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Congruence relations on pseudo-UP algebras

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Abstract

In this paper, we analyze the links between congruences, ideals, and homomorphisms in pseudo-UP algebras. Moreover, we establish a theorem which can be viewed as the First Isomorphism Theorem for these algebras. Furthermore, our results may serve as a base to construct other isomorphism theorems for pseudo-UP algebras.

1 Introduction

Let A be a nonempty set. A total function $w : A \times A \longrightarrow A$ is an internal binary operation on A. The system (A, w) is the simplest algebraic structure recognizable by the name "groupoid". If the structure (A, w) satisfies some additional conditions, then the question naturally arises about the properties of such complex structure. Examples of such complex structures include Hilbert, BCK/BCI/BE-algebras. In such and similar structures the focus, among other things, is not only on their properties but also on the properties

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of the substructures in them. In addition, many researchers have dealt with some recognizable relations in such algebraic structures such as the order relation equivalence. Often, the subject of study has been homomorphisms between such algebraic structures and their relation to some of the substructures in them.

In 2007, Kim and Kim [7] introduced the notion of BE-algebras which is another generalization of BCK-algebras. In 2013, Borzooei et al. [1] generalized the notion of BE-algebras and introduced the notion of pseudo BE-algebras. In 2014, Rezaei et al. [10] introduced the notion of congruence relation on pseudo BE-algebras and constructed the quotient pseudo BE-algebra via this congruence relation.

Back in 2009, Prabpayak and Leerawat [9] introduced a new algebraic structure which is called KU-algebras. As a generalization of that, in 2017, Iampan [2] introduced a new algebraic structure which is called UP-algebras, studied ideals and congruences in UP-algebras, introduced the concept of homomorphism of UP-algebras and investigated some related properties, and derived some properties of the relations between quotient UP-algebras and isomorphism (see, for example, [3, 8]). In the study of this algebraic structure, Romano took part as well ([11, 12, 13, 14, 15]).

In 2020, Romano [16] introduced the concept of pseudo-UP algebra and derived basic properties. In a forthcoming article [17], Romano introduced the concepts of pseudo-UP ideals and pseudo-UP filters in pseudo-UP algebras. As a continuation of these papers, Romano [18] introduced the concept of homomorphisms between pseudo-UP algebras. The notion of homomorphisms between pseudo-UP algebras is designed in the same way as it was done in [4, 5, 6] when analyzing pseudo-BCK and pseudo-BCI algebras.

in our paper, we discusses the links between congruences, ideals, and homomorphisms in pseudo-UP algebras using the idea in [10]. We show how a congruence θ_J can be constructed on a pseudo-UP algebra $\mathfrak{A}$ by the given pseudo-UP ideal J and the relation of that congruence with its quotient structure. In addition, we prove a theorem which can be viewed as the First Isomorphism Theorem for these algebras.

2 Preliminaries

In this section, we describe some elements of UP-algebras from the literature [2, 3, 16, 17, 18] that we are going to need.

Definition 2.1. *([2], Definition 1.3) An algebra* $A = (A, \cdot, 0)$ *of type* $(2, 0)$

is called a UP-algebra, *where A is a nonempty set, "$\cdot$" is a binary operation on A, and "0" is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:*

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, *and*
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot x = 0) \implies x = y)$.

Definition 2.2. *([16], Definition 3.1) A* pseudo-UP algebra *is a structure $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$, where "$\leqslant$" is a binary relation on a nonempty set A, "$\cdot$" and "$*$" are internal binary operations on A and "0" is an element of A, satisfying the following axioms:*

(pUP-1) $(\forall x, y, z \in A)(y \cdot z \leqslant (x \cdot y) * (x \cdot z) \ \wedge \ y * z \leqslant (x * y) \cdot (x * z))$,
(pUP-4) $(\forall x, y \in A)((x \leqslant y \ \wedge \ y \leqslant x) \implies x = y)$,
(pUP-5) $(\forall x, y \in A)((y \cdot 0) * x = x \ \wedge \ (y * 0) \cdot x = x)$, *and*
(pUP-6) $(\forall x, y \in A)((x \leqslant y \iff x \cdot y = 0) \wedge (x \leqslant y \iff x * y = 0))$.

