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Principally Self Injective Modules

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Abstract

A module A is called *principally self injective* if for each principal left ideal L of R and each homomorphism $f : L \longrightarrow A$ with $\ker f \supseteq \mathrm{ann}(a)$ for some $a \in A$, there is a homomorphism $R \longrightarrow A$ extending f. We extend certain properties of (quasi)-injective, principally-injective, absolutely (self) pure and finitely R-injective modules are extended to principally self injective modules. We describe Von Neumann regular and left pp-ring rings by this concept. For example, the homomorphic image of any principally-injective R-module is principally self injective if and only if R is left pp-ring.

1 Introduction

In what follows, R is an associative ring with identity, all modules are left unitary R-module, and all homomorphism are R-homomorphisms. Recall

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that a module M is said to be *injective* ([6], p. 60), if for each monomorphism $h : A \longrightarrow B$ of modules and every homomorphism $f : A \longrightarrow M$, there exists a homomorphism $g : B \longrightarrow M$ such that $f = h \circ g$. This is equivalent to *Baer condition*; that is, a left module A is injective if and only if for any left ideal I of R, any homomorphism $f : I \longrightarrow A$ can be extended to a homomorphism $h : R \longrightarrow A$. A module M is called *quasi-injective* [5] if for any submodule $N \subseteq M$, any homomorphism $f : N \longrightarrow M$ can be extended to an endomorphism of M. The smallest (quasi)-injective module that contains M as a submodule is called the (quasi)-injective *envelope* of M [5], ([6], p 75).
On the other hand, Fuchs [1] proved that an R-module A is quasi-injective if and only if for any left ideal I of R and any homomorphism $f : I \longrightarrow A$ with $\ker f \in \Omega(A)$, ($\Omega\,(A)$ = set of all left ideals I of R such that $I \supseteq \operatorname{ann}(a)$ for some $a \in A$), there exists a homomorphism $h : R \longrightarrow A$ that extends f. This is analogous to Baer condition for injective modules.
A module A is called *principally-injective* [9] (or p-injective for short) if each homomorphism from a principal left ideal of R to A can be extended to a homomorphism from $R \longrightarrow A$. It is clear that every injective module is principally injective and quasi-injective, but the converse is not true.
A submodule M of a module N is called a *pure* submodule [12] if for any commutative diagram with a finitely generated submodule L of a free module F

$$\begin{array}{ccc} L & \hookrightarrow & F \\ \downarrow & \swarrow & \downarrow \\ M & \hookrightarrow & N \end{array}$$

there is a homomorphism $F \longrightarrow M$ making the upper triangle commutative.
A module A is called *absolutely-pure* [7] if it is pure in each module containing it as a submodule; equivalently, if A is pure in some injective module or A is pure in its injective envelope [8].
In [2], the second author defined a module A to be *absolutely self pure* if for any finitely generated left ideal I of R and any homomorphism $f : I \longrightarrow A$ with $\ker f \in \bar{\Omega}(A)$, ($\bar{\Omega}(A)$ = A filter generated by $\Omega(A)$ in the lattice of left ideals of R), there is an extension $h : R \longrightarrow A$ of f. An R-module A is *finitely R-injective* [11] if any homomorphism from a finitely generated left ideal of $R \longrightarrow A$ can be extended to a homomorphism $R \longrightarrow A$. As a result, each absolutely pure R-module is finitely R-injective, and the concept of absolute self purity generalizes absolute purity, finite R-injectivity, and quasi-injectivity [2].

A module M is called *neat* in N, exactly when each diagram:

$$\begin{array}{ccc} L & \hookrightarrow & R \\ {\scriptstyle f}\downarrow & & \downarrow \\ M & \hookrightarrow & N \end{array}$$

with L being a maximal left ideal of R, there is a homomorphism $R \longrightarrow M$ extending f, and that a module M is called *absolutely self neat* [4] if for every a homomorphism $f : L \longrightarrow M$, where L is a maximal left ideal of R and $\ker f \in \bar{\Omega}(M)$, there is a homomorphism $h : R \longrightarrow M$ extending f.
In this paper, we generalize the concepts of absolute self-purity and principal injectivity in a way analogous to absolutely self-neatness. A module A is called *principally self injective* (or ps-injective for short) if for each principal left ideal L of R and each homomorphism $f : L \longrightarrow A$ with $ker f \supseteq \operatorname{ann}(a)$ for some $a \in A$, there is a homomorphism $R \longrightarrow A$ extending f. Using this concept, we characterize Von Neumann regular and left pp-ring rings.

