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Non-static solutions of the Einstein equation in the vacuum

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Abstract

We find the non-static solutions of Einstein's equation in the general theory of gravitation. We obtain a numerical solution and a generalized analytical approximation. In the same way, there is a set of families of solutions that are of great interest to researchers. From the solutions, we observe that time in the general theory of gravitation plays an important role in the curvature of space-time.

1 Introduction

From experimental data, it is well-known that our universe expanded isotropically, at a very early time, around $t = 10^{-36}s$. At this time, it is known that

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the general solution of Einstein's equation in vacuum $T_{\mu,\nu} = 0$ near the singularity (Big Bang) is isotropic [1, 7, 8].

By chance, the symmetric initial conditions led to isotropy leaving the singularity and presenting itself as a very small possibility. That is why at very early times in the evolution of the universe some process must have happened that led to a rapid isotropy and expansion. In many cases, classical mechanics is not effective enough to give an explanation of isotropy with respect to the time [2]. One of the possible solutions to this problem occurs in the field of inflationary cosmological theory where the possibility of isotropy is analyzed taking into account the effective bonding of space, originating particles from the vacuum [3].

Consider Einstein's equations for the free gravitational field [1, 9]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k T_{\mu\nu}, \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, R is the curvature scalar, and $T_{\mu\nu}$ is the energy-impulse tensor of the substance.

2 Non-static solutions

In a space-time, the metric adopts the following structure:

$$ds^2 = a_0^2(t) dt^2 - a_i^2(t)(dx^i)^2, \tag{2.2}$$

where $a(t)$ is a non-negative and time-dependent function.

From the metric (2.2) when $T_{\mu\nu} = 0$, we have $R_{\mu\nu} = 0$. Solving the Einstein's equation in the vacuum (1.1), the nonzero components are (simplification of the equations obtained in [5, 6]):

$$\begin{gathered}
\frac{a_0'(t)\left(\frac{a_1'(t)}{a_1(t)} + \frac{a_2'(t)}{a_2(t)} + \frac{a_3'(t)}{a_3(t)}\right)}{a_0(t)} - \frac{a_1''(t)}{a_1(t)} - \frac{a_2''(t)}{a_2(t)} - \frac{a_3''(t)}{a_3(t)} = 0 \\
-\frac{a_0'(t)a_1'(t)}{a_0(t)} + \frac{a_2'(t)a_1'(t)}{a_2(t)} + \frac{a_3'(t)a_1'(t)}{a_3(t)} + a_1''(t) = 0 \\
-\frac{a_0'(t)a_2'(t)}{a_0(t)} + \frac{a_1'(t)a_2'(t)}{a_1(t)} + \frac{a_3'(t)a_2'(t)}{a_3(t)} + a_2''(t) = 0 \\
-\frac{a_0'(t)a_3'(t)}{a_0(t)} + \frac{a_1'(t)a_3'(t)}{a_1(t)} + \frac{a_2'(t)a_3'(t)}{a_2(t)} + a_3''(t) = 0
\end{gathered} \tag{2.3}$$

From the metric $ds^2 = c^2dt^2 - t^{p_i}(dx^i)^2$ and the solution of equations (2.3), we obtain the following conditions for the coefficients [9]:

$$\sum_{i=1}^{3} p_i = 1$$

$$\sum_{i=1}^{3} p_i^2 = 1.$$

If the coefficients $p_1 = p_2 = p_3 = 0$, then we obtain the Minkowski metric $ds^2 = c^2dt^2 - (dx^i)^2$.

The empty particular solution of equations (2.3) is a generalization of the Kasner metric [4, 9] according to the relationship:

$$a_\alpha(t) = t^{p_\alpha} \tag{2.4}$$

$$ds^2 = c^2t^{p_0}dt^2 - t^{p_i}(dx^i)^2 \tag{2.5}$$

with the following conditions for p_α from the solution of Einstein's equation (1.1) for the vacuum

$$\sum_{i=1}^{3} p_i = p_0 + 1 \text{ and } \sum_{i=1}^{3} p_i^2 = (p_0 + 1)^2 \tag{2.6}$$

To deduce the Minkowski metric from Kasner's generalization (2.5), the coefficients are set to zero $p_0 = p_1 = p_2 = p_3 = 0$, causing a contradiction in the conditions (2.6) derived from the solution of Einstein's equation (2.3). This leads to the deduction that in the early days of the origin of the universe, time gives rise to a Kasner's singularity that curves space, establishing that in this early epoch of the universe space-time was curved.

