a tree could be achieved by optimizing the number of colors used in the palms and so this motivates us to study the locating chromatic number of a palm.

In general, we will follow the terminologies introduced in [13]. A *palm* $S_n(a_1, a_2, \cdots, a_n)$ for $n \geq 2$, is a graph obtained from a star S_n on $n+1$ vertices by subdividing the i^{th} edge in $a_i - 1$ times. Let us denote by

$$V = \{a_0\} \cup \{a_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq a_i\}, \text{ and}$$
$$E = \{a_0 a_{i,1} \mid 1 \leq i \leq n\} \cup \{a_{i,j} a_{i,j+1} \mid 1 \leq i \leq n, 1 \leq j \leq a_i - 1\}$$

the vertex- and edge- sets of $S_n(a_1, a_2, \cdots, a_n)$.

The k^{th} *level* is the set of vertices of distance k to the *hub* vertex a_0, and the k^{th} *end-path* is the subgraph induced by the set $\{a_0\} \cup \{a_{k,j} : 1 \leq j \leq a_k\}$. If $a_i = i$ for every i then this palm tree is called an *olive* tree and denoted by O_n, namely $O_n = S_n(1, 2, \cdots, n)$. Figure 1 is an example of an olive tree O_5. If $a_i = k$ for some k then the palm tree is called *regular*, and it is denoted by $S_n(k) := S_n(k, k, \cdots, k)$.

In the second section, we derive a 'better' upper bound of the locating chromatic number of trees by using the locating coloring of its composing palms. In the third section, we study the relation between the locating chromatic number of a graph and its maximum degree. In the fourth section, we discuss a tight upper and lower bound for the locating chromatic number of palms and we also prove that, for every integer k between the bounds, there is a palm having the locating chromatic number equal to k. In the last

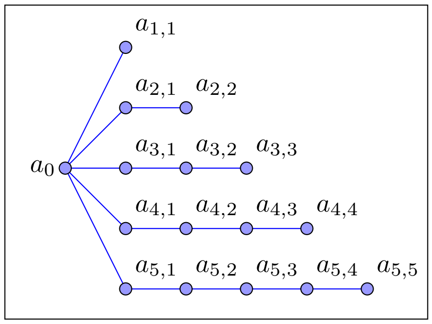

Figure 1: Graph $O_5 = S_5(1, 2, 3, 4, 5)$

section, we take an asymptotic approach to study the locating chromatic number of regular palms and we prove that $\chi_L(S_n(k)) = \Theta(n^{1/k})$. This leads to the observation that $\chi_L(S_n(k))$ is decreasing and goes to $\lceil \log_3 \left(\frac{n}{4}\right) \rceil + 3$ as a function of k but it is increasing and unbounded as a function of n.

2 Better upper bound

In this section, we propose an algorithm to build a locating coloring for any given tree. This algorithm requires a locating coloring on each of its palms which will be provided in the next sections. We also compare our algorithm with the algorithm given in [14] and a combined result in [11] and [17].

Let T be a tree. Any vertex u of T with degree at least 3 is called a *branch* in T. An *end-path* in a tree T is a path connecting a leaf with its nearest branch. If there are at least two end-paths starting from a branch vertex u, then the vertex u is called a *local-end branch*. A *palm* on vertex u of a tree T is a subtree constructed from u and its end-paths starting from u.

The following Algorithm 1 gives a locating coloring of a given tree T utilizing a locating coloring of all of its palms.

Theorem 2.1. *If T is a tree with b palms $P_1, P_2, \cdots, P_b$, then*

$$\chi_L(T) \leq 2 - 2b + \sum_{i=1}^{b} \chi_L(P_i).$$

To prove Theorem 2.1, we need the following lemma

Algorithm 1 Building a locating coloring of a tree T.

procedure A LOCATING COLORING OF A TREE $c(T, n)$
  Input :Tree T, locating coloring of all of its palms
  Output: A locating coloring c of T
  Fix a vertex w and let $c(w) = 1$.
  for every vertex u **do** $c(u) = d(u, w) \pmod 2 + 1$,
  $m \leftarrow 0$
  for every palm P of T **do**
    Let $v :=$ branch vertex of palm P, $v_1 \sim v$ with $v_1 \in V(P)$
    Let c' be a locating coloring of this palm P with $c'(v) = c(v)$ and $c'(v_1) = c(v_1)$
    for every vertex u in this palm **do**
      if $c'(u) \leq 2$ **then**
        $c(u) \leftarrow c'(u)$
      else
        $c(u) \leftarrow m + c'(u)$
    $m \leftarrow m - 2 + \max\{c'(u) : u \text{ in this palm}\}$

Lemma 2.2. *[6] Let G be a graph and let xy be a bridge of G. Let G_x and G_y be the components of $G - xy$ containing x and y, respectively. Let c be a coloring of G. If there exist i and j such that $c^{-1}(i) \subseteq V(G_x)$ and $c^{-1}(j) \subseteq V(G_y)$, then ,for any two vertices $u \in V(G_x)$ and $v \in V(G_y)$, their color codes are different.*

Proof of Theorem 2.1. Let T be a tree with b palms. Color such a tree T by using Algorithm 1. Then any two vertices in the same palm of T will be distinguished by the existence of the locating coloring in that palm and any other two vertices will be distinguished by Lemma 2.2. □

Theorem 2.1 makes us realize the importance of studying the locating chromatic number of a specific tree called a palm. If we could optimize the locating coloring in each palm, then by Theorem 2.1 we obtain a better upper bound of the tree composed by such palms. In section 4, we will pay attention to find a locating coloring of a palm.

2.1 Algorithm comparison

In this section, we will compare the upper bounds obtained by various algorithms. Let T be a tree with l leaves and β branches with at least one end-path. If $dim(T)$ is the metric dimension of T, then, by combining the result in [17],

$$\chi_L(T) \leq dim(T) + \chi(T), \tag{2.1}$$

and the result in [11],

$$dim(T) = l - \beta, \tag{2.2}$$

we obtain the following theorem.

