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Solution of Duffing's differential equation by means of elementary functions and its application to a non-linear electrical circuit

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Abstract

In the present work, we obtain the exact solution of a non-linear oscillator by means of elementary functions. We study the third order Duffing oscillator and find an analytical solution which we apply in the study of a non-linear electrical circuit. We compare our obtained solution with the solution obtained by means of elliptical functions of Jacobi and the numerical solution.

1 Introduction

This work aims to introduce the reader in a didactic way to the study of non-linear oscillators and their applications to physics and engineering. Non-

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linear oscillators are a model that arises in various branches of physics and engineering and has been used to study various problems such as the oscillations of a large-amplitude physical pendulum, non-linear electrical circuits, image processing, open states of DNA, the movement of satellites, Bose-Einstein condensates, among others [1, 2, 3, 4, 5, 6]. Oscillations and nonlinear effects are of great importance. In basic physics textbooks [15, 16] and articles like [17] the analysis of harmonic oscillations gives it great relevance. This is done for simplicity of the model and because this is the theoretical foundation to analyze more complicated situations such as propagation of electromagnetic waves [18], the model harmonic oscillator quantum [19] and the quantization of the electromagnetic field [20].

It is well known that to treat nonlinear problems, one must resort to the study of elliptic functions. Jacobi elliptic functions were named in honor of Carl Gustav Jacob Jacobi (1804–1851) who published a classic treatise on elliptic functions [7] almost two centuries ago. However, the study of elliptical functions disappeared from the mathematical literature of science and even though they were important decades ago as evidenced in 1937 by the publication of Whittaker's physics text on the study of the dynamics of the particle and the solid body [8]. It should be noted that in the classical texts on analytical mechanics, for example in Landau's [9] or Goldstein's [10], various solutions of nonlinear oscillator problems were obtained by means of elliptic functions. However, in these texts and in most of the modern texts that deal with the subject, analytical solutions allowing a better analysis and understanding of the problem were not obtained.

Motivated by the applications of the non-linear oscillator and the search for analytical solutions is the objective of this work.

The work is organized as follows:
In the first part, we treat the third order nonlinear Duffing oscillator and we find a good analytical approximation by means of the elementary functions. In the second part, we give an application to a non-linear electrical circuit. In the third part, we solve the Duffing equation by means of the Jacobi's elliptical functions with the objective of comparing our obtained solution with the solution obtained by means of the elliptical functions.

2 Solution of the third order Duffing differential equation by means of elementary functions

We describe the Duffing nonlinear oscillator by means of the following third order nonlinear differential equation:

$$\frac{d^2q}{dt^2} + \alpha q + \beta q^3 = 0, \tag{2.1}$$

where α and β are constants and $q = q(t)$ represents a generalized coordinate.

The Duffing Chaotic Oscillator was designed by the German electrical engineer Georg Duffing (1864-1944) in the beginning of the 20th century in order to study the buckling motion of a beam. It is a classic model that arises in many branches of physics and engineering and has been used to study everything from the oscillations of a physical pendulum of great amplitude to image processing and many more. It has provided us with a useful paradigm to study non-linear oscillations and chaotic systems and it led to the development of new approximate analytical methods based on ideas such as perturbative methods and to the development of new numerical methods for the quantitative analysis of chaotic systems.

There are a variety of works in which equation (2.1) has been solved. Analytical solutions by means of the elliptical functions of Jacobi or Weierstrass can be found for example in [6],[11].

Moving away from the traditional procedure, let's look for the solution of (2.1) those with the initial conditions $q(t = 0) = q_o$, $\frac{dq}{dt}\mid_{t=0} = q'(0) = q'_o = 0$. The main objective of this work is to find the solution with the help of the following equation:

$$q(t) = \frac{q_o \cos(kt)}{\sqrt{1 + \lambda \sin^2(kt)}}, \tag{2.2}$$

where k and λ are indeterminate constants.

