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Application and efficiency evaluation of 4-point Newton Explicit Group to solve 2D porous medium equation

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Abstract

In this paper, a linearized implicit finite difference method is used to approximate the solution of a two-dimensional nonlinear porous medium equation. A large and sparse nonlinear system is iteratively

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solved using the Newton method and the 4-point explicit group technique. The efficiency of the applied 4-point NEG method is evaluated based on the number of iterations and elapsed time recorded from the simulation with different grid points. The accuracy of the method is measured using the error of absolute of the numerical solution against the exact solution of the proposed problems. A comparative analysis is made using the control method, the Newton-Gauss-Seidel method. The numerical finding showed that the 4-point NEG method is faster than the Newton-Gauss-Seidel method by 36.75%, and the number of iterations is successfully reduced by 45.63%. The accuracy of the 4-point NEG method to solve two-dimensional porous medium equation is better than the Newton-Gauss-Seidel method.

1 Introduction

The nonlinear partial differential equation (NPDE) plays an important role in developing mathematical models of many complex physical phenomena. One of these NPDEs is a porous medium equation (PME). PME is commonly used to describe mass transfer in porous media. Nonlinearity in PME appears due to the nonlinear behaviour in the transfer process. This property has made PME successfully extend the theory and application of the classic linear diffusion equation. PME can be modelled in a higher dimension to describe the mass transfer fully. For instance, foam flows in porous media, which take both longitudinal coordinate and bubble area at a particular time as a subject [1]. However, high-dimensional NPDEs, including PME, are difficult to solve efficiently because of the dimension's exponential growth of the computational complexity.

Different methods have been introduced to the literature to solve high-dimensional PME, particularly in a two-dimension situation, such as finite difference method (FDM) [2, 3, 4, 5, 6, 7], finite element method [8], finite volume method [9], and Adomian Decomposition method [10]. FDM is necessary to be used as an alternative approximation when the formulated PME has no exact solution. Although the FDM can only approximate the solution, the efficiency of a particular numerical method derived from FDM needs to be investigated. Therefore, the capability of such a numerical method to compute approximate solutions within the specified error with a small computational complexity must be evaluated.

Several iterative techniques have been proposed to improve the iteration convergence rate in solving equations. One of these is the explicit-group technique which uses several small groups of grid point strategy to reduce the computational complexity in solving a large and sparse system. Chew, Sulaiman, and Muhiddin [11] studied and evaluated the efficiency of 2-point and 4-point explicit group techniques to solve the one-dimensional PME. The result showed that the 4-point explicit group technique is significantly more efficient than the 2-point explicit group technique to obtain the numerical solution of several one-dimensional PME problems. This paper extends the work in [11] by applying the 4-point explicit group technique in a higher dimensional PME, particularly in two-dimension. In this paper, we evaluate the efficiency of the 4-point explicit group technique with the Newton method to numerically solve several two-dimensional PME (2D PME) problems. The method is developed based on the second-order accurate implicit FDM and uses the Newton method and the 4-point explicit group technique to solve the generated nonlinear system. The comparative analysis in terms of the number of iterations, elapsed time and absolute errors is made. We compare the results of the proposed 4-point Newton explicit group (4-point NEG) with the developed Newton-Gauss-Seidel (NGS) method in [12].

2 Numerical Method

2.1 Implicit FDM for 2D PME

In this paper, we consider the following 2D PME [13]:

$$\frac{\partial u}{\partial t} = \alpha \left[\frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y}\left(u^m \frac{\partial u}{\partial y} \right) \right], \tag{2.1}$$

where m is a positive integer and α is an arbitrary constant. The solution of equation 2.1 is denoted by $u = u(x, y, t)$, where $0 \leq x \leq L, 0 \leq y \leq W$ and $0 < t < T$. By taking the exponent $m > 0$, equation 2.1 will be uniformly parabolic anywhere and in which u is bounded away from zero and degenerate at any neighborhood point $u = 0$. Meanwhile, $m < 0$ in equation 2.1 gives a fast diffusion equation in which propagation speed becomes infinite. Furthermore, equation 2.1 becomes a well-known two-dimensional heat equation when $m = 0$ [14]. Equation 2.1 can be derived into

