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Maximal Intra-regular Submonoids and Relationship between Some Regular Submonoids of $Hyp_G(n)$

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Abstract

A generalized hypersubstitution of type τ is a mapping from the set of any operation symbols of type τ to the set of all terms which does not necessarily preserve the arity. The set of all generalized hypersubstitutions of type τ with a binary operation defined on this set forms a monoid. In this paper, we characterize the set of all intra-regular elements of the monoid of all generalized hypersubstitutions of type τ. Moreover, we determine all maximal intra-regular submoniods of this monoid. Furthermore, we show that all maximal intra-regular submonoids and all maximal completely regular submonoids of this monoid are the same.

1 Introduction

Let S be a semigroup and let a be an element of S. An element a is called a *regular element* of S if there is an element b of S such that $a = aba$, and a is called a *completely regular element* of S if a is regular such that $a = aba$ and $ab = ba$ for some $b \in S$. We see that a completely regular element is a regular element, but a regular element need not to be a completely regular

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element. So the set of all completely regular elements of S is a subset of the set of all regular elements of S.

Let a be an element of a semigroup S. Then

$$\begin{aligned} a \text{ is called } \textit{left regular} &\quad \text{iff} \quad \exists\, b \in S \text{ such that } a = ba^2 \ (a \in Sa^2), \\ a \text{ is called } \textit{right regular} &\quad \text{iff} \quad \exists\, b \in S \text{ such that } a = a^2 b \ (a \in a^2 S), \\ a \text{ is called } \textit{intra-regular} &\quad \text{iff} \quad \exists\, b, c \in S \text{ such that } a = ba^2 c \ (a \in Sa^2 S). \end{aligned}$$

A semigroup S is said to be a *regular semigroup* if every element of S is regular and a regular semigroup with identity is called a *regular monoid*. A *completely regular* [*left regular*, *right regular*, *intra−regular*] *monoid* can be defined in a similar manner.

In semigroup theory, we know that an element a of a semigroup S is completely regular if and only if a is both left regular and right regular. If an element a is either left regular or right regular, then a need not be completely regular. If an element a of S is both left regular and right regular such that $a = ba^2$ and $a = a^2c$ for some $b, c \in S$, then $a = baa = baa^2c$ for some $ba, c \in S$, and then a is intra-regular. It follows that every completely regular element is an intra-regular element. On the other hand, an intra-regular element need not be a completely regular element. Therefore, the set of all completely regular elements of a semigroup S is a subset of the set of all intra-regular elements of a semigroup S. The main result of this paper is to characterize the set of all intra-regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$. In general, we know that the set of all intra-regular elements of the monoid is not its submonoid. In this paper, we determine all maximal intra-regular submonoids of the monoid of all generalized hypersubstitutions of type $\tau = (n)$. Moreover, we show that all maximal intra-regular submonoids and all maximal completely regular submonoids of this monoid are the same.

2 Monoid of all Generalized Hypersubstitutions

In this section, we recall the concept of the monoid of all generalized hypersubstitutions of type τ and introduce some notations and some theorems which will be used throughout this paper.

The concept of a generalized hypersubstitution of type τ was introduced by Leeratanavalee and Denecke [6]. A generalized hypersustitution of type

τ is a mapping σ which maps any operation symbol f_i to the term $\sigma(f_i)$ which does not necessarily preserve the arity. Each generalized hypersubstitution of type τ can be extended to a mapping defined on the set of all terms of type τ. Leeratanavalee and Denecke used the extension of a generalized hypersubstitution to define a binary operation on the set of all generalized hypersubstitution of type τ and showed that the set of all generalized hypersubstitutions of type τ together with this binary operation and the identity generalized hypersubstitution forms the monoid.