From the previous definition, the following lemma follows immediately.

Lemma 2.3 ([16]). *In a pseudo-UP algebra $\mathfrak{A}$, the following hold:*

(1) $(\forall x \in A)(x \cdot 0 = 0 \ \wedge \ x * 0 = 0)$,
(2) $(\forall x \in A)(0 \cdot x = x \ \wedge \ 0 * x = x)$, *and*
(3) $(\forall x \in A)(x \cdot x = 0 \ \wedge \ x * x = 0)$.

In the following definitions, we introduce the concept of pseudo-UP ideals and pseudo-UP filters in pseudo-UP algebras.

Definition 2.4. *([17]) A nonempty subset J of a pseudo-UP algebra $\mathfrak{A}$ is called a* pseudo-UP ideal *of $\mathfrak{A}$ if it satisfies the following properties:*

(pJ1) $0 \in J$,
(pJ2) $(\forall x, y, z \in A)((x \cdot (y * z) \in J \ \wedge \ y \in J) \implies x \cdot z \in J)$, *and*
(pJ3) $(\forall x, y, z \in A)((x * (y \cdot z) \in J \ \wedge \ y \in J) \implies x * z \in J)$.

The following theorem describes the characteristic features of these substructures.

Theorem 2.5 ([16], Theorem 3.1). *Let J be a pseudo-UP ideal in a pseudo-UP algebra $\mathfrak{A}$. Then*

(4) $(\forall x, y \in A)((x * y \in J \ \wedge \ x \in J) \implies y \in J)$,
(5) $(\forall x, y \in A)(y \in J \implies x * y \in J)$,
(6) $(\forall x, y \in A)((x \cdot y \in J \ \wedge \ x \in J) \implies y \in J)$, *and*
(7) $(\forall x, y \in A)(y \in J \implies x \cdot y \in J)$.

Corollary 2.6 ([16], Corollary 3.1). *Let J be a pseudo-UP ideal in a pseudo-UP algebra $\mathfrak{A}$. Then*
(8) $(\forall x, y \in A)((x \leqslant y \wedge x \in J) \implies y \in J)$.

Definition 2.7. *([17]) A nonempty subset F of a pseudo-UP algebra $\mathfrak{A}$ is called a* pseudo-UP filter *of $\mathfrak{A}$ if it satisfies the following properties:*
(pF1) $0 \in F$,
(pF2) $(\forall x, y \in A)((x \in F \wedge x \cdot y \in F \implies y \in F)$, *and*
(pF3) $(\forall x, y \in A)((x \in F \wedge x * y \in F \implies y \in F)$.

The pseudo homomorphisms between pseudo-BCK algebras were studied by Jun et al. in [4] and Lee and Park in [6]. The pseudo homomorphisms between pseudo-BCI algebras were studied by Jun et al. in [5].

Definition 2.8. *([18], Definition 3.1) Let $\mathfrak{A} = ((A, \leqslant_A), \cdot_A, *_A, 0_A)$ and $\mathfrak{B} = ((B, \leqslant_B), \cdot_B, *_B, 0_A)$ be pseudo-UP algebras. A mapping $f : A \longrightarrow B$ is called a* pseudo-UP homomorphism *if*

$$(\forall x, y \in A)(f(x \cdot_A y) = f(x) \cdot_B f(y))$$

and

$$(\forall x, y \in A)(f(x *_A y) = f(x) *_B f(y)).$$

Lemma 2.9. *If $f : A \longrightarrow B$ is a pseudo-UP homomorphism, then $f(0_A) = 0_B$ holds.*

Proof. The assertion of this lemma follows directly from the Definition 2.8 with respect to equality (10) in [16]. □

Corollary 2.10. *If $f : A \longrightarrow B$ is a pseudo-UP homomorphism, then*
(9) $(\forall x, y \in A)(x \leqslant_A y \implies f(x) \leqslant_B f(y))$.

