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Fourth order iterative methods for solving nonlinear equations

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Abstract

In this paper, we present two new iteration methods to find the solutions of nonlinear equations, which were developed from a concept of Obadah [10] and Taylor's Series to estimate the second derivative. Analysis of its convergence shows that the order of convergence of the modified iterative method is four, for a simple root of the equation. Finally, to illustrate the efficiency and performance of the proposed method, we give some numerical experiments and comparison.

1 Introduction

Solving a nonlinear equation

$$f(x) = 0, \tag{1.1}$$

where f is a real-valued function whose domain is an open connected set, is one of the most important problems in Numerical Analysis. We assume that α is a simple zero of equation (1.1) and x_0 is an initial guess which is

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sufficiently close to α. Using Taylor's series around x_0 for equation (1.4), we have

$$f(x_0) + (x - x_0)f(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + ... = 0. \tag{1.2}$$

If $f(x_0) \neq 0$, then we can evaluate the above expression as follows:

$$f(x_0) + (x - x_0)f'(x_0) = 0. \tag{1.3}$$

If we choose a root x_{n+1} of the equation, then we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, ... \tag{1.4}$$

This is called Newton's method [6] for root-finding of nonlinear functions, which converges quadratically. From equation (1.2), one can evaluate

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, \quad n = 0, 1, 2, ... \tag{1.5}$$

This is called Halley's method [3] for root-finding of nonlinear functions, which converges cubically. Simplification of equation (1.2) yields another iterative method as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)}, \quad n = 0, 1, 2, ... \tag{1.6}$$

This is known as Householder's method [4] for solving nonlinear equations in one variable, which converges cubically. Many researchers attempted to improve Halley's method by increasing the order of convergence or reducing the estimation of functions in each round of iteration without the second order derivative for more precise and effective results. Among these methods, we are interested in the concept of Obadah [10], referred to as an improvement from Halley's method.

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{2f^2(x_0)f'^2(x_0)f''(x_0)}{4f'^4(x_0) - 4f(x_0)f'^2(x_0)f''(x_0) + f^2(x_0)f''^2(x_0)}. \tag{1.7}$$

In 2015, Sharma and Bahl [8] presented a new fourth order method to find simple roots of a nonlinear equation $f(x) = 0$. In terms of computational cost, per iteration, the method uses one evaluation of the function and two evaluations of its first derivative. In 2021, Kong-ied [7] introduced two new eighth and twelfth order iterative methods for solving nonlinear equations.

The new methods have eight order convergence with the efficiency index at 1.5157, and twelfth order convergence with the efficiency index at 1.5131. In this paper, we present two methods with three step iterations each. We estimate the second order derivative by using Taylor's Series to reduce functions in each round of iteration. We implement the concept of Obadah [10], equation (1.7), and then rewrite in the form of Newton's method. For the first method, we increase another step of iteration by Newton's method. The second method has one more step by Householder's method [4]. After that, the order of convergence of these two new methods was analyzed. The effectiveness was tested based on numerical examples with other iteration methods, which had an equal order of convergence to each of the new methods.

2 Proposed Methods

Let $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}$, be a scalar function. Using the basic idea of modified homotopy perturbation method, Obadah [10] derived the following method:

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{2f^2(x_0)f'^2(x_0)f''(x_0)}{4f'^4(x_0) - 4f(x_0)f'^2(x_0)f''(x_0) + f^2(x_0)f''^2(x_0)}. \tag{2.8}$$

In iterative form (1.7):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f^2(x_n)f'^2(x_n)f''(x_n)}{4f'^4(x_n) - 4f(x_n)f'^2(x_n)f''(x_n) + f^2(x_n)f''^2(x_n)}. \tag{2.9}$$

After that, the second derivative $f''(x_n)$ is estimated by Taylor's Series expansion of function $f(y_n)$ around $x = x_n$:

$$f(y_n) \approx f(x_n) + f'(x_n)(y_n - x_n) + \frac{f''(x_n)(y_n - x_n)^2}{2!} \tag{2.10}$$

which implies

$$f''(x_n) \approx \frac{2f(y_n)f'^2(x_n)}{f^2(x_n)}. \tag{2.11}$$

Using equation (2.11) in equation (2.9), one can get the following iterative method free from second derivatives:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f(y_n)}{f'(x_n)(f(x_n) - f(y_n))^2}. \tag{2.12}$$

Algorithm 2.1. *For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:*

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.13}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(x_n)f(y_n)}{f'(x_n)(f(x_n)-f(y_n))^2} \tag{2.14}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{2.15}$$

Algorithm 2.1 is a new three step iteration method (AL1) with order of convergence four.