Theorem 2.3. *[11, 17] Let T be a tree with l leaves and β branch with at least one end-path, then $\chi_L(T) \leq l - \beta + 2$.*

Another upper bound is given in the following theorem.

Theorem 2.4. *[14] Let T be a tree having l leaves and b branch with at least two end-paths, then $\chi_L(T) \leq l - b + 2$.*

The upper bound for the locating chromatic of any tree T in Theorem 2.1 is better than the upper bounds in Theorems 2.3 and 2.4.

Theorem 2.5. *Let T be a tree having l leaves, β branch with at least one end-path, and b branch with at least two end-paths. If $P_1, P_2, \cdots, P_b$ are the palms of T, then*

$$\chi_L(T) \leq 2 - 2b + \sum_{i=1}^{b} \chi_L(P_i) \leq l - \beta + 2 \leq l - b + 2.$$

Proof. We will only prove the second inequality. Let l_i be the number of leaves in a palm P_i. There are $\beta - b$ end branch(es) with exactly one end-path. So $l = \beta - b + \sum_{i=1}^{b} l_i$. By Theorem 2.3, $\chi_L(P_i) \leq l_i + 1$ and the result follows. □

3 Maximum degree

In this section, we derive a relation between the maximum degree of any graph with its locating chromatic number. The relation between the maximum degree of a graph with its metric dimension is used to characterize infinite graphs with finite metric dimension [9]. The maximum degree of a graph having certain locating chromatic number is also needed to characterize infinite graphs with finite locating chromatic number. In particular, we show that any graph G with locating chromatic number $k \geq 3$ must have maximum degree $\Delta(G)$ at most $4 \cdot 3^{k-3}$.

Theorem 3.1. *If G is a graph with $\chi_L(G) = k \geq 3$, then $\Delta(G) \leq 4 \cdot 3^{k-3}$.*

Proof. Let G be a graph with $\chi_L(G) = k \geq 3$ and $c : V(G) \to \{1, 2, \cdots, k\}$ be a locating coloring of G. Consider the color code of any vertex v; i.e., $r_c(v) = (a_1, a_2, \cdots, a_k)$. Without loss of generality, by permuting the colors, we may assume that $c(v) = 1$ and so $a_1 = 0$ and $0 < a_2 \leq \cdots \leq a_k$. Let u be a neighbor of v and $r_c(u) = (b_1, b_2, \cdots, b_k)$. By the triangle inequality, $|a_i - b_i| \leq 1$ for all i and so $b_i \in \{a_i - 1, a_i, a_i + 1\}$ for all i.

We now prove that $d(v) \leq 4 \cdot 3^{k-3}$ for any vertex v. Suppose, to the contrary, that $d(v) \geq 4 \cdot 3^{k-3} + 1$. First, group all the neighbors of v depending to the distances to colors $4, 5, \cdots, k-1$ and k. All neighbors of v with the same distances to colors $4, 5, \cdots, k-1$ and k will be in the same group. This means that the color codes of all members in the group will have the same ordinates in positions $4, 5, \cdots, k-1$, and k. Since the distance of any neighbor of v to $c^{-1}(i)$ is either $a_i - 1$, a_i, or, $a_i + 1$, there will be at most 3^{k-3} groups. Since v has $d(v) \geq 4 \cdot 3^{k-3} + 1$ neighbors, by the pigeon hole principle, there exists a group containing at least 5 vertices; say, u_1, u_2, u_3, u_4, u_5. The color codes of all the members of such a group will be $(1, *, *, x_4, x_5, \cdots, x_k)$, for some fixed nonnegative integers $x_4, x_5, \cdots, x_k$.

If there exists a vertex u in $U = \{u_1, u_2, \cdots, u_5\}$ with $c(u) \geq 4$, then $c(u_i) = c(u)$, for all $i \in \{1, 2, \cdots, 5\}$. Therefore, $0 = a_1 < a_2 \leq a_3 \leq \cdots \leq a_{c(u)} = 1$ and so $a_2 = a_3 = 1$. This implies that, for every $u \in U$, $d(u, c_j)$ is 1 or 2, for $j = 2, 3$. Since there are 5 vertices in U with 4 possible representations, there will be two distinct vertices with the same color code, a contradiction.

Now, the only possibility is that the color of each vertex $u \in U$ is either 2 or 3; it cannot be color 1 because u is adjacent to v and $c(v) = 1$. If all vertices in U have the same color; say, $c(u) = x$ for every $u \in U$ with $x = 2$ or $x = 3$, let $y \in \{2, 3\} - \{x\}$ (the other color). Then $d(u, c^{-1}(1)) = 1$, $d(u, c^{-1}(x)) = 0$, and $d(u, c^{-1}(y)) \in \{a_y - 1, a_y, a_y + 1\}$. This means that there are 5 vertices in U with 3 possible representations. Therefore , there will be two vertices with the same color code, a contradiction.

So U must contain vertices of colors 2 and 3 only, and as a result $a_2 = a_3 = 1$. Let $u \in U$. If $c(u) = 2$, then $d(u, c^{-1}(1)) = 1$, $d(u, c^{-1}(2)) = 0$, and $d(u, c^{-1}(3)) \in \{1, 2\}$. In addition, if $c(u) = 3$, then $d(u, c^{-1}(1)) = 1$, $d(u, c^{-1}(2)) \in \{1, 2\}$, and $d(u, c^{-1}(3)) = 0$. Again, we have 4 possible representations for at least 5 vertices. Therefore, there will be two vertices with the same color codes, a contradiction.

Therefore, $deg(v) \leq 4 \cdot 3^{k-3}$ for any vertex v. Consequently, $\Delta(G) \leq 4 \cdot 3^{k-3}$. □

The tightness of this bound will be discussed in the next section. A different proof of Theorem 3.1 was given in [7]. In [11], Chartrand et al. gave the following result.