$$\frac{\partial u}{\partial t} = \alpha \left[u^m \frac{\partial^2 u}{\partial x^2} + m u^{m-1} \left(\frac{\partial u}{\partial x} \right)^2 + u^m \frac{\partial^2 u}{\partial y^2} + m u^{m-1} \left(\frac{\partial u}{\partial y} \right)^2 \right], \tag{2.2}$$

and we will focus on equation 2.2 from now on.

Let $u(x,y,t)$ be discretized uniformly on the two dimensional spatial and temporal grids: $u(x,y,t) = U(ih, jh, nk), 1 \leq i,j \leq M-1$, and $n > 0$. Here, h and k are the spatial and temporal steps by $h = L/M = W/M$ and $k = T/n$. By approximating the derivative terms in equation 2.2 with the implicit FDM, the finite difference approximation equation can be written as

$$\begin{aligned} F_{i,j}^n &= U_{i,j}^n - A_1(U_{i,j}^n)^m \left(U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n\right) \\ &- A_2 m(U_{i,j}^n)^{m-1}\left((U_{i+1,j}^n)^2 - 2U_{i+1,j}^n U_{i-1,j}^n + (U_{i-1,j}^n)^2\right) \\ &- A_3(U_{i,j}^n)^m \left(U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n\right) \\ &- A_4 m(U_{i,j}^n)^{m-1}\left((U_{i,j+1}^n)^2 - 2U_{i,j+1}^n U_{i,j-1}^n + (U_{i,j-1}^n)^2\right) - U_{i,j}^{n-1}, \end{aligned} \tag{2.3}$$

where $A_1 = A_3 = \alpha k/h^2$, $A_2 = A_4 = \alpha k/4h^2$. Based on the nonlinear approximation 2.3, a highly complex nonlinear system can be formed as

$$\underset{\sim}{F}^n(\underset{\sim}{U}^n) = 0, \tag{2.4}$$

where $\underset{\sim}{F}^n(.) = Transpose(F_{1,1}^n, F_{2,1}^n, \ldots, F_{1,2}^n, F_{2,2}^n, \ldots, F_{M-1,M+1}^n)$ and $\underset{\sim}{U}^n = Transpose(U_{1,1}^n, U_{2,1}^n, \ldots, U_{1,2}^n, U_{2,2}^n, \ldots, U_{M-1,M+1}^n)$.

2.2 4-Point Newton Explicit Group for 2D PME Approximation

In this section, we show the formulation of the 4-point NEG method. From the nonlinear system 2.4, the solution can be computed iteratively using the Newton method as follows:

$$(\underset{\sim}{U}^n)^{(\ell+1)} = (\underset{\sim}{U}^n)^{(\ell)} - \left[J_F(\underset{\sim}{U}^n)^{(\ell)}\right]^{-1} \underset{\sim}{F}^n\left[(\underset{\sim}{U}^n)^{(\ell)}\right], \tag{2.5}$$

where

$$J_F(\underset{\sim}{U}^n) = \frac{d}{d\underset{\sim}{U}^n}\left[\underset{\sim}{F}^n(\underset{\sim}{U}^n)\right], \tag{2.6}$$

and the iterative index is represented by (ℓ).