In the second method, we changed the last step by Householder's method and thus the new iteration method is obtained as Algorithm 2.2:

Algorithm 2.2. *For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:*

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.16}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(x_n)f(y_n)}{f'(x_n)(f(x_n)-f(y_n))^2} \tag{2.17}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} - \frac{f^2(z_n)f''(z_n)}{2f'^3(z_n)} \tag{2.18}$$

Algorithm 2.2 is a new three step iteration method (AL2) with order of convergence four.

3 Convergence Analysis

In this section, we discuss the convergence order of the proposed iterative methods.

Theorem 3.1. *Suppose that α is a root of the equation $f(x) = 0$. If $f(x)$ is sufficiently smooth in a neighborhood of α, then the order of convergence of Algorithm 2.1 is four.*

Proof. To analyze the convergence of Algorithm 2.1, suppose that α is a root of the equation $f(x) = 0$ and e_n is the error at the nth iteration. Then $e_n = x_n - \alpha$ and, using Taylor's series expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + ...] \quad (3.19)$$

$$f'(x_n) = f'(\alpha)[1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5+7c_7e_n^6+...], \quad (3.20)$$

where $c_n = \frac{f^{(n)}(\alpha)}{n!f'(\alpha)}$.

With the help of equations (3.19)) and (3.20, we get

$$y_n = \alpha + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 - (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + ... \quad (3.21)$$

$$f(y_n) = f'(\alpha)[c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + ...] \quad (3.22)$$

$$f'(y_n) = f'(\alpha)[1+2c_2^2e_n^2+4(c_3c_2-c_2^3)e_n^3+(6c_2c_4-11c_3c_2^2+8c_2^4)e_n^4+...] \quad (3.23)$$

Using equations (3.19) - (3.23), we get

$$z_n = \alpha - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (c_2^3 + 6c_2c_3 - 3c_4)e_n^4 + ... \quad (3.24)$$

$$f(z_n) = f'(\alpha)[-c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (c_2^3 + 6c_2c_3 - 3c_4)e_n^4 + ...] \quad (3.25)$$

$$f'(z_n) = f'(\alpha)[1-2c_2^2e_n^2+4(c_2^3-c_2c_3)e_n^3+(2c_2^4+15c_2^2c_3-6c_2c_4)e_n^4+...]. \quad (3.26)$$

Using equations (3.24) - (3.26), we obtain

$$x_{n+1} = \alpha + 2c_2^3e_n^4 + O(e_n^5), \quad (3.27)$$

which implies that

$$e_{n+1} = \alpha + 2c_2^3e_n^4 + O(e_n^5). \quad (3.28)$$

The above equation shows that the order of convergence of Algorithm 2.1 is four.

Theorem 3.2. *Suppose that α is a root of the equation $f(x) = 0$. If $f(x)$ is sufficiently smooth in a neighborhood of α, then the order of convergence of Algorithm 2.2 is four.*

Proof. With the help of equations (3.19) - (3.26) along with the same assumptions of the previous theorem, we have

$$\begin{aligned} f''(z_n) =& f'(\alpha)[2c_2 - 6c_2c_3e_n^2 + (12c_2^2c_3 - 12c_3^2)e_n^2 \\ &+ (6c_2^3c_3 + 36c_2c_3^2 + 12c_2^2c_4 - 18c_3c_4)e_n^4 + ...]. \end{aligned} \tag{3.29}$$

Using equations (3.24) - (3.29), we get

$$x_{n+1} = \alpha + c_2^3e_n^4 + O(e_n^5), \tag{3.30}$$

which implies

$$e_{n+1} = \alpha + c_2^3e_n^4 + O(e_n^5). \tag{3.31}$$

The above equation shows that the order of convergence of Algorithm 2.2 is four.

4 Numerical Experiments

In this section, we compare the number of iterations to obtain an approximated root of our proposed methods with the other methods that had an equal order of convergence. Algorithm 2.1 (AL1) and Algorithm 2.2 (AL2) fourth order convergence compare with Cordero et al. method (COR)[2], Chun et al. method (CHU) [1], Junjua et al. (JUN) [5] and Li (LI) [9]. Considering the following numerical examples:

$$\begin{aligned} f_1(x) &= 4x^4 - 4x^2 \\ f_2(x) &= (x-2)^{23} - 1 \\ f_3(x) &= e^x \sin x + \ln(x^2+1) \\ f_4(x) &= (x+2)e^x - 1 \\ f_5(x) &= e^{x^2+7x-30} - 1 \\ f_6(x) &= x^3 - 2x^2 - 5 \\ f_7(x) &= (x-1)e^{-x} \\ f_8(x) &= \cos x - x \\ f_9(x) &= \sin^2 x - x^2 + 1 \\ f_{10}(x) &= xe^{x^2} - \sin^2 x + 3\cos x + 5 \end{aligned}$$

using Maple with 100 digit decimals and under the condition that the program will stop when $|x_n - x_{n-1}| < \epsilon$ and $|f(x_n)| < \epsilon$, where $\epsilon = 10^{-15}$.