Expanding (2.2) in Taylor series up to a fourth order polynomial in t, we have

$$q(t) = q_o - \frac{1}{2}(q_o(\lambda k^2 + k^2))t^2 + \frac{1}{24}q_0(9\lambda^2 k^4 + 10\lambda k^4 + k^4)t^4 + o(t^5). \tag{2.3}$$

Substituting (2.3) into (2.1), we get

$$q_o(\alpha+\beta q_o^2-k^2(1+\lambda))t^0+\frac{1}{2}(q_o k^2(1+\lambda)(-\alpha-3q_o^2\beta+k^2(1+9\lambda))t^2=0. \quad (2.4)$$

Since the coefficients of the polynomial (2.4) must be equal to zero, we obtain the following system of algebraic equations for k and λ.

$$\{ \begin{array}{c} q_o(\alpha+\beta q_o^2-k^2(1+\lambda))=0 \\ \frac{1}{2}(q_o k^2(1+\lambda)(-\alpha-3q_o^2\beta+k^2(1+9\lambda))=0 \end{array} \quad (2.5)$$

Solving (2.5), we have

$$k=\frac{1}{2}\sqrt{4\alpha+3\beta q_o^2}\ ,\ \lambda=\frac{\beta q_o^2}{4\alpha+3\beta q_o^2} \quad (2.6)$$

Therefore, the solution of (2.1) is given by

$$q(t)=\frac{q_o\cos(\frac{1}{2}\sqrt{4\alpha+3\beta q_o^2}t)}{\sqrt{1+\frac{\beta q_o^2}{4\alpha+3\beta q_o^2}\sin^2(\frac{1}{2}\sqrt{4\alpha+3\beta q_o^2}t)}} \quad (2.7)$$

3 An application: A non-linear electrical circuit

Let us consider a capacitor of two terminals as a dipole in which a functional relationship between the electric charge , the voltage, and the time has the following form:

$$f(q,u,t)=0 \quad (3.8)$$

A nonlinear capacitor is said to be controlled by charge when is possible to express the tension as a function of charge

$$u=u(q). \quad (3.9)$$

As an example of a nonlinear electrical circuit, let us consider the circuit shown in Figure 1.

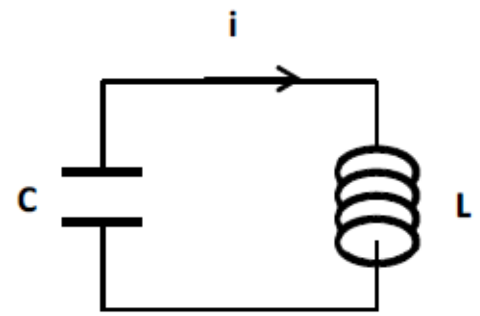

Figure 1. LC circuit

This circuit consists of a linear inductor in series with a nonlinear capacitor. The relationship between the charge of the nonlinear capacitor and the voltage drop across it may be approximated by the following cubic equations [12, 13, 14]:

$$u_c = sq + aq^3, \tag{3.10}$$

where u_c is the potential across the plates of the nonlinear capacitor q is the charge and s and a are constants.
The equation of the circuit may be written as

$$L\frac{di}{dt} + sq + aq^3 = 0, \tag{3.11}$$

where L is the inductance of the inductor. Dividing by L and taking into account that $i = \frac{dq}{dt}$, we obtain the cubic Duffing equation in the form:

$$\frac{d^2q}{dt^2} + \alpha q + \beta q^3 = 0, \tag{3.12}$$

where $\alpha = s/L = 1/LC$ and $\beta = a/L = 1/LCq_0^2$ are constants.

For $L = 2,81mH$, $C = 9pF$, $q_o = 10^{-10}C$, $i_0 = 0$, we have that $\alpha = 1/LC = 4 \times 10^{13}$ and $\beta = 1/LCq_0^2 = 4 \times 10^{33}$ and taking into account (2.7):

$$q(t) = \frac{\cos(10^6\sqrt{70}t)}{10^{10}\sqrt{1 + \frac{1}{7}\sin^2(10^6\sqrt{70}t}}. \tag{3.13}$$

The numerical solution for the Runge-Kutta method and the analytical solution given by (3.13) are compared graphically in Figure 2. The dashed red curve corresponds to the Runge-Kutta method and the thin blue curve to the solution (3.13)