Theorem 3.2. *(Theorem 4.3 in [11]) Let $k \geq 3$. If T is a tree for which $\Delta(T) > (k-1)2^{k-2}$, then $\chi_L(T) > k$.*

In other form, we have that if T is a tree with locating chromatic number $\chi_L(T) = k$ (≥ 3), then $\Delta(T) \leq (k-1)2^{k-2}$. This result is true only for $k = 3$ and $k = 4$. For $k \geq 5$, Theorem 3.1 corrects the upper bound of the maximum degree of such tree T; namely, $\Delta(T) \leq 4 \cdot 3^{k-3}$. Figure 2 gives a locating coloring with $k = 5$ colors for a tree with $\Delta(T) = 36$.

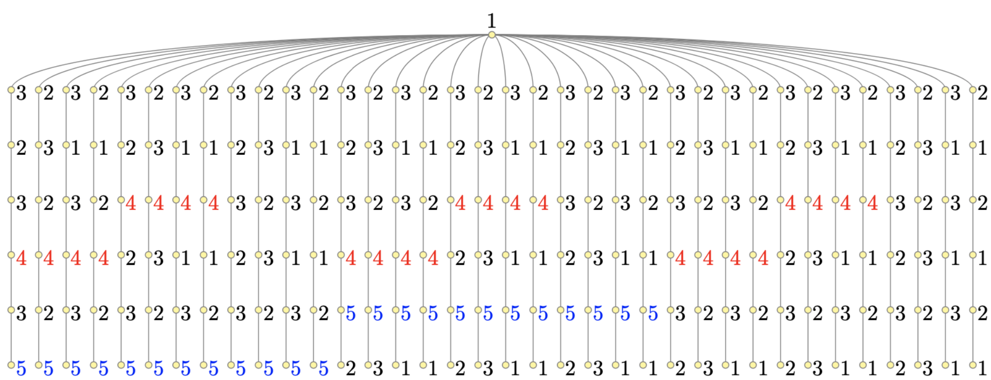

Figure 2: A tree T with $\Delta(T) = 36$ and $\chi_L(T) = 5$.

4 General palms

In this section, we give tight upper and lower bounds of the locating chromatic number of a palm $S_n(a_1, a_2, \cdots, a_n)$. We show that the upper bounds of the maximum degree in Theorems 3.1 are tight not only for a general graph but also for trees. We also show that for every integer $k \geq 3$, there is a palm with locating chromatic number k and $\Delta = 4 \cdot 3^{k-3}$.

Theorem 4.1. *For a palm $G = S_n(a_1, a_2, \cdots, a_n)$ with $n \geq 2$, we have*

$$\left\lceil \log_3\left(\frac{n}{4}\right) \right\rceil + 3 \leq \chi_L(G) \leq n + 1. \tag{4.3}$$

Moreover, for every k *with* $\lceil \log_3\left(\frac{n}{4}\right)\rceil + 3 \leq k \leq n+1$*, there exists a palm* G *with* $\chi_L(G) = k$*.*

Proof. We will prove the first part of Theorem 4.1 with the second part proven after Corollary 4.4. The lower bound is a direct consequence of Theorem 3.1. To prove the upper bound, let $c : V \to \{1, 2, \cdots, n+1\}$ with $c(a_0) = n+1$, $c(a_{i,j}) = i$ if j is odd, and $c(a_{i,j}) = n+1$ if j is even. Any two vertices in the same end-path; say, the i^{th} end-path, are resolved by $c^{-1}(j)$ for every $j \neq i$, and any two vertices in different end-paths, say the i^{th} and j^{th} end-paths, are resolved by $c^{-1}(i)$ and $c^{-1}(j)$. Thus, c is a locating $(n+1)$-coloring of G and the result follows. □

Next, we shall study the locating chromatic number of palms with a special property. A palm tree $S_n(a_1, a_2, \cdots, a_n)$ with $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ is said to have property $\mathbb{O}$ if $a_3 \geq 2$, $a_{4.3^k+1} \geq 2k+3$ and $a_{8.3^k+1} \geq 2k+4$ for all non negative integers $k \leq \log_3\left(\frac{n}{4}\right)$. Palms with property $\mathbb{O}$ are counter examples of Theorem 4.3 in [11]. The tightness of the upper bound of the maximum degree in Theorem 3.1 and the lower bound in Theorem 4.1 are achieved by palms with property $\mathbb{O}$. The upper bound in Theorem 4.1 is achieved by star S_n.

We give an example illustrating Algorithm 2. If $n = 108$, then $k = 6$; write $i = 57 = 4 \times 14 + 1$ with $(14)_{10} = (112)_3$, $i = 80 = 4 \times 19 + 4$ with $(19)_{10} = (201)_3$, and $i = 100 = 4 \times 24 + 4$ with $(24)_{10} = (220)_3$. So $57 = 4 \times (112)_3 + 1$, $80 = 4 \times (201)_3 + 4$, and $100 = 4 \times (220)_3 + 4$.

The sequences $A_i := \{c(a_{i,j})\}$ for $i = 1, 57, 80, 100$ are as follows.

$$A_{4\times(000)_3+1} = \{2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \cdots\}$$
$$A_{4\times(112)_3+1} = \{2, 1, 2, 4, 5, 1, 6, 1, 2, 1, 2, \cdots\}$$
$$A_{4\times(201)_3+4} = \{3, 2, 4, 2, 3, 2, 3, 6, 3, 2, 3, \cdots\}$$
$$A_{4\times(220)_3+4} = \{3, 2, 3, 2, 3, 5, 3, 6, 3, 2, 3, \cdots\}$$

Theorem 4.2. *Let* G *be an* $\mathbb{O}$ *palm. Then* $\chi_L(G) = \lceil \log_3\left(\frac{n}{4}\right)\rceil + 3$*.*

Proof. Since $\Delta(G) = n$, by Theorem 4.1, we have $\chi_L(G) \geq \lceil \log_3\left(\frac{n}{4}\right)\rceil + 3$. Now, construct a locating coloring on G with $k = \lceil \log_3\left(\frac{n}{4}\right)\rceil + 3$ colors by using Algorithm 2. We will prove that the color codes of all vertices are different.