With $n \geq 1$, define $X_n := \{x_1, \ldots, x_n\}$ as an n-element set of variables and $X := \{x_1, x_2, \ldots\}$ as a countably infinite set of variables. Let $\tau = (n_i)_{i\in I}, n_i \in \mathbb{N}$, be a type with operation symbols f_i for each $i \in I$ and let $W_\tau(X)$ be the set of all terms of type τ built up by operation symbols from $\{f_i : i \in I\}$ and variables from $X := \{x_1, x_2, \ldots\}$. A generalized hypersubstitutions of type τ is a mapping $\sigma : \{f_i : i \in I\} \to W_\tau(X)$ which does not necessarily preserve the arity. The set of all generalized hypersubstitutions of type τ is denoted by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, first we introduce the concept of a generalized superposition of term $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$, which is defined inductively as follows:

For each $t_1, \ldots, t_m \in W_\tau(X)$,

(i) $S^m(x_j, t_1, \ldots, t_m) = t_j$ for all $j \in \{1, \ldots, m\}$,

(ii) $S^m(x_j, t_1, \ldots, t_m) = x_j$ for all $j \in \mathbb{N}$ and $j > m$,

(iii) if $t = f_i(s_1, \ldots, s_{n_i}) \in W_\tau(X)$, then
$S^m(t, t_1, ..., t_m) := f_i(S^m(s_1, t_1, ..., t_m), ..., S^m(s_{n_i}, t_1, ..., t_m))$.

Each generalized hypersubstitution σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ which is defined by the following steps:

(i) $\widehat{\sigma}[x] := x$, for all $x \in X$,

(ii) for each n_i-ary operation symbol f_i and $f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X)$,

$$\widehat{\sigma}[f_i(t_1, ..., t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_{n_i}]),$$

where $\widehat{\sigma}[t_j]$ are already defined for all $j \in \{1, \ldots, n_i\}$.

Let $\sigma_1, \sigma_2 \in Hyp_G(\tau)$ and let $\circ$ denote the usual composition of mappings. Define a binary operation of $\circ_G$ on the set $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$. Leeratanavalee and Denecke [6] showed that $(Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid, where σ_{id} is a generalized hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$.

In this paper, we study the monoid of all generalized hypersubstitutions of type $\tau = (n)$. So, we fix the type $\tau = (n)$; i.e., we only have one n-ary operation symbol, say f. Let $t \in W_{(n)}(X)$ and

$\sigma_t :=$ the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t,

$\sigma_t^2 :=$ the generalized hypersubstitution $\sigma_t \circ_G \sigma_t(f)$ of type $\tau = (n)$,

$var(t) :=$ the set of all variables occurring in the term t.

For each $t \in W_{(n)}(X) \setminus X$ where $t = f(t_1, ..., t_n)$ for some $t_1, ..., t_n \in W_{(n)}(X)$, let $\pi_{i_l} : W_{(n)}(X) \setminus X \to W_{(n)}(X)$ with $\pi_{i_l}(t) = \pi_{i_l}(f(t_1, ..., t_n)) = t_{i_l}$. Maps π_{i_l} are defined for $i_l = 1, ..., n$. Let $x_j \in var(t)$ and let $x_j^{(l)}$ be a variable x_j occurring in the l^{th} order of t (from the left). We denote the sequence of $x_j^{(l)}$ in t by $seq^t(x_j^{(l)})$ and denote the depth of of $x_j^{(l)}$ in t by $depth^t(x_j^{(l)})$. If $x_j^{(l)} = \pi_{i_m} \circ ... \circ \pi_{i_1}(t)$ for some $m \in \mathbb{N}$, then

$$seq^t(x_j^{(l)}) = (i_1, ..., i_m) \quad \text{and} \quad depth^t(x_j^{(l)}) = m.$$

We denote the set of all a sequences of x_j in term t by $seq^t(x_j)$; i.e.,

$$seq^t(x_j) = \{seq^t(x_j^{(l)}) | \, l \in \mathbb{N}\}.$$

Theorem 2.1. [2] *Let $t, s \in W_{(n)}(X) \setminus X$ and $x_i \in var(t)$. Let $x_i^{(j)}$ be a variable x_i occurring in the j^{th} order of t (from the left) such that $seg^t(x_i^{(j)}) = (i_1, ..., i_m)$ for some $i_1, ..., i_m \in \{1, .., n\}$. Then $x_i^{(j)} \in var(\hat{\sigma}_s[t])$ if and only if $x_{i_k} \in var(s)$ for all $1 \leq k \leq m$. Let $x_i^{(j,h)}$ be a variable $x_i^{(j)}$ occurring in the h^{th} order of $\hat{\sigma}_s[t]$ (from the left). Then*