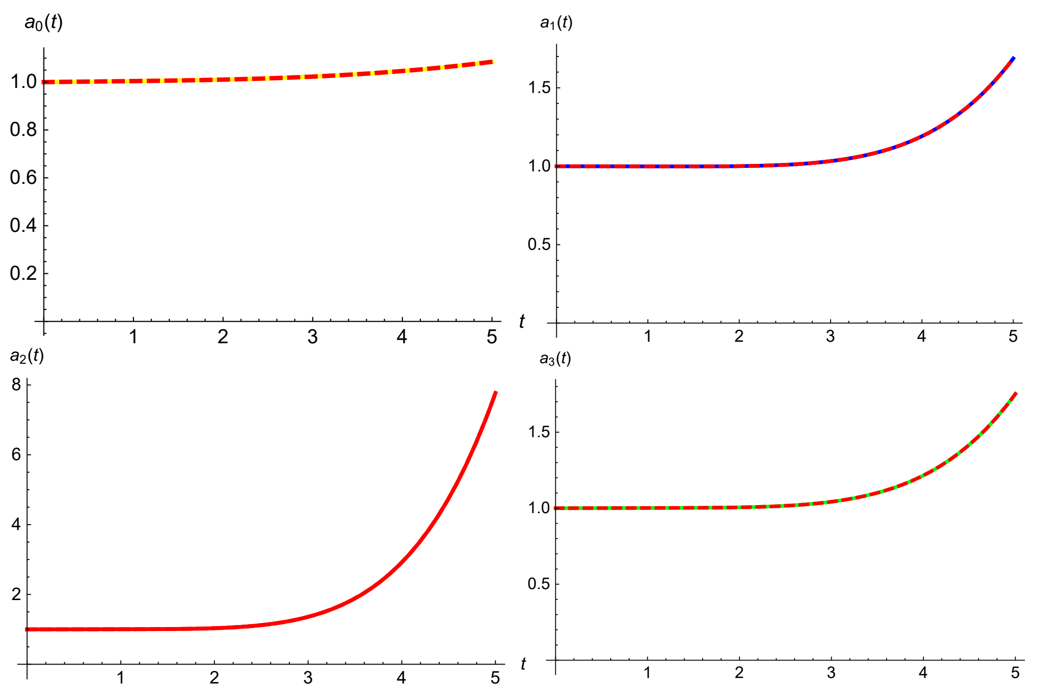
The system of equations (2.3) is a coupled system of differential equations that was solved numerically and then particular analytically approximated solutions were found, expressed as a polynomial of fifth degree in t:

$$a_\alpha(t) = c_{\alpha 0}t^0 + c_{\alpha 1}t + c_{\alpha 2}t^2 + c_{\alpha 3}t^3 + c_{\alpha 4}t^4 + c_{\alpha 5}t^5 = \sum_{i=0}^{5} c_{\alpha i}t^i, \tag{2.7}$$

where the coefficient matrix is given as $c_{\alpha i}$

$$c_{\alpha i} = \begin{array}{cccccc} 0.99999 & 0.00523 & 0.00023 & 0.00019 & 0.00001 & 2.15 \times 10^{-6} \\ 1 & -0.00129 & -0.00004 & 0.00011 & -0.00015 & 0.00008 \\ 0.99999 & 0.00493 & -0.00072 & 0.00210 & -0.00292 & 0.00161 \\ 0.99999 & 0.00178 & -0.00010 & 0.00030 & -0.00042 & 0.00023 \end{array} \tag{2.8}$$

The graphs of the numerical solution and the analytical solution given by (8) are shown below. The solid line corresponds to the numerical solution and the discrete one to the analytical solution.



The initial conditions when $t \to 0$ are given by:

$$\begin{aligned} g_{00} &= a_0^2(t) \to 1 \\ g_{11} &= a_1^2(t) \to 1 \\ g_{22} &= a_2^2(t) \to 1 \\ g_{33} &= a_3^2(t) \to 1 \end{aligned} \tag{2.9}$$

which leads to a flat space-time, with metric $ds^2 = c^2dt^2 - (dx^i)^2$.