Note that vertex a_0 is the only vertex whose color 1 and has neighbors with colors 2 and 3. So its color code is different from the color codes of the other vertices. Let $a_{i,j}$ and $a_{p,q}$ be two different vertices and write $i =$

Algorithm 2 Building a locating coloring of an $\mathbb{O}$ palm

procedure A LOCATING COLORING OF AN $\mathbb{O}$ PALM
  Input: An $\mathbb{O}$ palm $S_n(a_1, a_2, \cdots, a_n)$
  Output: c, a locating coloring of $S_n(a_1, a_2, \cdots, a_n)$
  $c(a_0) \leftarrow 1$
  $k \leftarrow \left\lceil \log_3\left(\frac{n}{4}\right)\right\rceil + 3.$
  for $i = 1, 2, \cdots, n$ **do**
    Define l and r such that $i = 4l + r$ with $r \in \{1, 2, 3, 4\}$ and $l \in \mathbb{Z}$,
    Write $l = (l_k l_{k-1} \cdots l_5 l_4)_3$, a $(k-3)$-digit number in base 3;
    if $r = 1$ **then** $(x, y) \leftarrow (2, 1)$
    if $r = 2$ **then** $(x, y) \leftarrow (3, 1)$
    if $r = 3$ **then** $(x, y) \leftarrow (2, 3)$
    if $r = 4$ **then** $(x, y) \leftarrow (3, 2)$
    for $j = 1, 2, \cdots, a_i$ **do**
      if j is odd **then**
        $c(a_{i,j}) \leftarrow x$
      else
        $c(a_{i,j}) \leftarrow y$
      for $t = 4, 5, \cdots, k$ **do**
        if $l_t \neq 0$ **then**
          $c(a_{i,2t+l_t-6}) \leftarrow t$

$4 \times (\overline{i_k i_{k-1} \cdots i_5 i_4}) + r_1$ and $p = 4 \times (\overline{p_k p_{k-1} \cdots p_5 p_4}) + r_2$ as in lines 7 and 8 in Algorithm 2. Let (w, x) and (y, z) be the alternating coloring for A_i and A_p as in lines 9-12 in the Algorithm. Consider the following cases:

Case I: $\mathbf{j \neq q}$. In this case, $a_{i,j}$ and $a_{p,q}$ are in different levels. Without loss of generality, let $j > q$. If $\{w, x\} = \{y, z\}$, then the color $s \in \{1, 2, 3\} - \{w, x\}$ is not used in A_i and A_p. Since $j \neq q$, $d(a_{i,j}, c^{-1}(s)) \neq d(a_{p,q}, c^{-1}(s))$. If $\{w, x\} \neq \{y, z\}$, then there is a color used in A_i but not in A_p and vice versa. Let $s \in \{y, z\} - \{w, x\}$; s is the color used in A_p but not in A_i. Note that either $a_{p,1}$ or $a_{p,2}$ is colored by s. So $d(a_{p,q}, c^{-1}(s)) < q < j < d(a_{i,j}, c^{-1}(s))$.

Case II: $\mathbf{j = q}$. Since $a_{i,j}$ and $a_{p,q}$ are in the same level, they must be in different end-paths; $i \neq p$. If there is a t $(4 \leq t \leq k)$ with $i_t \neq p_t$, then the position of vertex with color t is different in A_i and A_p. So $d(a_{i,j}, c^{-1}(t)) \neq d(a_{p,q}, c^{-1}(t))$. If $i_t = p_t$ for all t, then $r_1 \neq r_2$ which means that A_i and A_p have different alternating colorings. If $w \neq y$, then these two colors will distinguish $r_c(a_{i,j})$ and $r_c(a_{p,q})$ because they are in the same level. A similar argument can be applied if $x \neq z$.

Thus, we have constructed a locating coloring of G with $k = \left\lceil \log_3 \left(\frac{n}{4}\right) \right\rceil + 3$ colors. Therefore, $\chi_L(G) = \left\lceil \log_3 \left(\frac{n}{4}\right) \right\rceil + 3$. □

The following corollaries are special cases of Theorem 4.2 when $a_i = i$ and $a_i = k$ for all i.

Corollary 4.3. *For any olive O_n with $n \geq 2$, $\chi_L(O_n) = \left\lceil \log_3 \left(\frac{n}{4}\right) \right\rceil + 3$.*

Corollary 4.4. *Let $n \geq 3$ and $k \geq 2\left\lceil \log_3 \left(\frac{n}{4}\right) \right\rceil + 4$, then $\chi_L(S_n(k)) = \left\lceil \log_3 \left(\frac{n}{4}\right) \right\rceil + 3$.* □

To prove the second part of Theorem 4.1, we need to prove the following lemma:

Lemma 4.5. *Let $G = S_n(a_1, \cdots, a_n)$ and $G' = S_n(a_1, \cdots, a_i + 1, \cdots, a_n)$, then*

$$\chi_L(G') \geq \chi_L(G) - 1.$$

Proof. Let $\chi_L(G') = p$ and c' be a locating p-coloring of G'. We will construct a locating $(p+1)$-coloring of G. By doing so, we will have $\chi_L(G) \leq p+1$ and the result follows.

Let w be the only vertex in G' but not in G and let z be the only neighbor of w. Define $c : V(G) \to \{1, 2, \cdots, p+1\}$ with $c(z) = p+1$ and $c(v) = c'(v)$ if $v \neq z$. Without loss of generality let $c'(w) = 1$. Suppose there are two different vertices u and v in G with $r_c(u) = r_c(v)$; that means, $d_G(u, c^{-1}(k)) =$

$d_G(v, c^{-1}(k))$ for $k = 1, 2, \cdots, p+1$. Since $d_G(u, c^{-1}(k)) = d_{G'}(u, c^{-1}(k))$ for $k = 2, 3, \cdots, p$ and $r_{c'}(u) \neq r_{c'}(v)$, we have $d_{G'}(u, c^{-1}(1)) \neq d_{G'}(v, c^{-1}(1))$.