Corollary 2.11. *Let $f : A \longrightarrow B$ be a pseudo-UP homomorphism of pseudo-UP algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then the kernel $Ker(f) = \{x \in A : f(x) = 0_B\}$ of f is a pseudo-UP ideal of $\mathfrak{A}$.*

3 Congruence relations on pseudo-UP algebras

Since there are two internal binary operations in a pseudo-UP algebra $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$, we can determine compatibility of an equivalence θ an A in three ways:

Definition 3.1. *Let $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$ be a pseudo-UP algebra and θ be an equivalence relation on the set A.*

(i) θ is a congruence of type 1 *on $\mathfrak{A}$ if the following holds*

(10) $(\forall x, y, z \in A)((x, y) \in \theta \implies ((x \cdot z, y \cdot z) \in \theta \wedge (z \cdot x, z \cdot y) \in \theta)$.

(ii) θ is a congruence of type 2 *on $\mathfrak{A}$ if the following holds*

(11) $(\forall x, y, z \in A)((x, y) \in \theta \implies ((x * z, y * z) \in \theta \wedge (z * x, z * y) \in \theta)$.

We define the notion of congruence relations on pseudo-UP algebras.

Definition 3.2. *Let $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$ be a pseudo-UP algebra and θ be an equivalence relation on the set A. Then θ is called a* congruence relation of common type *(or briefly* congruence*, if it will not lead to misunderstanding) on $\mathfrak{A}$ if it is a congruence of type 1 and type 2 at the same time.*

Lemma 3.3. *The condition $(10) \wedge (11)$, which appears in the Definition 3.1, is equivalent to the following implication*

$$(\forall x, y, u, v \in A)(((x, y) \in \theta \wedge (u, v) \in \theta) \implies ((x \cdot u, y \cdot v) \in \theta \wedge (x * u, y * v) \in \theta)).$$

Proposition 3.4. *Let θ be a congruence relation on a pseudo-UP algebra $\mathfrak{A}$. Then the set $C_0 = \{x \in A : (x, 0) \in \theta\}$ is a pseudo-UP ideal in $\mathfrak{A}$.*

Proof. Since $(0, 0) \in \theta$, it is obvious that $0 \in C_0$ holds.

Let $x, y, z \in A$ be such that $x \cdot (y * z) \in C_0$ and $y \in C_0$. Then $(x \cdot (y * z), 0) \in \theta$ and $(y, 0) \in \theta$. From $(y, 0) \in \theta$ it follows $(y * z, 0 * z) \in \theta$ and $(y * z, z) \in \theta$ by (9). Thus $(x \cdot (y * z), x \cdot z) \in \theta$. Now, from $(x \cdot (y * z), 0) \in \theta$ and $(x \cdot (y * z), z \cdot z)$, it follows $(0, x \cdot z) \in \theta$ because θ is a transitive relation. So, $x \cdot z \in C_0$.

Proof of the Second Implication $(x * (y \cdot z) \in C_0 \wedge y \in C_0) \implies x * z \in C_0$ is similar to that of a previous implication. □

Theorem 3.5. *Let $f : A \longrightarrow B$ be a pseudo-UP homomorphism and*

$$\theta_f = \{(x, y) \in A \times A : f(x) = f(y)\}.$$

Then θ_f is a congruence relation of common type on $\mathfrak{A}$.

Proof. The proof follows by direct verification. □

Analysis 3.6. *Let θ be a congruence on a pseudo-UP algebra $\mathfrak{A}$. Define operations "$\odot$" and "$\otimes$" on A/θ by*

$$(\forall x, y \in A)([x] \odot [y] = [x \cdot y] \text{ and } [x] \otimes [y] = [x * y]).$$

Let us check if this factor structure is a pseudo-UP algebra. Define the relation " $\leqq$ " on A/θ by

$$(\forall x, u \in A)([0] = [x] \odot [y] \Longleftrightarrow [x] \leqq [y] \Longleftrightarrow [x] \otimes [y] = [0]).$$

Then

(pUP-1): $$\begin{aligned}([y] \odot [z]) \otimes (([x] \odot [y]) \otimes ([x] \odot [z])) &= [y \cdot z] \otimes ([x \cdot y] \otimes [x \cdot z]) \\ &= [y \cdot z] \otimes [(x \cdot y) * (x \cdot z)] \\ &= [(y \cdot z) * ((x \cdot y) * (x \cdot z))] \\ &= [0].\end{aligned}$$

The second equality in (pUP-1) can be proved similar to this proof.