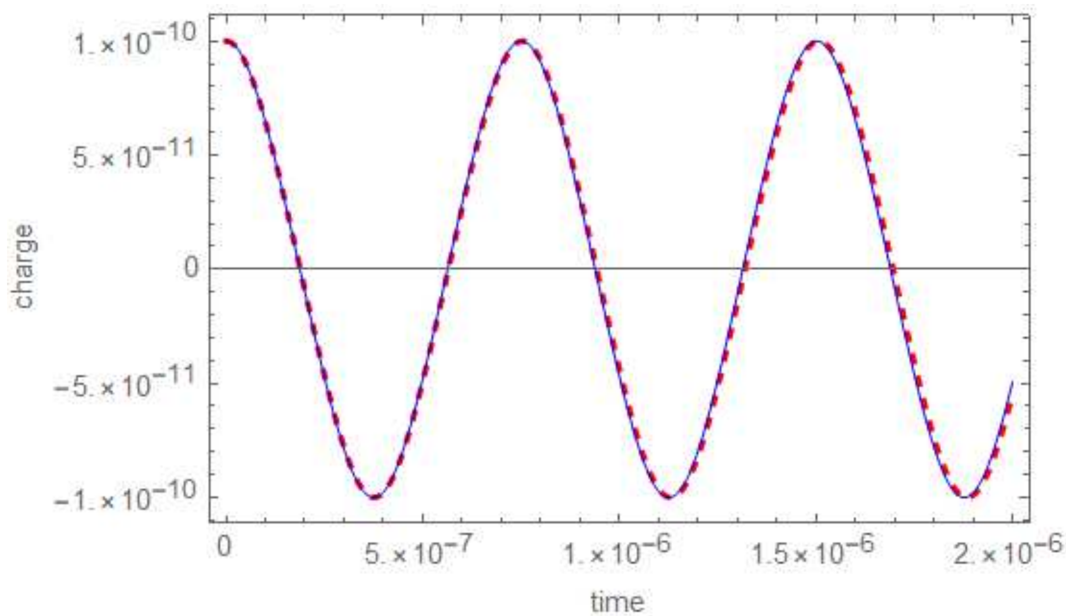

Figure 2. Analytical and numerical solution

4 Solution by means of elliptical functions

We are going to find the analytical solution of the equation (2.1) subject to initial conditions $q(0) = q_0$ and $\frac{dq}{dt} |_{t=0}= q_0' = 0$.

We seek a solution of the form.

$$q = q(t) = q_0 \text{ cn}(\omega t, m), \tag{4.14}$$

where $cn(\omega t, m)$ represents the elliptic Jacobi function cn. Substituting equation (4.14) into equation (2.1) and taking in account that

$$\frac{d^2q}{dt^2} = q_0 m\omega^2 \text{cn}(\omega t, m) \text{ sn}^2(\omega t, m) - \omega^2 q_0 \text{cn}(\omega t, m)\text{dn}^2(\omega t, m),$$

we obtain

$$q_0 m\omega^2 \mathrm{cn}(\omega t, m)\,\mathrm{sn}^2(\omega t, m) - \omega^2 q_0 \mathrm{cn}(\omega t, m)\mathrm{dn}^2(\omega t, m) + \alpha q_0 \mathrm{cn}(\omega t, m) + \beta q_0^3 \mathrm{cn}^3(\omega t, m) = 0 \tag{4.15}$$

On the other hand, we know that

$$\mathrm{sn}^2(\omega t, m) = 1 - \mathrm{cn}^2(\omega t, m) \;\; \mathrm{y} \;\; \mathrm{dn}^2 \omega t, m) = 1 - m\ \mathrm{sn}^2(\omega t, m) \tag{4.16}$$

Making use of these relationships (4.16), we deduce that

$$m\omega^2 q_0 x\left(1 - x^2\right) - \omega^2 q_0\left(1 - m\left(1 - x^2\right)\right)x + \alpha q_0 x + \beta q_0^3 x^3 = 0, \;\text{where } x = \mathrm{cn}(\omega t, m) \tag{4.17}$$

Equation (4.17) can be written as

$$q_0\left(\beta q_0^2 - 2m\omega^2\right)x^3 + q_0 \mathrm{cn}\left(2m\omega^2 + \alpha - \omega^2\right)x = 0 \tag{4.18}$$

As $q_0 \neq 0$ and the equation (4.18) must be validated for any x, we have

$$\beta q_0^2 - 2m\omega^2 = 0 \quad \text{and} \quad 2m^2\omega^2 + \alpha - \omega^2 = 0. \tag{4.19}$$

The system (4.19) has the following solution

$$m = \frac{\beta q_0^2}{2(\alpha + \beta q_0^2)}, \;\; \omega = \sqrt{\alpha + \beta q_0^2}\ . \tag{4.20}$$

Therefore, the solution to the problem

$$\frac{d^2 q}{dt^2} + \alpha q + \beta q^3 = 0,\ q(0) = q_0,\;\; q'(0) = i_0 \tag{4.21}$$

Is

$$q(t) = q_0\ \mathrm{cn}\left(\sqrt{\alpha + \beta q_0^2}\, t, \frac{\beta q_0^2}{2(\alpha + \beta q_0^2)}\right) \tag{4.22}$$