Table 1 shows the number of iterations, the absolute value of the functions $|f(x_n)|$, and the absolute difference $|x_n - x_{n-1}|$ to the approximation of the roots of the iteration method with order of convergence four, respectively.

Table 1 Numerical experiments and comparison of different iterative methods.

Method	N	x_n	$\|f(x_n)\|$	$\|x_n - x_{n-1}\|$
$f_1(x)$, $x_0 = 0.75$				
COR	30	1.00000000000000000000	$1.48e-56$	$2.38e-15$
CHU	24	1.00000000000000000000	$6.82e-35$	$1.39e-12$
JUN	11	1.00000000000000000000	$8.47e-32$	$4.28e-9$
LI	4	1.00000000000000000186	$1.49e-17$	$2.04e-5$
AL1	4	1.00000000000000000000	$3.71e-25$	$2.33e-7$
AL2	4	0.99999999999999999999	$5.66e-92$	$1.87e-16$
Method	N	x_n	$\|f(x_n)\|$	$\|x_n - x_{n-1}\|$
$f_2(x)$, $x_0 = 2.9$				
COR	57286	3.00000000000000000000	$5.70e-36$	$8.62e-11$
CHU	153	3.00000000000000000000	$8.71e-20$	$3.97e-8$
JUN	50	3.00000000000000000000	$1.17e-29$	$3.85e-9$
LI	5	3.00000000000000000000	$5.25e-57$	$8.28e-16$
AL1	5	3.00000000000000000000	$7.31e-47$	$2.21e-13$
AL2	4	2.99999999999999999999	$7.21e-54$	$1.06e-10$
$f_3(x)$, $x_0 = 2.9$				
COR	4	3.23756298402392131325	$1.24e-42$	$1.11e-11$
CHU	4	3.23756298402392131545	$6.00e-17$	$1.71e-6$
JUN	4	3.23756298402392131325	$2.39e-62$	$1.52e-16$
LI	3	3.23756298402392131325	$2.04e-29$	$3.41e-8$
AL1	3	3.23756298402392131325	$1.07e-30$	$1.48e-8$
AL2	3	3.23756298402392131325	$8.20e-94$	$1.71e-16$
$f_4(x)$, $x_0 = -0.9$				
COR	4	−0.4428544010023885831	$1.53e-56$	$9.51e-15$
CHU	4	−0.4428544010023885831	$5.25e-28$	$1.09e-9$
JUN	3	−0.4428544010023885831	$3.78e-21$	$7.90e-6$
LI	3	−0.4428544010023885831	$9.78e-39$	$4.62e-10$
AL1	3	−0.4428544010023885831	$4.17e-33$	$9.32e-9$
AL2	2	−0.4428544010023885909	$1.27e-17$	$1.80e-3$
$f_5(x)$, $x_0 = 3.1$				
COR	4	3.00000000000000000000	$1.12e-51$	$1.73e-14$