We also observe that when the exponent of the function t increases the components of the metric tensor in (2.9) above tends to one leading to a flat space-time at infinity, since the coefficients $c_{\alpha 0} \to 1$ and the $c_{\alpha\infty} \to 0$, which makes us think that in the early and late periods of the universe, space-time was flat and in an intermediate age, such as the current age of approximately 14 billion years, it was curved. The equations given in (2.7) can be generalized as:

$$a_\alpha(t) = \sum_{i=0}^{\infty} c_{\alpha i} t^i. \tag{2.10}$$

3 Other non-static solutions

The system of coupled differential equations (2.3) is overdetermined which leads to $a_i(t) = f(a_3(t), a_3'(t))$, $i = 0, 1, 2$. The solution of the system (2.3) of coupled differential equations is given by:

$$\begin{aligned} a_0(t) &= \frac{a_3(t)^{\frac{c_1^2}{1+c_1}}}{a_3'(0)} a_3'(t) \\ a_1(t) &= a_3(t)^{-1+\frac{1}{1+c_1}} \\ a_1(t) &= a_3(t)^{c_1} \end{aligned} \tag{3.11}$$

where c_1 is an integer.

The solutions found for the metric (2.2) are the following families:
First, polynomial family:

$$\begin{aligned} a_0 &= \frac{((mt+1)^p)^{\frac{n^2}{n+1}+1}}{mt+1} \\ a_1 &= ((mt+1)^p)^{\frac{1}{n+1}-1} \\ a_2 &= ((mt+1)^p)^n \\ a_3 &= (mt+1)^p \end{aligned} \tag{3.12}$$

where m, n, p are integers.

Secondly, exponential family:

$$\begin{aligned} a_0 &= l^{1-p}(l+mt)^{p-1} e^{q((l+mt)^p - l^p)} \left(e^{q(l+mt)^p}\right)^{\frac{n^2}{n+1}} \\ a_1 &= \left(e^{q(l+mt)^p}\right)^{\frac{1}{n+1}-1} \\ a_2 &= \left(e^{q(l+mt)^p}\right)^n \\ a_3 &= e^{q(l+mt)^p} \end{aligned} \tag{3.13}$$

where p, q, l, m are integers.

Thirdly, harmonic family 1:

$$\begin{aligned} a_0 &= \frac{\sec(lp)(m+2nt)\cos(p(l+t(m+nt)))\sqrt{q\sin(p(l+t(m+nt)))}}{m} \\ a1 &= \frac{1}{\sqrt{q\sin(p(l+t(m+nt)))}} \\ a_2 &= q\sin(p(l+t(m+nt))) \\ a_3 &= q\sin(p(l+t(m+nt))) \end{aligned} \tag{3.14}$$

where m, n, l, p, q are integers.

Harmonic Family 2:

$$\begin{aligned} a_0 &= \frac{\csc(lp)(m+2nt)\sin(p(l+t(m+nt)))\sqrt{q\cos(p(l+t(m+nt)))}}{m} \\ a_1 &= \frac{1}{\sqrt{q\cos(p(l+t(m+nt)))}} \\ a_2 &= q\cos(p(l+t(m+nt))) \\ a_3 &= q\cos(p(l+t(m+nt))) \end{aligned} \tag{3.15}$$

where m, n, l, p, q are integers.

Elliptical family:

$$\begin{aligned} a_0 &= \mathrm{cn}(qt|m)\mathrm{dn}(qt|m)\sqrt{p\mathrm{sn}(qt|m)} \\ a_1 &= \frac{1}{\sqrt{p\mathrm{sn}(qt|m)}} \\ a_2 &= p\mathrm{sn}(qt|m) \\ a_3 &= p\mathrm{sn}(qt|m) \end{aligned}$$

where p, q are integers and $0 < m < 1$.

4 Conclusions

Einstein's equation for the gravitational field in vacuum was solved. A numerical solution and an analytical approximation were found in polynomial form, several solutions were found by classifying them into families. From the solutions, we observed that the gravitational (cosmological) time behaved like a fluid in a vacuum that generated the deformation of space-time. The origin of time in the universe suffered a singularity that led to the appearance of gravitational time, deforming the structure of space-time.

The Minkowski plane space-time appears in the solutions when gravitational time tends to zero or its coefficients to infinity. This methodology can be of great help to researchers of the general theory of relativity.

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