The iteration process of equation 2.5 can be modified by implementing the Gauss-Seidel iterative method. The resultant NGS method is generally stated as:

$$\begin{aligned} &(\varphi_{i,j}^n)^{(\ell+1)} = (ac_{i,j})^{-1}(F_{i,j}^n - aa_{i,j}(\varphi_{i,j-1}^n)^{(\ell+1)} - ab_{i,j}(\varphi_{i-1,j}^n)^{(\ell+1)} \\ &- ad_{i,j}(\varphi_{i+1,j}^n)^{(\ell)} - ae_{i,j}(\varphi_{i,j+1}^n)^{(\ell)}), i = 1, 2, \ldots, M-1; \\ &j = 1, 2, \ldots, M-1; n = 1, 2, ..., N, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} aa_{i,j} &= \frac{d}{dU^n_{i,j-1}}(F^n_{i,j}), ab_{i,j} = \frac{d}{dU^n_{i-1,j}}(F^n_{i,j}), ac_{i,j} = \frac{d}{dU^n_{i,j}}(F^n_{i,j}), \\ ad_{i,j} &= \frac{d}{dU^n_{i+1,j}}(F^n_{i,j}), ae_{i,j} = \frac{d}{dU^n_{i,j+1}}(F^n_{i,j}), \varphi^n_{i,j} = (U^n_{i,j})^{(\ell+1)} - (U^n_{i,j})^{(\ell)}. \end{aligned} \tag{2.8}$$

Then, the 4-point NEG method can be formulated by considering a system of four equations based on equation 2.7, in a matrix form, as follows:

$$\begin{bmatrix} ac_{i,j} & ad_{i,j} & ae_{i,j} & 0 \\ ab_{i+1,j} & ac_{i+1,j} & 0 & ae_{i+1,j} \\ aa_{i,j+1} & 0 & ac_{i,j+1} & ad_{i,j+1} \\ 0 & aa_{i+1,j+1} & ab_{i+1,j+1} & ac_{i+1,j+1} \end{bmatrix} \begin{bmatrix} \varphi^n_{i,j} \\ \varphi^n_{i+1,j} \\ \varphi^n_{i,j+1} \\ \varphi^n_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} S^n_{i,j} \\ S^n_{i+1,j} \\ S^n_{i,j+1} \\ S^n_{i+1,j+1} \end{bmatrix}, \tag{2.9}$$

where

$$\begin{aligned} S^n_{i,j} &= F^n_{i,j} - aa_{i,j}\varphi^n_{i,j-1} - ab_{i,j}\varphi^n_{i-1,j}, \\ S^n_{i+1,j} &= F^n_{i+1,j} - aa_{i+1,j}\varphi^n_{i+1,j-1} - ab_{i+1,j}\varphi^n_{i+2,j}, \\ S^n_{i,j+1} &= F^n_{i,j+1} - ab_{i,j+1}\varphi^n_{i-1,j+1} - ae_{i,j+1}\varphi^n_{i,j+2}, \\ S^n_{i+1,j+1} &= F^n_{i+1,j+1} - ad_{i+1,j+1}\varphi^n_{i+2,j+1} - ae_{i+1,j+1}\varphi^n_{i+1,j+2}. \end{aligned} \tag{2.10}$$

From the system of equations shown in equation 2.9, the iterative form of the 4-point NEG method for solving equation 2.2 can be derived into

$$\begin{bmatrix} \varphi^n_{i,j} \\ \varphi^n_{i+1,j} \\ \varphi^n_{i,j+1} \\ \varphi^n_{i+1,j+1} \end{bmatrix}^{(\ell+1)} = \begin{bmatrix} ac_{i,j} & ad_{i,j} & ae_{i,j} & 0 \\ ab_{i+1,j} & ac_{i+1,j} & 0 & ae_{i+1,j} \\ aa_{i,j+1} & 0 & ac_{i,j+1} & ad_{i,j+1} \\ 0 & aa_{i+1,j+1} & ab_{i+1,j+1} & ac_{i+1,j+1} \end{bmatrix}^{-1} \begin{bmatrix} S^n_{i,j} \\ S^n_{i+1,j} \\ S^n_{i,j+1} \\ S^n_{i+1,j+1} \end{bmatrix}, \tag{2.11}$$