Without loss of generality, let $d_{G'}(u, c^{-1}(1)) < d_{G'}(v, c^{-1}(1))$. Since $c'(w) = 1$, we have $d_{G'}(u, c^{-1}(1)) \leq d_{G'}(u, w)$. Consider the following cases:

Case I : $d_{G'}(u, c^{-1}(1)) < d_{G'}(u, w)$.
In this case, $d_G(u, c^{-1}(1)) = d_{G'}(u, c^{-1}(1))$. This means that $d_{G'}(v, c^{-1}(1)) \leq d_G(v, c^{-1}(1)) = d_G(u, c^{-1}(1)) = d_{G'}(u, c^{-1}(1))$, a contradiction.

Case II : $d_{G'}(u, c^{-1}(1)) = d_{G'}(u, w)$.
In this case, $d_{G'}(v, w) \geq d_{G'}(v, c^{-1}(1)) > d_{G'}(u, c^{-1}(1)) = d_{G'}(u, w)$. This implies $d_G(v, c^{-1}(p+1)) = d_G(v, z) = d_{G'}(v, w) - 1 > d_{G'}(u, w) - 1 = d_G(v, z) = d_G(v, c^{-1}(p+1))$, a contradiction. Thus c is a locating $(p+1)$-coloring of G. $\square$

Now we will prove the second part of Theorem 4.1.

Proof of Theorem 4.1 (2). Define

$$S_n = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_z = O_n = S_n(1, 2, \cdots, n)$$

where $|G_{i+1}| = |G_i| + 1$. Note that the previous sequence is not unique. Consider the sequence

$$n + 1 = \chi_L(G_0), \chi_L(G_1), \chi_L(G_2), \cdots, \chi_L(G_z) = \left\lceil \log_3 \left(\frac{n}{4}\right) \right\rceil + 3.$$

From Lemma 4.5, we have $\chi_L(G_{i+1}) \geq \chi_L(G_i) - 1$. Note that the previous sequence is decreasing in some of its terms because $n + 1 \geq \left\lceil \log_3 \left(\frac{n}{4}\right) \right\rceil + 3$. Since each term is an integer and when it decreases it can only decrease by 1, the sequence will pass every integer between $\left\lceil \log_3 \left(\frac{n}{4}\right) \right\rceil + 3$ and $n+1$. This means that for every integer k between $\left\lceil \log_3 \left(\frac{n}{4}\right) \right\rceil + 3$ and $n+1$, there exists a palm G_i with $\chi_L(G_i) = k$. $\square$

Theorem 4.6. *Let $G = S_n(a_1, a_2, \cdots, a_n)$ be a palm. Then $\chi_L(G) = n + 1$ if and only if G is a star.*

Proof. $(\Leftarrow)$: This is trivial.
$(\Rightarrow)$: Let G be a palm which is not a star. Without loss of generality, let $a_1 > 1$. Let $c : V \to \{1, 2, \cdots, n\}$ with $c(a_0) = 1$, $c(a_{1,j}) = 2$ if j is odd, $c(a_{1,j}) = 3$ if j is even, $c(a_{i,j}) = i$ if j is odd, and $c(a_{1,j}) = 1$ if j is even. It is not hard to see that c is a locating n-coloring of G and the result follows. $\square$

5 Regular palms

In this section, we will study the order of the locating chromatic number of regular palms. If we fix n and consider $\chi_L(S_n(k))$ as a function of n, then $\chi_L(S_n(k)) \to \left\lceil \log_3\left(\frac{n}{4}\right)\right\rceil + 3$ as $k \to \infty$ from Corollary 4.4. However, if we fix k and consider $\chi_L(S_n(k))$ as a function of n, then $\chi_L(S_n(k))$ is increasing and unbounded (Lemma 5.4).

Let f and g be two function with integer variable n. We say that $f = O(g)$ if there exist $c > 0$ such that $|f(n)| \leq c\,|g(n)|$ for large values of n, $f = \Omega(g)$ if $g = O(f)$, and $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. We also denote $f = o(g)$ if $\lim \frac{f}{g} = 0$. Note that $\lim \frac{f}{g} = 1$ is equivalent to $f = (1 + o(1))g$.

The order of $\chi_L(S_n(k))$ is given in the following theorem:

Theorem 5.1. *For every fixed positive integer k, $\chi_L(S_n(k)) = \Theta\left(n^{\frac{1}{k}}\right)$.*

In the following theorems, we give the exact value of $\chi_L(S_n(k))$ for $k = 1, 2, 3$.

Theorem 5.2. *For $n \geq 2$, $\chi_L(S_n(1)) = n + 1$ and $\chi_L(S_n(2)) = \lceil\sqrt{n}\rceil + 1$.*

Theorem 5.3. *For integers $p \geq 3$, let $f(p) = (p-1)\left\lfloor \frac{p^2}{4}\right\rfloor - \left\lfloor \frac{p^2-2p}{4}\right\rfloor$. Then*

$$\chi_L(S_n(3)) = p \iff f(p-1) < n \leq f(p); \tag{5.4}$$

or simply $\chi_L(S_n(3)) = (1 + o(1))\sqrt[3]{4n}$.