$$seq^{\hat{\sigma}_s[t]}(x_i^{(j,h)}) = (a_{i_1}, ..., a_{i_m}),$$

where a_{i_k} is a sequence of natural number $k_1, ..., k_z$ such that $(k_1, ..., k_z) \in seq^s(x_{i_k})$ for all $k \in \{1, .., m\}$. Moreover,

$$depth^{\hat{\sigma}_s[t]}(x_i^{(j,h)}) = depth^s(x_{i_1}^{l_1}) + ... + depth^s(x_{i_m}^{l_m}),$$

for some $l_1, ..., l_m \in \mathbb{N}$ where $x_{i_k}^{l_k}$ is a variable x_{i_k} occurring in the l_k^{th} order of s (from the left) for all $k \in \{1, ..., m\}$.

3 Main Results

Let $\sigma_t \in Hyp_G(n)$ and

$R_1 := \{\sigma_{x_i} | x_i \in X\};$

$R_2 := \{\sigma_t | t \in W_{(n)}(X) \setminus X$ and $var(t) \cap X_n = \emptyset\}$;

$R_3 := \{\sigma_t | t \in W_{(n)}(X) \setminus X$ such that $t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and for distinct $j_1, \ldots, j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}\}$;

$CR(R_3) := \{\sigma_t | t \in W_{(n)}(X) \setminus X$ such that $t = f(t_1, ..., t_n)$. Then there exist $t_{i_1}, \ldots, t_{i_m} \in \{t_1, \ldots, t_n\}$ such that $t_{i_1} = x_{\pi(i_1)}, ..., t_{i_m} = x_{\pi(i_m)}$, where π is a bijective map on $\{i_1, ..., i_m\}$ and $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}\}$.

It is clear that $CR(R_3) \subset R_3$. In 2010, Puninagool and Leeratanavalee [7] showed that $R_1 \cup R_2 \cup R_3$ is the set of all regular elements in $Hyp_G(n)$. In 2013, Boonmee and Leeratanavalee [1] showed that $CR(Hyp_G(n)) := R_1 \cup R_2 \cup CR(R_3)$ is the set of all completely regular elements in $Hyp_G(n)$. In general, we know that $CR(Hyp_G(n))$ is a subset of the set all intra-regular elements in $Hyp_G(n)$.

In this section, we will show that every $\sigma_t \in Hyp_G(n) \setminus CR(Hyp_G(n))$ is not intra-regular. So $CR(Hyp_G(n))$ is the set of all intra-regular elements in $Hyp_G(n)$.

From now on, we construct some tools which will be used in this section.

Definition 3.1. Let $t \in W_{(n)}(X) \setminus X$ and $m \in \mathbb{N}$. Define $var(t)^{d(m)}_{X_n}$ as the set of all distinct variables $x_i \in var(t) \cap X_n$ such that $depth^t(x_i^{(l)}) = m$, where $x_i^{(l)}$ is a variable x_i occurring in the l^{th} order of t (from the left) for some $l \in \mathbb{N}$; i.e.,

$$var(t)^{d(m)}_{X_n} = \{x_i \in var(t) \cap X_n | depth^t(x_i^{(l)}) = m \text{ for some } l \in \mathbb{N}\}.$$

Example 3.2. Let $\tau = (4)$ and

$$t = f(x_2, f(x_5, x_1, f(x_3, x_1, x_5, x_1), x_6), x_4, f(f(x_1, x_2, x_6, x_7), x_3, x_1, x_7)).$$

Then $var(t)^{d(1)}_{X_n} = \{x_2, x_4\}$, $var(t)^{d(2)}_{X_n} = \{x_1, x_3\}$, $var(t)^{d(3)}_{X_n} = \{x_1, x_2, x_3\}$ and $var(t)^{d(m)}_{X_n} = \emptyset$ if $m \geq 4$.

Let $\tau = (3)$ and

$$s = f(f(x_3, x_4, f(x_3, x_5, x_2)), x_5, f(x_1, x_5, x_3)).$$

Then $var(s)^{d(1)}_{X_n} = \emptyset$, $var(s)^{d(2)}_{X_n} = \{x_1, x_3\}$, $var(s)^{d(3)}_{X_n} = \{x_2, x_3\}$ and $var(s)^{d(m)}_{X_n} = \emptyset$ if $m \geq 4$.