where

$$\begin{aligned} S^n_{i,j} &= F^n_{i,j} - aa_{i,j}(\varphi^n_{i,j-1})^{(\ell+1)} - ab_{i,j}(\varphi^n_{i-1,j})^{(\ell+1)}, \\ S^n_{i+1,j} &= F^n_{i+1,j} - aa_{i+1,j}(\varphi^n_{i+1,j-1})^{(\ell+1)} - ab_{i+1,j}(\varphi^n_{i+2,j})^{(\ell)}, \\ S^n_{i,j+1} &= F^n_{i,j+1} - ab_{i,j+1}(\varphi^n_{i-1,j+1})^{(\ell+1)} - ae_{i,j+1}(\varphi^n_{i,j+2})^{(\ell)}, \\ S^n_{i+1,j+1} &= F^n_{i+1,j+1} - ad_{i+1,j+1}(\varphi^n_{i+2,j+1})^{(\ell)} - ae_{i+1,j+1}(\varphi^n_{i+1,j+2})^{(\ell)}. \end{aligned} \tag{2.12}$$

The iteration process of equation 2.11 can be facilitated using the developed algorithm as in Algorithm 1. The convergence criterion for the implementation of the 4-point NEG method uses a tolerance error of $\varepsilon = 1 \times 10^{-10}$.

Algorithm 1 The 4-point NEG method

Define the value of $m, A_1, A_2, A_3, A_4, (U_{i,j}^0)^{(0)} = 1.0, (\varphi_{i,j}^0)^{(0)} = 0$ for $1 \leq i, j \leq M-1$;
Define the initial and boundaries;
while $n \leq N$ **do**
  Initialize $\ell = 0$;
  while $\left|(U_{i,j}^n)^{(\ell+1)} - (U_{i,j}^n)^{(\ell)}\right| > \varepsilon$ **do**
    while $\left|(\varphi_{i,j}^n)^{(\ell+1)} - (\varphi_{i,j}^n)^{(\ell)}\right| > \epsilon$ **do**
      Iterate equation 2.11;
    end while
    $(U_{i,j}^n)^{(\ell+1)} = (U_{i,j}^n)^{(\ell)} + (\varphi_{i,j}^n)^{(\ell+1)}$;
    $iter++$;
  end while
  $n++$;
end while
Display numerical solution, number of iterations, elapsed time, and error of absolute;

3 Numerical Experiment

To examine the performance of the 4-point NEG method, the number of iterations and elapsed time are recorded from the simulation with a different number of grid points. The accuracy of the method is measured using the error of absolute of the numerical solution against the exact solution of the proposed problem. A comparative analysis is made using the developed NGS method (equation 2.7), which is the control method. Three 2D PME examples are tested for the numerical experiment:

Example 1 [13]:

$$\frac{\partial u}{\partial t} = 0.2\left[\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right)\right]. \tag{3.13}$$

The exact solution is $u(x, y, t) = x + y + 0.4t$ for $0 \leq x, y \leq 1$ and $0 < t < 1$.

Example 2 [13]:

$$\frac{\partial u}{\partial t} = 0.2\left[\frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u^2\frac{\partial u}{\partial y}\right)\right]. \tag{3.14}$$

The exact solution is $u(x, y, t) = \sqrt{5x + 5y + 5t}$ for $0 \leq x, y \leq 1$ and $0 < t < 1$.

Example 3 [15]:

$$\frac{\partial u}{\partial t} = \left[\frac{\partial}{\partial x}\left(u^5\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u^5\frac{\partial u}{\partial y}\right)\right]. \tag{3.15}$$

The exact solution is $u(x,y,t) = \sqrt[4]{0.8x + 0.8y + 1.6t}$ for $0 \le x, y \le 1$ and $0 < t < 1$.