Method	N	x_n	$\|f(x_n)\|$	$\|x_n - x_{n-1}\|$
CHU	4	3.00000000000000000000	$3.80e-26$	$5.13e-10$
JUN	4	3.00000000000000000000	$1.10e-64$	$1.15e-17$
LI	3	3.00000000000000000000	$1.11e-25$	$9.81e-8$
AL1	3	3.00000000000000000000	$4.14e-27$	$3.25e-8$
AL2	3	3.00000000000000000000	$6.66e-84$	$1.77e-15$
$f_6(x)$, $x_0 = 2.0$				
COR	4	2.69064744802861375035	$3.34e-33$	$4.68e-9$
CHU	6	2.69064744802861375035	$1.21e-33$	$8.96e-12$
JUN	4	2.69064744802861375035	$5.66e-31$	$1.96e-8$
LI	3	2.69064744802861375035	$5.14e-22$	$4.45e-6$
AL1	3	2.69064744802861375035	$5.26e-31$	$2.30e-8$
AL2	3	2.69064744802861375035	$4.47e-94$	$2.77e-16$
$f_7(x)$, $x_0 = 0.25$				
COR	4	0.99999999999999999999	$1.16e-50$	$3.08e-13$
Method	N	x_n	$\|f(x_n)\|$	$\|x_n - x_{n-1}\|$
CHU	4	0.99999999999999999999	$2.51e-26$	$5.15e-9$
JUN	4	0.99999999999999999999	$1.50e-62$	$3.85e-16$
LI	3	0.99999999999999999999	$1.46e-23$	$2.98e-6$
AL1	3	0.99999999999999999999	$2.76e-26$	$5.23e-7$
AL2	3	1.00000000000000000000	$8.64e-76$	$3.40e-13$
$f_8(x)$, $x_0 = 1.7$				
COR	3	0.73908513321516064165	$7.35e-46$	$9.33e-12$
CHU	3	0.73908513321516064165	$9.80e-29$	$1.33e-9$
JUN	3	0.73908513321516064165	$3.27e-48$	$2.68e-12$
LI	3	0.73908513321516064165	$5.31e-48$	$3.33e-12$
AL1	3	0.73908513321516064165	$1.05e-65$	$1.55e-16$
AL2	2	0.73908513321516064165	$3.25e-33$	$1.01e-5$
$f_9(x)$, $x_0 = -2.5$				
COR	3	−1.40449164821534122566	$7.58e-17$	$6.37e-5$
CHU	4	−1.40449164821534122260	$6.44e-34$	$9.45e-12$
JUN	3	−1.40449164821534122262	$4.92e-19$	$2.08e-5$
LI	3	−1.40449164821534122260	$1.15e-21$	$5.80e-6$
AL1	3	−1.40449164821534122260	$1.38e-27$	$1.84e-7$
AL2	3	−1.40449164821534122260	$1.95e-78$	$1.06e-13$
$f_{10}(x)$, $x_0 = -1.0$				
COR	3	−1.20764782713091892771	$2.57e-18$	$1.04e-5$
CHU	4	−1.20764782713091892770	$4.83e-39$	$5.95e-14$
JUN	3	−1.20764782713091892770	$6.07e-26$	$1.55e-7$
LI	3	−1.20764782713091892770	$3.73e-55$	$1.48e-14$

AL1	3	-1.2076478271309189270	$1.76e-34$	$1.26e-9$
AL2	2	-1.2076478271309189269	$6.72e-19$	$3.96e-4$

5 Conclusion

In this paper, we proposed two iteration methods with three steps each for nonlinear equations. The new methods have order of convergence four. Using some test examples, the performance of the proposed algorithms was also discussed. The numerical results uphold the analysis of the convergence which can be seen in Table 1. The ten nonlinear functions were considered to illustrate that AL2 is better than some other methods.

References

[1] C. Chun, M. Y. Lee, B. Neta, J. Duni, On optimal fourth-order iterative methods free from second derivative and their dynamics, Applied Mathematics and Computation, **218,** no. 11, (2012), 6427–6438.

[2] A. Cordero, J. L. Hueso, E. Martínez, J. R. Torregrosa, New modifications of Potra-Pták's method with optimal fourth and eighth orders of convergence, Journal of Computational and Applied Mathematics, **234,** no. 10, (2010), 2969–2976.

[3] E. Halley, A new exact and easy method for finding the roots of equations generally and that without any previous reduction, Philo. Royal Soc. London, **18,** (1964), 136–148.

[4] A. S. Householder, The Numerical Treatment of a Single Nonlinear Equation, McGraw-Hill, New York, NY, USA, 1970.

[5] M. D. Junjua, S. Akram, N. Yasmin, F. Zafar, A New Jarratt-Type Fourth-Order Method for Solving System of Nonlinear Equations and Applications, Journal of Applied Mathematics, (2015), 14 pages.

[6] D. E. Kincaid, E. W. Cheney, Numerical Analysis, Brooks/Cole Publishing Company, Pacific Grove, CA, USA, 1990.

[7] B. Kong-ied, Two new eighth and twelfth order iterative methods for solving nonlinear equations, International Journal of Mathematics and Computer Science, **16,** no. 1, (2021), 333–344.

[8] R. Sharma, A. Bah, An Optimal Fourth Order Iterative Method for Solving Nonlinear Equations and Its Dynamics, Journal of Complex Analysis, **6,** no. 25, (2015), 1235–1250.

[9] Shengfeng Li, Fourth-order iterative method without calculating the higher derivatives for nonlinear equation, Journal of Algorithms & Computational Technology, (2019), 9 pages.

[10] O. S. Solaiman, T. Hashim, Two new efficient sixth order iterative methods for solving nonlinear equations, Journal of King Saud University-Science, **31,** no. 4, (2019), 701–705.