2 Principally Self Pure Submodules

We now introduce the definition of a principally self-pure submodule.

Definition 2.1. *A submodule A of an R-module B is called a principally self-pure submodule of B (or $A \leq^{ps-p} B$ for short) if the following is true: For any principal left ideal L of R and any homomorphism $f : L \longrightarrow A$ with $ker\ f \supseteq \operatorname{ann}(a)$ for some $a \in A$, if there is a homomorphism $h : R \longrightarrow B$ making the following square commutative:*

$$\begin{array}{ccc} L & \hookrightarrow & R \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle h} \\ A & \hookrightarrow & B, \end{array}$$

then there is a homomorphism $g : R \longrightarrow A$ making the upper triangle commutative.

In the following proposition, we give some properties of principally self-pure submodules:

Proposition 2.2. *Let $A \subseteq B \subseteq C$ be R-modules.*
1. If $A \leq^{ps-p} B$ and $B \leq^{ps-p} C$, then $A \leq^{ps-p} C$.
2. If $A \leq^{ps-p} C$, then $A \leq^{ps-p} B$.

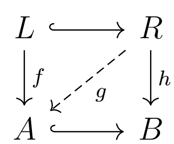
Proof. 1. Consider the following diagram:

$$\begin{array}{ccc} L & \hookrightarrow & R \\ \downarrow{\scriptstyle f} & & \downarrow \\ A & \hookrightarrow & C \end{array}$$

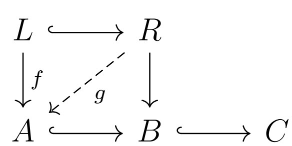
where L is a principal left ideal of R and f is a homomorphism, with $\ker f \supseteq \mathrm{ann}(a)$ for some $a \in A$ and $R \longrightarrow C$ is a homomorphism making the diagram commutative. Considering f as a homomorphism $L \longrightarrow B$, we get the commutative diagram:

$$\begin{array}{ccc} L & \hookrightarrow & R \\ \downarrow{\scriptstyle f} & & \downarrow \\ B & \hookrightarrow & C \end{array}$$

As $B \leq^{ps-p} C$, there is a homomorphism $h : R \longrightarrow B$ of f so that the following diagram is commutative:



and since $A \leq^{ps-p} B$, there is a $g : R \longrightarrow A$ extending f.
2. Consider the following commutative diagram:



Since $A \leq^{ps-p} C$, there is a homomorphism $g : R \longrightarrow A$ extending f. □

We now introduce the main definition.

Definition 2.3. *A module A is called principally self injective (or ps-injective for short) if for each principal left ideal L of R and each homomorphism $f : L \longrightarrow A$ with $\ker f \supseteq \mathrm{ann}(a)$ for some $a \in A$, there is a homomorphism $R \longrightarrow A$ extending f.*

Clearly, a ring R is ps-injective if and only if R is p-injective and R is absolutely self pure if and only if R is absolutely pure.
Now we can give some examples of ps-injective rings.

Examples 2.4. *(1) Any p-injective module is ps-injective.*
(2) Any absolutely self pure module and hence any quasi-injective and any absolutely pure is a ps-injective module.
(3) There is an 1-injective ring R (p-injective) [9] that is not 2-injective; i.e., there is a left ideal L of R generated by two elements and a homomorphism $L \longrightarrow R$ with no extension to $R \longrightarrow R$ (an example of a ps-injective module which is not absolutely self pure).
(4) The $\mathbb{Z}$-modules $\mathbb{Z}_n$ are ps-injective because they are all quasi-injective. However, they are not p-injective.