From the numerical result tabulated in Tables 1, 2 and 3, we observe that the 4-point NEG method requires fewer iterations and shorter computation time in computing the approximate solutions of Examples 1, 2 and 3 compared to the NGS method. In the numerical study, we found that the 4-point NEG method is faster than the NGS method by 36.75%, and the number of iterations is successfully reduced by 45.63%. Regarding the accuracy of the methods from the three tested Examples, the absolute errors by the 4-point NEG method are much smaller than the NGS method. The accuracy of the solution has been improved by using the proposed 4-point NEG method.

In terms of the computational complexity, which has been analyzed based on the implementation of the 4-point NEG and NGS methods (see Table 4), we found that the proposed 4-point NEG method uses a lesser number of addition and multiplication operations for every iteration toward the exact solution of equation 2.1 compared to NGS method. Since the computational complexity is lesser by the 4-point NEG method, the number of iterations and computation time to solve equation 2.1 is less. As a result, the 4-point NEG method has a better efficiency in solving 2D PME than the NGS method.

Table 1: Methods' efficiency comparison from solving Example 1

M	Method	ℓ	s	error
16	NGS	136	0.76	8.86×10^{-11}
	4-point NEG	84	0.44	1.05×10^{-11}
32	NGS	436	2.93	3.25×10^{-10}
	4-point NEG	244	2.64	1.36×10^{-10}
64	NGS	1525	19.59	1.90×10^{-9}
	4-point NEG	818	8.55	6.59×10^{-10}
128	NGS	5462	360.89	9.02×10^{-9}
	4-point NEG	2903	214.01	3.96×10^{-9}
256	NGS	19404	4586.69	3.86×10^{-8}
	4-point NEG	10338	3717.50	1.80×10^{-8}

Table 2: Methods' efficiency comparison from solving Example 2

M	Method	ℓ	s	error
16	NGS	130	0.97	7.57×10^{-11}
	4-point NEG	80	0.62	2.09×10^{-11}
32	NGS	400	2.70	2.31×10^{-9}
	4-point NEG	223	1.93	8.45×10^{-10}
64	NGS	1380	18.26	1.31×10^{-8}
	4-point NEG	738	8.56	6.39×10^{-9}
128	NGS	4901	248.82	4.95×10^{-8}
	4-point NEG	2596	187.24	2.79×10^{-8}
256	NGS	17458	4243.79	1.75×10^{-7}
	4-point NEG	9279	2295.15	9.92×10^{-8}

Table 3: Methods' efficiency comparison from solving Example 3

M	Method	ℓ	s	error
16	NGS	739	0.98	1.10×10^{-9}
	4-point NEG	406	0.86	1.95×10^{-10}
32	NGS	2630	9.51	6.69×10^{-9}
	4-point NEG	1393	7.75	2.50×10^{-9}
64	NGS	9478	113.63	3.56×10^{-8}
	4-point NEG	4996	61.96	1.51×10^{-8}
128	NGS	34098	1653.85	1.72×10^{-7}
	4-point NEG	18034	763.70	7.45×10^{-8}
256	NGS	121649	29234.80	7.78×10^{-7}
	4-point NEG	64894	14890.79	3.56×10^{-7}

Table 4: Computational complexity by NGS and 4-point NEG methods

Method	Addition per iteration	Multiplication per iteration
NGS	$4(M-1)^2$	$5(M-1)^2$
4-point NEG	$8(\frac{M}{2}-1)^2+8(\frac{M}{2}-1)+2$	$12(\frac{M}{2}-1)^2+12(\frac{M}{2}-1)+3$

4 Conclusion

We have presented the linearized implicit FDM for approximating the solution of 2D PME. Newton method has been applied to solve the high complexity nonlinear system and the iteration efficiency has been improved using the derived 4-point NEG method. The numerical experiment finding showed that the proposed 4-point NEG method is more efficient and accurate in solving 2D PME than the NGS method. For future research, we intend to investigate the efficiency of the 4-point NEG method to solve NPDE by considering a three-dimensional PME and a system of PME.

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