Lemma 3.3. *Let $t, s \in W_{(n)}(X) \setminus X$. Then $|var(\widehat{\sigma}_t[s])^{d(1)}_{X_n}| \leq |var(t)^{d(1)}_{X_n}|$.*

Proof. Let $t, s \in W_{(n)}(X) \setminus X$, where $t = f(t_1, \ldots, t_n)$ and $s = f(s_1, \ldots, s_n)$. If $var(t)^{d(1)}_{X_n} = \emptyset$, then $var(\widehat{\sigma}_t[s])^{d(1)}_{X_n} = \emptyset$. So $|var(t)^{d(1)}_{X_n}| = 0 = |var(\widehat{\sigma}_t[s])^{d(1)}_{X_n}|$. If $var(t)^{d(1)}_{X_n} \neq \emptyset$, then there are $t_{i_1}, \ldots, t_{i_m} \in \{t_1, \ldots, t_n\}$ such that $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$, where $var(t)^{d(1)}_{X_n} = \{x_{j_1}, \ldots, x_{j_m}\}$. Then $|var(t)^{d(1)}_{X_n}| = m$. Consider $(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[s] = f(u_1, \ldots, u_n)$, where $u_i = S^n(t_i, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], \ldots, \widehat{\sigma}_t[s_n])$ for all $i \in \{1, \ldots, n\}$. Then $var(\widehat{\sigma}_t[s])^{d(1)}_{X_n} \subseteq \{u_{i_1}, \ldots, u_{i_m}\}$. So $|var(\widehat{\sigma}_t[s])^{d(1)}_{X_n}| \leq m$. Therefore, $|var(\widehat{\sigma}_t[s])^{d(1)}_{X_n}| \leq |var(t)^{d(1)}_{X_n}|$. □

Theorem 3.4. *Let $t = f(t_1, \ldots, t_n) \in W_{(n)}(X) \setminus X$, where there exist $t_{i_1}, \ldots, t_{i_m} \in \{t_1, \ldots t_n\}$ such that $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ and $var(t)^{d(1)}_{X_n} = \{x_{j_1}, \ldots, x_{j_m}\}$. If there exists $i_l \in \{i_1, \ldots, i_m\}$ such that $x_{i_l} \notin var(t)^{d(1)}_{X_n}$, then σ_t is not intra-regular.*

Proof. Using the hypothesis, $var(\widehat{\sigma}_t[t])^{d(1)}_{X_n} \subseteq var(t)^{d(1)}_{X_n} = \{x_{j_1}, \ldots, x_{j_m}\}$ and $|var(\widehat{\sigma}_t[t])^{d(1)}_{X_n}| \leq |var(t)^{d(1)}_{X_n}|$. First, we show that if $x_{j_l} \notin var(\widehat{\sigma}_t[t])^{d(1)}_{X_n}$, then we have $|var(\widehat{\sigma}_t[t])^{d(1)}_{X_n}| < |var(t)^{d(1)}_{X_n}|$. Consider $(\sigma_t \circ_G \sigma_t)(f) = \widehat{\sigma}_t[t] = f(u_1, \ldots, u_n)$, where $u_i = S^n(t_i, \widehat{\sigma}_t[t_1], \ldots, \widehat{\sigma}_t[t_n])$ for all $i \in \{1, \ldots, n\}$. If $x_{j_l} \in var(\widehat{\sigma}_t[t])^{d(1)}_{X_n}$, then $u_k = x_{j_l}$ for some $k \in \{1, \ldots, n\}$. So there exists $k \in \{i_1, \ldots, i_m\}$ such that $t_k = x_{i_l}$ and so $x_{i_l} \in var(t)^{d(1)}_{X_n}$, which contradicts $x_{i_l} \notin var(t)^{d(1)}_{X_n}$. Hence $x_{j_l} \notin var(\widehat{\sigma}_t[t])^{d(1)}_{X_n}$. Then $var(\widehat{\sigma}_t[t])^{d(1)}_{X_n} \subset var(t)^{d(1)}_{X_n}$. Therefore, $|var(\widehat{\sigma}_t[t])^{d(1)}_{X_n}| < |var(t)^{d(1)}_{X_n}|$.
Next, we show that $\sigma_t \neq \sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v$ for all $\sigma_u, \sigma_v \in Hyp_G(n)$. Obviously, if $\sigma_u \in R_1 \cup R_2$ or $\sigma_v \in R_1 \cup R_2$, then $\sigma_t \neq \sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v$. Let $\sigma_u, \sigma_v \in Hyp_G(n) \setminus (R_1 \cup R_2)$. Then $var(\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v)^{d(1)}_{X_n} \subseteq var(\sigma_t^2 \circ_G \sigma_v)^{d(1)}_{X_n}$ So $|var(\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v)^{d(1)}_{X_n}| \leq |var(\sigma_t^2 \circ_G \sigma_v)^{d(1)}_{X_n}|$. By Lemma 3.3, we have $|var(\sigma_t^2 \circ_G \sigma_v)^{d(1)}_{X_n}| \leq |var(\sigma_t^2)^{d(1)}_{X_n}|$. Hence $|var(\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v)^{d(1)}_{X_n}| < |var(t)^{d(1)}_{X_n}|$. So $\sigma_t \neq \sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v$. It follows that σ_t is not intra-regular. □