We end this section with the following conjecture:

Conjecture 5.1. *For $k \geq 4$, $\chi_L(S_n(k)) = (1 + o(1))\left(\frac{k-1}{2}\right)\sqrt[k]{4n}$.*

Proof of Theorems 5.1, 5.2, and 5.3

To prove Theorem 5.1 we need to prove the following lemma:

Lemma 5.4. *Let k be a positive integer and $n \geq 3$. Then $\chi_L(S_n(k))$ is increasing and unbounded as a function of n.*

Proof. The case of $k = 1$ is clear. Let $k \geq 2$ and $n \geq 3$. We will prove that $\chi_L(S_n(k)) \geq \chi_L(S_{n-1}(k))$. Let $\chi_L(S_n(k)) = p$. Then $p \leq n$ by Theorems 4.1 and 4.6. Let c be a locating p-coloring of $S_n(k)$ with p as the color of the center. For $i = 1, 2, \cdots, p-1$, choose one vertex for color i with the smallest distance to a_0. If there is more than one vertex, then choose one arbitrary. Such a vertex is called the reference vertex of color i. Therefore, there are $p-1$ chosen reference vertices. Since $p \leq n$, there exists an end-path not containing any reference vertex. If we remove this end-path, then the remaining graph (with the remaining coloring) is a locating p-coloring of $S_{n-1}(k)$ because the color code for all vertices does not change. Therefore, $\chi_L(S_{n-1}(k)) \leq p = \chi_L(S_n(k))$. The unbounded property comes from Theorem 4.1. □

Proof of Theorem 5.1. We will prove that there exist $A, B > 0$ such that

$$An^{\frac{1}{k}} \leq \chi_L(S_n(k)) \leq Bn^{\frac{1}{k}}, \tag{5.5}$$

for large values of n.

First, we prove the second inequality in (5.5). Let $n \geq 4 \cdot 3^{2k+1}$ and $\chi_L(S_n(k)) = p+2 \geq 2k+4$ from Theorem 4.1. For $i = 1, 2, \cdots, p$; let $A_i = \{a \in \{1, 2, \cdots, p\} \mid a \equiv i \bmod k\}$. Let $m = \prod_{i=1}^{k} |A_i|$. We will construct a locating $(p+1)$-coloring c of $S_m(k)$ as follows:

1. Let $c(a_0) = p+1$.
2. Arrange the elements in $\mathbb{A} = A_1 \times A_2 \times \cdots \times A_k$ by their lexicographic order.
3. For $i = 1, 2, \cdots, m$; let $(a_0, a_{i,1}, a_{i,2}, \cdots, a_{i,k})$ be the i^{th} end-path of $S_m(k)$, and $(\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ik})$ be the i^{th} element in $\mathbb{A}$. then define $c(a_{i,j}) = \alpha_{ij}$.

It is easy to verify that the previous coloring is a locating $(p+1)$-coloring of $S_m(k)$. Thus $\chi_L(S_m(k)) \leq p+1 < \chi_L(S_n(k))$. By Lemma 5.4, we have $n > m$. Therefore,

$$n > \prod_{i=1}^{k} |A_i| \geq \prod_{i=1}^{k} \left\lfloor \frac{p}{k} \right\rfloor \geq \left(\frac{p}{k} - 1\right)^k \geq \left(\frac{p+2}{2k}\right)^k,$$

which is equivalent to $p+2 \leq (2k)n^{\frac{1}{k}}$. Thus the bound in (5.5) is satisfied for $n \geq 4 \cdot 3^{2k+1}$ and $B = 2k$.

Now, we only need to prove the first inequality in (5.5). Let $q = \chi_L(S_n(k))$. There are k vertices in an end-path without the center of $S_n(k)$. In a resolving q-coloring, each vertex has q possible colors. Therefore, there are at most q^k possible ways to color an end-path. Since two different end-paths cannot have the same coloring, there are at most q^k end-paths and thus $n \leq q^k$ which is equivalent to $n^{\frac{1}{k}} \leq q = \chi_L(S_n(k))$. Hence (5.5) is true for $A = 1$. $\square$

Proof of Theorem 5.2. For $k = 1$, $S_n(k) = S_n$ and the result follows. Let $\lceil\sqrt{n}\rceil = p$. We will prove that $\chi_L(S_n(2)) = p + 1$.

First, we construct a locating $(p+1)$-coloring of $S_n(2)$. Let $A = \{(x, y) \mid x \in \{1, 2, \cdots, p\}, y \in \{1, 2, \cdots, p+1\} - \{x\}\}$. Then $|A| = p^2 \geq n$. Define a coloring c with $c(a_0) = p + 1$, $c(a_{i,1}) = x_i$ and $c(a_{i,2}) = y_i$, where (x_i, y_i) is the i^{th} element in A (based on lexicographic order). Note that c is a locating $(p + 1)$-coloring of $S_n(2)$ because vertices in the same level are resolved by the color of their neighbors and vertices in different levels are resolved by $c^{-1}(p + 1)$.

To prove that $(p + 1)$ is minimum, let c' be a q-coloring of $S_n(2)$ with $c(a_0) = q$ and $q \leq p$. We will prove that c' is not a locating coloring. Let $A' = \{(c'(a_{i,1}), c'(a_{i,2})) \mid i = 1, 2 \cdots, n\}$. Note that $A' \subseteq \{(x, y) \mid x \in \{1, 2, \cdots, p-1\}, y \in \{1, 2, \cdots, p\} - \{x\}\}$. So $|A'| \leq (p-1)^2 < n$. This means that there are two different indices i and j such that $(c'(a_{i,1}), c'(a_{i,2})) = (c'(a_{j,1}), c'(a_{j,2}))$. Therefore, $a_{c'}(a_{i,1}) = a_{c'}(a_{j,1})$ and thus c' is not a locating coloring of $S_n(2)$. $\square$

Proof of Theorem 5.3. Note that $f(p) = (p-1)\left\lfloor\frac{p^2}{4}\right\rfloor - \left\lfloor\frac{p^2-2p}{4}\right\rfloor = \left\lceil\frac{p}{2}\right\rceil(p-1)\left\lfloor\frac{p}{2} - 1\right\rfloor + \left\lceil\frac{p}{2}\right\rceil^2$ is a strictly increasing function for $p \geq 3$. So, for $n \geq 2$. there is a unique p such that $f(p-1) < n \leq f(p)$. Proving (5.4) is equivalent to proving $\chi_L(S_n(3)) = p$, where p is the smallest integer such that $n \leq f(p)$.