(pUP-5): $([y] \odot [0]) \otimes [x] = [y \cdot 0] \otimes [x] = [(y \cdot 0) * x] = [x]$.

The second equality in (pUP-5) can be proved similar to this proof.

Thus, the axioms (pUP-1), (pUP-5) and (pUP-6) that determine pseudo-UP algebras are valid formulas on the carrier A/θ with operations " $\odot$ " and " $\otimes$ " and the relation " $\leqq$ ". The axiom (pUP-4) in A/θ remains to be checked. Let $[x] \leqq [y]$ and $[y] \leqq [x]$ be valid relations. Then $[x] \odot [y] = [0]$ and $[y] \odot [x] = [0]$. Thus $[x \cdot y] = [0]$ and $[y \cdot x] = [0]$. From here, it follows $(x \cdot y, 0) \in \theta$ and $(y \cdot x, 0) \in \theta$. Hence $(x \cdot y, y \cdot x) \in \theta$. To have $[x] = [y]$, it is sufficient that the following is true $(x, x \cdot y) \in \theta$ and $(y, y \cdot x) \in \theta$ true; i.e., it is sufficient that $[x] = [x \cdot y]$ and $[y] = [y \cdot x]$ are valid. As expected, the formula (pUP-4) is not a valid formula in $((A/\theta, \leqq), \odot, \otimes, [0])$.

4 Congruence relations induced by pseudo-UP ideals

We start this section with one important technical lemma.

Lemma 4.1. *Let J be a pseudo-UP ideal of a pseudo-UP algebra $\mathfrak{A}$. Then*
(12) $(\forall x, y \in A)(x \cdot y \in J \Longleftrightarrow x * y \in J)$.

Proof. Suppose $x \cdot y \in J$. From the left side of the formula (pUP-1), it follows immediately that $(z \cdot x) * (z \cdot y) \in J$ for any $z \in A$, in accordance with (8). For $z = 0$, we get $x * y \in J$ according to (2).

Conversely, let $x * y \in J$. From the right side of the formula (pUP-1), it follows immediately that $(z * x) \cdot (z * y) \in J$ for any $z \in A$ by (8). For $z = 0$, we get $x \cdot y \in J$ according to (2). $\square$

Let J be a pseudo-UP ideal of a pseudo-UP algebra $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$. Let us define the relation θ_J on A as follows

(13) $(\forall x, y \in A)((x, y) \in \theta_J \iff (x \cdot y \in J \wedge y \cdot x \in J))$.

Theorem 4.2. *The relation θ_J is a congruence relation on $\mathfrak{A}$.*

Proof. (1) Since $0 \in J$, we have $x \cdot x = 0 \in J$ for any $x \in A$. This means that θ_J is a reflexive relation.

(2) It is obvious that the relation θ_J is a symmetric relation.

(3) Let $x, y, z \in A$ such that $(x, y) \in \theta_J$ and $(y, z) \in \theta_J$. Then $x \cdot y \in J$, $y \cdot x \in J$ (and $x * y \in J$, $y * x \in J$ by (12)) and $z \cdot y \in J$, $y \cdot z \in J$ (and $z * y \in J$, $y * z \in J$ by (12)). Now, from $y \cdot z \in J$ and $y \cdot z \leqslant (x \cdot y) * (x \cdot z)$ it follows $(x \cdot y) * (x \cdot z) \in J$ by Corollary 2.6. Hence, from $(x \cdot y) * (x \cdot z) \in J$ and $x \cdot y \in J$, we have $x \cdot z \in J$ by the claim (1) in Theorem 2.5. On the other hand, from $y \cdot x \in J$ and $y \cdot x \leqslant (z \cdot y) * (z \cdot x)$, it follows that $(z \cdot y) * (z \cdot x) \in J$ by Corollary 2.10. Thus, from $(z \cdot y) * (z \cdot x) \in J$ and $z \cdot y \in J$, we get $z \cdot x \in J$ by Theorem 2.5(1). So, the relation θ_J is an equivalence relation on A.