Theorem 3.5. *If $\sigma_t \in R_3 \setminus CR(R_3)$, then σ_t is not intra-regular.*

Proof. The proof follows from Theorem 3.4. □

Lemma 3.6. *Let $t, s \in W_{(n)}(X) \setminus X$ and $x_j \in var(t)$. Fix a variable $x_j^{(l)}$ occurring in the l^{th} order of t (from the left) for some $l \in \mathbb{N}$ such that $seg^t(x_j^{(l)}) = (i_1, \ldots, i_m)$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$. Let $x_j^{(l)} \in var(\widehat{\sigma}_s[t])$ and let $x_j^{(l,h)}$ be a variable $x_j^{(l)}$ occurring in the h^{th} order of $\widehat{\sigma}_s[t]$ (from the left) for some $h \in \mathbb{N}$. If there is $i_k \in \{i_1, \ldots, i_m\}$ such that $x_{i_k} \notin var(s)^{d(1)}_{X_n}$, then $depth^{\widehat{\sigma}_s[t]}(x_j^{(l,h)}) > depth^t(x_j^{(l)})$.*

Proof. Assume that the hypothesis holds. Then $depth^t(x_j^{(l)}) = m$. By Theorem 2.1, we have $x_{i_1}, \ldots, x_{i_k}, \ldots, x_{i_m} \in var(s)$ and

$$depth^{\widehat{\sigma}_s[t]}(x_j^{(l,h)}) = depth^s(x_{i_1}^{q_1}) + \ldots + depth^s(x_{i_k}^{q_k}) + \ldots + depth^s(x_{i_m}^{q_m})$$

for some $q_1, \ldots, q_k, \ldots, q_m \in \mathbb{N}$, where $x_{i_p}^{q_p}$ is a variable x_{i_p} occurring in the q_p^{th} order of s (from the left) for all $p \in \{1, \ldots, m\}$. Since $x_{i_k} \notin var(s)_{X_n}^{d(1)}$, we have $depth^s(x_{i_k}^{q_k}) > 1$. Then $depth^{\widehat{\sigma}_s[t]}(x_j^{(l,h)}) > m$. Therefore $depth^{\widehat{\sigma}_s[t]}(x_j^{(l,h)}) > depth^t(x_j^{(l)})$. □

Corollary 3.7. *Let $t = f(t_1, \ldots, t_n) \in W_{(n)}(X) \setminus X$ and $x_j \in (var(t) \cap X_n) \setminus var(t)_{X_n}^{d(1)}$. For each $x_j^{(l)} \in var(t_i)$ for some $i \in \{1, \ldots, n\}$ such that $x_i \notin var(t)_{X_n}^{d(1)}$, where $x_j^{(l)}$ is a variable occurring in the l^{th} order of t (from the left) for some $l \in \mathbb{N}$. If $x_j^{(l)} \in var(\widehat{\sigma}_t[t])$ and $x_j^{(l,h)}$ is a variable occurring in the h^{th} order of $\widehat{\sigma}_t[t]$ (from the left) for some $h \in \mathbb{N}$, then $depth^{\widehat{\sigma}_t[t]}(x_j^{(l,h)}) > depth^t(x_j^{(l)})$.*

Let $t \in W_{(n)}(X) \setminus X$, where $t = f(t_1, \ldots, t_n)$. Consider
$I_1^t = \{i \in \{1, \ldots, n\} | t_i \in X \setminus X_n\}$,
$I_2^t = \{i \in \{1, \ldots, n\} | t_i \in X_n\}$, $I_3^t = \{i \in \{1, \ldots, n\} | t_i \in W_{(n)}(X) \setminus X\}$.
Clearly, I_1^t, I_2^t and I_3^t are partitions of $\{1, \ldots, n\}$.