Let $n \geq 2$ and $\chi_L(S_n(3)) = p$. First, we will prove that $n \leq f(p)$. Let $V(S_n(3)) = \{v\} \cup \{x_i, y_i, z_i \mid i = 1, 2, \cdots, n\}$ and for $i = 1, 2, \cdots, n$; the subrgaph induced by $\{v, x_i, y_i, z_i\}$ is a path. Let c be a locating p-coloring of $S_n(3)$ with $c(v) = p$. Let $A = \{j \mid d(v, c^{-1}(j)) = 1\}$, $B = \{j \mid d(v, c^{-1}(j)) = 2\}$, and $C = \{j \mid d(v, c^{-1}(j)) = 3\}$; also $|A| = \alpha$, $|B| = \beta$, and $|C| = \gamma$.

Now, we will count the number of possible color codes for x_i. We know that $c(x_i) \in A$ and $c(y_i) \in A \cup B \cup \{p\}$. Let $a_c(x_i) = (a_1, a_2, \cdots, a_p)$. Then $a_p = 1$, $a_j = 2$ for $j \in A\backslash\{c(x_i), c(y_i)\}$, $a_j = 3$ for $j \in B\backslash\{c(y_i), c(z_i)\}$, and $a_j = 4$ for $j \in C\backslash\{c(z_i)\}$. So, for fixed A, B, and C, the color code of x_i depends only on $c(x_i)$, $c(y_i)$, and $c(z_i)$.

(i) If $c(y_i) \in B$ and $c(z_i) \in A \cup \{p\}$, then there are α possible choices for $c(x_i)$ and β possible choices for $c(y_i)$. So there are $\alpha\beta$ possible values for $a_c(x_i)$; note that the value of $a_c(x_i)$ does not change for different values of $c(z_i) \in A \cup \{p\}$. **(ii)** If $c(y_i) \in B$ and $c(z_i) \in (B \cup C)\backslash\{c(y_i)\}$, then there are α possible choices for $c(x_i)$, β possible choices for $c(y_i)$, and $\beta + \gamma - 1$ possible choices for $c(z_i)$. So there are $\alpha\beta(\beta+\gamma-1)$ possible values for $a_c(x_i)$. **(iii)** If $c(y_i) \in (A \cup \{p\})\backslash\{c(x_i)\}$ and $c(z_i) \in (A \cup \{p\})\backslash\{c(y_i)\}$, then there are α possible choices for $c(x_i)$ and α possible choices for $c(y_i)$. So there are α^2 possible values for $a_c(x_i)$; note that the value of $a_c(x_i)$ does not change for different values of $c(z_i) \in A \cup \{p\}\backslash\{c(y_i)\}$. **(iv)** If $c(y_i) \in (A \cup \{p\})\backslash\{c(x_i)\}$ and $c(z_i) \in (B \cup C)$, then there are α possible choices of $c(x_i)$, α possible choices for $c(y_i)$, and $\beta+\gamma$ possible values for $c(z_i)$. So there are $\alpha^2(\beta+\gamma)$ possible values for $a_c(x_i)$.

Since $\alpha+\beta+\gamma = p-1$, the total number of possible values for $a_c(x_i)$ is $\alpha\beta + \alpha\beta(\beta+\gamma-1) + \alpha^2 + \alpha^2(\beta+\gamma) = \alpha(\alpha+\beta)(\beta+\gamma) + \alpha^2 = \alpha(p-1-\gamma)(p-1-\alpha) + \alpha^2$. This value is maximized by taking $\gamma = 0$. So the number of possible values for $a_c(x_i)$ is at most $\alpha(p-1)(p-1-\alpha)+\alpha^2$ which is maximized when $\alpha = \frac{p}{2} + \frac{1}{2p-4}$. Since α must be an integer and the closest integer to $\frac{p}{2} + \frac{1}{2p-4}$ is $\left\lceil \frac{p}{2} \right\rceil$, the number of possible values for $a_c(x_i)$ is at most

$$\begin{aligned}\left\lceil \frac{p}{2} \right\rceil (p-1) \left\lfloor \frac{p}{2} - 1 \right\rfloor + \left\lceil \frac{p}{2} \right\rceil^2 &= (p-1)\left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor - \left\lceil \frac{p}{2} \right\rceil \left(p - 1 - \left\lceil \frac{p}{2} \right\rceil\right) \\ &= (p-1)\left\lfloor \frac{p^2}{4} \right\rfloor - \left\lfloor \frac{p^2-2p}{4} \right\rfloor .\end{aligned}$$

Therefore, $n \leq f(p)$.

Now, we prove that p is the smallest integer such that $n \leq f(p)$. The proof is by contradiction. Let k be an integer such that $n \leq f(k)$ and $k < p$. We will construct a locating k-coloring of $S_n(3)$ which will contradict $\chi_L(S_n(3)) = p$.

Let $A = \left\{1, 2, \cdots, \left\lceil \frac{k}{2} \right\rceil\right\}$ and $B = \left\{\left\lceil \frac{k}{2} \right\rceil + 1, \left\lceil \frac{k}{2} \right\rceil + 2, \cdots, k-1\right\}$. Let $S_1 = \{(a,b,a) \mid a \in A; b \in B\}$, $S_2 = \{(a,b,c) \mid a \in A; b,c \in B; b \neq c\}$, $S_3 = \{(a,b,a) \mid a,b \in A \cup \{k\}; b \neq a \neq k\}$, $S_4 = \{(a,b,c) \mid a,b \in A \cup \{k\}; c \in B; b \neq a \neq k\}$, and $S = S_1 \cup S_2 \cup S_3 \cup S_4$. Note that $|S| = \alpha\beta + \alpha\beta(\beta-1) + \alpha^2 + \alpha^2\beta$, where $\alpha = |A|$ and $\beta = |B|$, so $|S| = f(k) \geq n$. Let c be a coloring of $S_n(3)$ as follows:

1. Color v with k.
2. Arrange the elements of S from S_1 to S_4.
3. For $i = 1, 2, \cdots, n$; let (a_i, b_i, c_i) be the i^{th} element in S (based on the previous ordering), and color x_i with a_i, color y_i with b_i, and color z_i with c_i.