(4) Let $x, y, z \in A$ such that $(x, y) \in \theta_J$. Then $x \cdot y \in J \wedge y \cdot x \in J$ (and $x * y \in J \wedge y * x \in J$). Thus

$$x \cdot y \leqslant (z \cdot x) * (z \cdot y) \wedge x \cdot y \in J \Longrightarrow (z \cdot x) * (z \cdot y) \in J$$

and

$$y \cdot x \leqslant (z \cdot y) * (z \cdot x) \wedge y \cdot x \in J \Longrightarrow (z \cdot y) * (z \cdot x) \in J.$$

From here it follows $(z \cdot x) \cdot (z \cdot y) \in J$ and $(z \cdot y) \cdot (z \cdot x) \in J$ according to (12).

The implication

$$x \cdot y \in J \Longrightarrow ((x \cdot z) \cdot (y \cdot z) \in J \wedge (y \cdot z) \cdot (x \cdot z) \in J)$$

can be proven analogously. Indeed, suppose $(x, y) \in \theta_J$. Then $x \cdot y \in J$ and $y \cdot x \in J$. From $(x \cdot z) \cdot ((y \cdot x) * (y \cdot z)) = 0 \in J$ (left side of the formula (pUP-1)) and $y \cdot x \in J$ it follows $(x \cdot z) \cdot (y \cdot z) \in J$ by (pJ2). On the other hand, from $(y \cdot z) \cdot ((x \cdot y) * (x \cdot z)) = 0 \in J$ and $x \cdot y \in J$ follows $(y \cdot z) \cdot (x \cdot z) \in J$ by (pJ2).

Therefore, θ_J is a congruence of common type on $\mathfrak{A}$. □

For this congruence θ_J we say that it is generated by the pseudo-UP ideal J.

Let J be a pseudo-UP ideal of a pseudo-UP algebra $\mathfrak{A}$ and θ_J be a congruence on $\mathfrak{A}$ generated by J. In the following theorem, we show that the quotient structure $((A/\theta_J, \leqq), \odot, \otimes, [0])$ of a pseudo-UP algebra $\mathfrak{A}$ constructed by the congruence θ_J, determined as in Analysis 3.6, is also a pseudo-UP algebra.

Theorem 4.3. *Let J be a pseudo-UP ideal of a pseudo-UP algebra $\mathfrak{A}$ and let θ_J be a congruence on $\mathfrak{A}$ induced by J. Then the family $((A/\theta_J, \leqq), \odot, \otimes, [0])$ is a pseudo-UP algebra.*

Proof. We have already showed that the structure $((A/\theta_J, \leqq), \odot, \otimes, [0])$ satisfies the axioms (pUP-1), (pUP-5) and (pUP-6) (in Analysis 3.6). It remains to show that this structure also satisfies the axiom (pUP-4).

Let $[x] \leqq [y]$ and $[y] \leqq [x]$. Then $[x] \odot [y] = [0]$ and $[y] \odot [x] = [0]$. Thus $[x \cdot y] = [0]$ and $[y \cdot x] = [0]$. It follows that $(x \cdot y, 0) \in \theta_J$ and $(y \cdot x, 0) \in \theta_J$. This means $x \cdot y = 0 \cdot (x \cdot y) \in J \wedge 0 = (x \cdot y) \cdot 0 \in J$ and $y \cdot x = 0 \cdot (y \cdot x) \in J \wedge 0 = (y \cdot x) \cdot 0 \in J$. Thus $(x, y) \in \theta_J$ and $[x] = [y]$ by definition of θ_J. □

We end this section with an east-to-verify theorem that can be viewed as the First Isomorphism Theorem between pseudo-UP algebras.

Theorem 4.4. *Let $f : A \longrightarrow B$ be a pseudo-UP homomorphism between pseudo-UP algebras. Then there is a unique epimorphism $\pi : A \longrightarrow A/\theta_f$, defined by $\pi(x) = [x]$ for any $x \in A$ and a unique monomorphism $g : A/\theta_f \longrightarrow B$, defined by $g([x]) = f(x)$ for any $x \in A$, such that $f = g \circ \pi$.*

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