Theorem 3.8. *If $\sigma_t \in Hyp_G(n) \setminus (R_1 \cup R_2 \cup R_3)$, then σ_t is not intra-regular.*

Proof. Let $\sigma_t \in Hyp_G(n) \setminus (R_1 \cup R_2 \cup R_3)$. Then $t \in W_{(n)}(X) \setminus X$. Assume that $t = f(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n \in W_{(n)}(X)$. Suppose that σ_t is not intra-regular. Then there exist $\sigma_u, \sigma_v \in Hyp_G(n) \setminus (R_1 \cup R_2)$ such that $\sigma_t = \sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v$. $(*)$
Consider
$u = f(u_1, \ldots, u_n)$,
$v = f(v_1, \ldots, v_n)$,
$w = (\sigma_t \circ_G \sigma_t)(f) = f(w_1, \ldots, w_n)$, where $w_i = S^n(t_i, \widehat{\sigma}_t[t_1], \ldots, \widehat{\sigma}_t[t_n])$,
$s = (\sigma_w \circ_G \sigma_v)(f) = f(s_1, \ldots, s_n)$, where $s_i = S^n(w_i, \widehat{\sigma}_w[v_1], \ldots, \widehat{\sigma}_w[v_n])$,
$r = (\sigma_u \circ_G \sigma_s)(f) = f(r_1, \ldots, r_n)$, where $r_i = S^n(u_i, \widehat{\sigma}_u[s_1], \ldots, \widehat{\sigma}_u[s_n])$
for all $i \in \{1, \ldots, n\}$. Since $\sigma_t \notin R_1 \cup R_2 \cup R_3$, there exists $t_i \in W_{(n)}(X) \setminus X$ for some $i \in \{1, \ldots, n\}$ such that $var(t_i) \cap X_n \neq \emptyset$. Then $I_3^t \neq \emptyset$.
Case I: I_1^t and I_2^t are non-empty sets. Since $I_2^t \neq \emptyset$, $var(t)_{X_n}^{d(1)} \neq \emptyset$. Let $var(t)_{X_n}^{d(1)} = \{x_{i_1}, \ldots, x_{i_m}\}$. Using Theorem 3.4 and the fact that σ_t is intra-regular, there exist $t_{i_1}, \ldots, t_{i_m} \in \{t_1, \ldots, t_n\}$ such that $t_{i_1} = x_{\pi(i_1)}, \ldots, t_{i_m} = x_{\pi(i_m)}$, where π is a bijective map on $\{i_1, ..., i_m\}$. Then $\{i_1, ..., i_m\} \subseteq I_2^t$.

Since $w = \widehat{\sigma}_t[t] = f(w_1, \dots, w_n)$, so $w_{i_1} = x_{\pi_w(i_1)}, \dots, w_{i_m} = x_{\pi_w(i_m)}$ where $\pi_w = \pi \circ \pi$ is a bijective map on $\{i_1, ..., i_m\}$ and so $var(t)^{d(1)}_{X_n} = var(w)^{d(1)}_{X_n}$. It is clear that $I^t_1 = I^w_1$, $I^t_2 = I^w_2$ and $I^t_3 = I^w_3$. By $(*)$, we have

$$v_{i_1} = x_{\pi_v(i_1)}, \dots, v_{i_m} = x_{\pi_v(i_m)} \text{ and } u_{i_1} = x_{\pi_u(i_1)}, \dots, u_{i_m} = x_{\pi_u(i_m)},$$