Now, we prove that c is a locating coloring. Note that $\{w \in V(S_n(3)) \mid c(w) = k\} \subseteq \{v, y_1, y_2, \cdots, y_n\}$ and $c(z_i) \in A \Rightarrow c(z_i) = c(x_i)$. By contradiction, let u, w be two vertices with $a_c(u) = a_c(w)$. Consider the following cases.

Case I : $d(u, c^{-1}(k))$ is even. Since $S_n(3)$ is bipartite and color k only appears in one partition, if $d(u, c^{-1}(k))$ is even, then $u, w \in \{v, y_1, y_2, \cdots, y_n\}$. Note that v is the only vertex with $d(v, r) = 1$ for all $r \in A$, so $v \notin \{u, w\}$. Let $u = y_i$ and $w = y_j$ with $i \neq j$. Since $a_c(u) = a_c(w)$, then $\{c(x_i), c(z_i)\} = \{c(x_j), c(z_j)\}$. If $c(z_i) = c(z_j)$, then $c(x_i) = c(x_j)$, which implies $i = j$. If $c(z_i) = c(x_j)$, then $c(z_i) \in A$, which implies $|\{c(x_i), c(z_i), c(x_j), c(z_j)\}| = 1$ and thus $i = j$.

Case II : $d(u, c^{-1}(k))$ is odd. Let $u \in \{x_i, y_i, z_i\}$ and $w \in \{x_j, y_j, z_j\}$. If $d(u, v) = d(w, v)$, then $i \neq j$ and the different colors in (a_i, b_i, c_i) and (a_j, b_j, c_j) will distinguish $a_c(u)$ and $a_c(w)$. If $d(u, v) < d(w, v)$, then $u = x_i$ and $w = z_j$. This implies $c(z_j) = c(x_i) \in A$, thus $c(z_j) = c(x_j)$. Since $d(z_j, c^{-1}(k)) = d(x_i, c^{-1}(k)) = 1$, then $c(y_j) = p$ and $d(z_j, r) = 5 > 3 \geq d(x_i, r)$ for every $r \in B$, a contradiction.

We have already proved that c is a locating k coloring of $S_n(3)$ which means $\chi_L(S_n(3)) \leq k$, but $\chi_L(S_n(3)) = p > k$, a contradiction. Therefore, p is the smallest integer such that $n \leq f(p)$ and as a result (5.4) follows.

From (5.4), we have

$$\lim_{n\to\infty} \frac{n}{f(p)} = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n}{p^3/4} = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{p}{\sqrt[3]{4n}} = 1.$$

Therefore $\chi_L\left(S_n(3)\right) = (1 + o(1))\sqrt[3]{4n}$. □

Acknowledgment

This research was supported by the World Class Research (WCR) Program, Ministry of Research and Technology/National Research and Innovation Agency, Indonesia, and the Program Riset ITB, Institut Teknologi Bandung, Indonesia.

References

[1] H. Assiyatun, B. Rahadjeng, E. T. Baskoro, Characterizing all trees with locating-chromatic number 3, *Electron. J. Graph Theory Appl.*, **2**, (2013), 109–117.

[2] H. Assiyatun, D. K. Syofyan, E. T. Baskoro, Calculating an upper bound of the locating chromatic number of trees, *Theoretical Computer Science*, **806**, (2020), 305–309.

[3] H. Assiyatun, D. K. Syofyan, E. T. Baskoro, Locating-chromatic number of the edge-amalgamation of trees, *Indones. J. Combin.*, **4**, no. 2, (2020), 125–131.

[4] E. T. Baskoro, A. Asmiati, Characterizing all trees with locating-chromatic number 3, *Electron. J. Graph Theory Appl.*, **I(2)**, (2013), 109–117.

[5] E. T. Baskoro, D. O. Haryeni, All graphs of order $n \geq 11$ and diameter 2 with partition dimension $n-3$, *Heliyon*, **6**, (2020), e03694.

[6] E. T. Baskoro, D. I. D. Primaskun, Improved algorithm for the locating-chromatic number of trees, *Theoretical Computer Science*, **856**, (2021), 165–168.

[7] A. Behtoei, M. Anbarloei, A Bound for The locating chromatic Number of Trees, *Transactions on Combinatorics*, **4**, (2015), 31–41.

[8] A. Behtoei, B. Omoomi, On the locating chromatic number of Kneser graphs, *Discrete Applied Mathematics*, **159**, (2011), 2214–2221.

[9] J. Cáceres, C. Hernando, M. Nora, I. M. Pelayo, M. L. Puertas, On the metric dimensions of infinite graphs, *Electron. Notes Discrete Math.*, **35**, (2009), 15–20.

[10] Glenn G. Chappell, John Gimbel, Chris Hartman, Bounds on the metric and partition dimensions of a graph, *Ars Comb.*, **88**, (2008).

[11] G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater, P. Zhang, The locating chromatic number of a graph, *Bull. Inst. Combin. Appl.*, **36**, (2002), 89–101.

[12] G. Chartrand, F. Okamoto, P. Zhang, The metric chromatic number of a graph, *Australasian Journal of Combinatorics*, **44**, (2009), 273–286.

[13] R. Diestel, Graph Theory, Fifth Edition, *Springer*, 2017.

[14] M. Furuya, N. Matsumoto, Upper bounds on the locating chromatic number of trees, *Discrete Applied Mathematics*, **257**, (2019), 338–341.

[15] H. Iswadi, E. T. Baskoro, A. N. M. Salman, R. Simanjuntak, The metric dimension of amalgamation of cycles, *Far East Mathematical Science Journal*, **41**, no. 1, (2010), 19–31.

[16] Ratih Suryaningsih, Edy Tri Baskoro, The locating-chromatic number and partition dimension of fibonacene graphs, *Indones. J. Combin.*, **3**, no. 2, (2019), 116–125.

[17] P. J. Slater, Leaves of Trees, *Proc. 6th S-E Conf. Combinatorics, Graph Theory and Computing*, (1975), 549–559.