The behavior of the solution (4.22) depends on the parameters α , β, and the initial condition q_0. This solution is periodic and limited if

$$\omega^2 = \alpha + \beta q_0^2 > 0. \tag{4.23}$$

If $\omega^2 = \alpha + \beta q_0^2 < 0$, then the solution (4.22) takes the following form:

$$q(t) = q_0\ \mathrm{cn}\left(\sqrt{-(\alpha + \beta q_0^2)}\, t, 1 - \frac{\beta q_0^2}{2(\alpha + \beta q_0^2)}\right) \tag{4.24}$$

If $m = \frac{\beta q_0^2}{2(\alpha+\beta q_0^2)} = 1$, we obtain $\alpha = -\frac{\beta q_0^2}{2}$ since $cn(\omega t, 1)$ =sech(ωt) the solution (4.22) becomes

$$q(t) = q_0 \operatorname{sech}(\sqrt{\frac{1}{2}\beta q_0^2} t). \tag{4.25}$$

We observe that (4.25) is bounded but aperiodic. This is a typical situation for solitons. Some important nonlinear differential equations admit soliton solutions that can be expressed in terms of sech.

For our non-linear electrical system and taking into account (4.22) $\alpha = 1/LC = 4 \times 10^{13}$ and $\beta = 1/LCq_0^2 = 4 \times 10^{33}$, we have

$$q(t) = 10^{-10} \mathrm{cn}(4 \times 10^6 \sqrt{5} t, \frac{1}{16}). \tag{4.26}$$

Graph 3 shows the elliptical solutions, the continuous blue curve and the elemental solution, the dashed red curve.

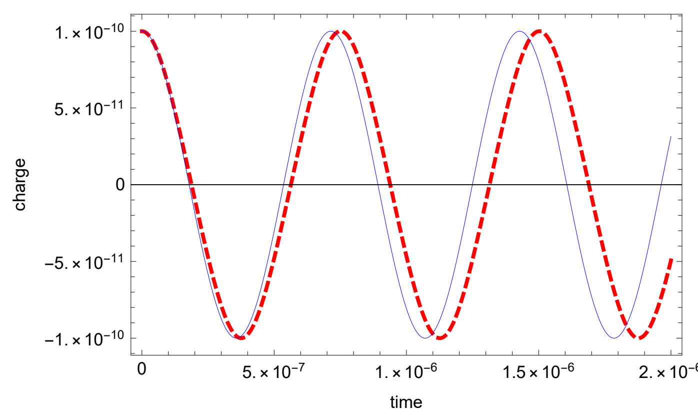

Figure 3. Elliptic solution and elementary solution

Graph 4 shows the numerical solutions, the dashed red curve, the elementary solution, the continuous blue curve, and the elliptic solution, yellow curve.

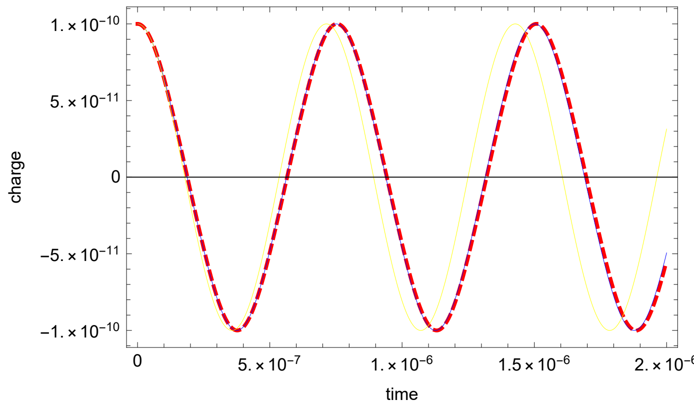

Figure 4. Numerical, elliptical, and elemental solution

5 Conclusions

The non-linear oscillator model, especially the Duffing Chaotic oscillator, is a classic model that arises in many branches of physics and engineering and has been used to study everything from the oscillations of a large-amplitude physical pendulum to image processing and many more. Solving it analytically is a difficult task since it involves the elliptical functions of Jacobi and Weirstrass which are generally not studied in mathematics courses for both physicists and engineers.

However, there was the possibility of approaching the solution in an analytical way through elementary functions through equation (2.7) that led us to a very good analytical solution in a simple and fast way. Of course, this has its limitations since there is the condition that $4\alpha + 3\beta q_o^2 > 0$.

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