where π_v and π_u are bijective maps on $\{i_1, ..., i_m\}$. So $\{x_{i_1}, \dots, x_{i_m}\}$ is a subset of $var(v)^{d(1)}_{X_n}$ and $var(u)^{d(1)}_{X_n}$. Since $s = \widehat{\sigma}_w[v] = f(s_1, \dots, s_n)$, we have $s_{i_1} = x_{\pi_s(i_1)}, \dots, s_{i_m} = x_{\pi_s(i_m)}$, where $\pi_s = \pi_w \circ \pi_v$ is a bijective map on $\{i_1, ..., i_m\}$ and so $var(s)^{d(1)}_{X_n} = var(w)^{d(1)}_{X_n} = \{i_1, \dots, i_m\}$. Then $I^s_1 = I^w_1 = I^t_1$, $I^s_2 = I^w_2 = I^1_2$ and $I^s = I^w_3 = I^t_3$. Since $\sigma_t \notin R_1 \cup R_2 \cup R_3$, we have $(var(t) \cap X_n) \setminus var(t)^{d(1)}_{X_n} \neq \emptyset$. Choose $x^{(l)}_j \in (var(t) \cap X_n) \setminus var(t)^{d(1)}_{X_n}$, where $x^{(l)}_j$ is a variable occurring in the l^{th} order of t (from the left) for some $l \in \mathbb{N}$ such that

$$\begin{aligned} depth^t(x^{(l)}_j) &= min\{depth^t(x^{(q)}_p) \mid x^{(q)}_p \in (var(t) \cap X_n) \setminus var(t)^{d(1)}_{X_n} \text{ where,} \\ & \quad x^{(q)}_p \text{ is a variable } x_p \text{ occurring in the } q^{th} \text{ order of t (from} \\ & \quad \text{the left) for some } l \in \mathbb{N}\}. \ (**) \end{aligned}$$

Since $\sigma_t = \sigma_u \circ_G \sigma_w \circ_G \sigma_v$ and $x_j \in (var(t) \cap X_n) \setminus \{x_{i_1}, \dots, x_{i_m}\}$, we have $x_j \in (var(v) \cap X_n) \setminus \{x_{i_1}, \dots, x_{i_m}\}$. Let $x^{(l_v)}_j$ be a variable x_j occurring in the l^{th}_v order of v (from the left) for some $l_v \in \mathbb{N}$ such that $seq^v(x^{(l_v)}_j) = (a_1, \dots, a_z)$ for some $a_1, \dots, a_z \in \{1, \dots, n\}$, where $x^{(l_v)}_j \in var(\sigma_u \circ_G \sigma_w \circ_G \sigma_v) = var(r)$. Then $x^{(l_v)}_j \in var(\sigma_w \circ_G \sigma_v) = var(s)$. Let $x^{(l_v, l_s)}_j$ be a variable $x^{(l_v)}_j$ occurring in the l^{th}_s order of s (from the left) for some $l_s \in \mathbb{N}$ and let $x^{(l_v, l_s, l_r)}_j$ be a variable $x^{(l_v, l_s)}_j$ occurring in the l^{th}_r order of r (from the left) for some $l_r \in \mathbb{N}$. By Theorem 2.1, there exist $x_{a_1}, \dots, x_{a_z} \in var(w) \cap X_n$ such that

$$depth^s(x^{(l_v, l_s)}_j) = depth^w(x^{(b_1)}_{a_1}) + \ldots + depth^w(x^{(b_z)}_{a_z}) \quad (***)$$

for some $b_1, \dots, b_z \in \mathbb{N}$, where $x^{(b_k)}_{a_k}$ is a variable x_{a_k} occurring in the b^{th}_k order of w (from the left) for all $k \in \{1, \dots, z\}$. Since $x^{(l_v)}_j \notin \{x_{i_1}, \dots, x_{i_m}\}$ and $v_{i_1} = x_{\pi_v(i_1)}, \dots, v_{i_m} = x_{\pi_v(i_m)}$, we have $x^{(l_v)}_j \in var(v_{a_1})$ such that $a_1 \notin \{i_1, \dots, i_m\}$ and so $x^{(b_1)}_{a_1} \notin \{x_{i_1}, \dots, x_{i_m}\} = var(w)^{d(1)}_{X_n}$. Since $x^{(b_1)}_{a_1} \in (var(w) \cap X_n) \setminus var(w)^{d(1)}_{X_n}$ and $w = \widehat{\sigma}_t[t]$, we have $x^{(b_1)}_{a_1} \in (var(t) \cap X_n) \setminus var(t)^{d(1)}_{X_n}$. Let $x^{(b_{1_t})}_{a_1}$ be a variable $x^{(b_1)}_{a_1}$ occurring in the $b^{th}_{1_t}$ order of t (from the left) for some $b_{1_t} \in \mathbb{N}$. By Corollary 3.7 and $(**)$, we have $depth^w(x^{(b_1)}_{a_1}) > depth^t(x^{(b_{1_t})}_{a_1}) \geq depth^t(x^{(l)}_j)$. By $(***)$, we have $depth^s(x^{(l_v, l_s)}_j) > depth^t(x^{(l)}_j)$. So $depth^r(x^{(l_v, l_s, l_r)}_j) \geq depth^s(x^{(l_v, l_s)}_j) > depth^t(x^{(l)}_j)$. It follows that every

$x_j^{(h)} \in var(r)$, where $x_j^{(h)}$ is a variable x_j occurring in the h^{th} order of r (from the left) for some $h \in \mathbb{N}$. We have $depth^r(x_j^{(h)}) > depth^t(x_j^{(l)})$, which contradicts $(*)$ and $(**)$.

Case II: I_1^t or I_2^t is empty. We skip the proof as it is similar to Case I.

Therefore, σ_t is not intra-regular. □

Theorem 3.9. *$CR(Hyp_G(n))$ is the set of all intra-regular elements in $Hyp_G(n)$*

Proof. Use Theorems 3.5 and 3.8. □

Definition 3.10. [8] Let $t \in W_{(n)}(X)$ and $i \in \mathbb{N}$, $1 \le i \le n$. An $i-most(t)$ is defined inductively as follows:

(i) if t is a variable, then $i-most(t) = t$,

(ii) if $t = f(t_1, \ldots, t_n)$, then $i-most(t) = i-most(t_i)$.

Let $\sigma_t \in Hyp_G(n)$ and let $\emptyset \neq I \subset \{1, \ldots, n\}$. Consider

$CR_1(R_3) := \{\sigma_t | t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$, where π is a bijective map on $\{1, \ldots, n\}\}$,

$E := \{\sigma_t | t = f(t_1, \ldots, t_n)$, where $t_{i_1} = x_{i_1}, \ldots, t_{i_m} = x_{i_m}$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and $var(t) \cap X_n = \{x_{i_1}, \ldots, x_{i_m}\}$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$, then $j-most(t_k) \neq x_{i_l}$ for all $j \neq i_l\}$,

$CR_I(R_3) := \{\sigma_t | t = f(t_1, \ldots, t_n)$, where $t_i = x_{\pi(i)}$ for all $i \in I$ and π is a bijective map on I, $var(t) \cap X_n = \{x_{\pi(i)} | i \in I\}\}$,

$CR'_I(R_3) := \{\sigma_t | t = f(t_1, \ldots, t_n)$, where $t_i = x_{\pi(i)}$; $\pi(i) \in I$ for all $i \in I$ and $t_k = x_{\pi(k)}$ for all $k \in \{1, \ldots, n\} \setminus I$ and π is a bijective map on$\{1, \ldots, n\}\}$

and define

$(MCR)_{Hyp_G(n)} := R_1 \cup R_2 \cup CR_1(R_3)$,

$(MCR_1)_{Hyp_G(n)} := R_1 \cup R_2 \cup E$,

$(MCR_I)_{Hyp_G(n)} := R_1 \cup R_2 \cup CR_I(R_3) \cup CR'_I(R_3) \cup \{\sigma_{id}\}\}$.

In 2019, Kunama and Leeratanavalee [5] showed that $(MCR)_{Hyp_G(n)}$, $(MCR_1)_{Hyp_G(n)}$ and $(MCR_I)_{Hyp_G(n)}$ are all maximal completely regular submonoids of $Hyp_G(n)$.

Theorem 3.11. *$(MCR)_{Hyp_G(n)}$, $(MCR_1)_{Hyp_G(n)}$ and $(MCR_I)_{Hyp_G(n)}$ are all maximal intra-regular submonoids of $Hyp_G(n)$.*

Proof. Use Theorem 3.9 and [5]. □

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