Proposition 2.5. *Any ps-pure submodule of a ps-injective module is again ps-injective.*

Proof. Consider the following diagram:

$$\begin{array}{ccc} L & \hookrightarrow & R \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ A & \hookrightarrow & B \end{array}$$

If B is ps-injective such that $\ker f \supseteq \operatorname{ann}(a)$ for some $a \in A$, then there exists a homomorphism $g : R \longrightarrow B$ making the diagram commutative. If $A \leq^{ps-p} B$, then there is a homomorphism $R \longrightarrow A$ making the upper triangle commutative and extending f. □

The following corollary follows from the previous proposition.

Corollary 2.6. *Any direct-summand of a ps-injective module is again ps-injective.*

In the following theorem, we characterize ps-injective in terms of ps-purity.

Theorem 2.7. *A module A is ps-injective if and only if it is principally self-pure in any module that contains it as a submodule.*

Proof. ($\Rightarrow$) Let B be any module containing A and let L be a principal left ideal L of R. Consider the following commutative diagram:

$$\begin{array}{ccc} L & \hookrightarrow & R \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ A & \hookrightarrow & B \end{array}$$

If A is ps-injective such that $\ker f \supseteq \operatorname{ann}(a)$ for some $a \in A$, then there exists a homomorphism $g : R \longrightarrow A$ extending f and making the upper triangle commutative. As a result, $A \leq^{ps-p} B$.
($\Leftarrow$) For any principal left ideal L of R and any homomorphism $f : L \longrightarrow A$ such that $\ker f \supseteq \operatorname{ann}(a)$ for some $a \in A$:

$$\begin{array}{ccc} L & \hookrightarrow & R \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ A & \hookrightarrow & Q(A) \end{array}$$

The homomorphism h exists by quasi-injectivity of $Q(A)$. Since A is ps-pure in $Q(A)$, there exists homomorphism $g : R \longrightarrow A$ making the upper triangle commutative and A ps-injective. $\square$

Remark 2.8. *By Proposition 2.2, if a module A is principally self-pure in some quasi-injective module, then it must be ps-pure in its quasi-injective envelope $Q(A)$.*
If A is a submodule of some other module B, then $Q(A) \leq^{ps-p} Q(B)$ and we must have $A \leq^{ps-p} B$.
From the proof of the above Theorem a module is ps-injective if and only if it is principally self-pure in some quasi-injective module if and only if it is principally self-pure in its quasi-injective envelope.

3 Characterization of Some Rings Using PS-Injectivity

Recall that a ring R is called *left principally projective ring* ([9], p 98) (or left pp-ring for short) if each of its principal left ideals is projective. We can characterize left pp-rings in the next Theorem following the Lemma.

Lemma 3.1. *[2] A left ideal L in a ring R is projective if and only if for any epimorphism $M \longrightarrow M'$ from an injective module M, any homomorphism $L \longrightarrow M$ with $\ker f \supseteq \operatorname{ann}(m)$ for some $m \in M$ can be lifted to a homomorphism $L \longrightarrow M$.*

Theorem 3.2. *For a ring R, the following statements are equivalent:*
(1) R is left pp-ring,
(2) The homomorphic image of any p-injective R-module is p-injective,
(3) The homomorphic image of any p-injective R-module is ps-injective.

Proof. (1) $\Rightarrow$ (2). Let M be p-injective and let $\beta : M \longrightarrow M'$ be any epimorphism. We will prove that M' is p-injective. Any principal left ideal L of R is projective, the inclusion map $\iota : L \longrightarrow R$ and $h : L \longrightarrow M'$. By assumption a ring R is left pp-ring. This implies that there is a homomorphism $f : R \longrightarrow M$ can be lifted to a homomorphism $g : R \longrightarrow M'$ and, as a result, there is g extending h. So M' is p-injective.
(2) $\Rightarrow$ (3). Obvious.
(3) $\Rightarrow$ (1). We will prove that any principal left ideal of R is projective. Consider the following diagram:

$$\begin{array}{ccc} R & \overset{\iota}{\hookleftarrow} & L \\ & & \downarrow h \\ M & \underset{\beta}{\longrightarrow} & M' \end{array}$$

with L a principal left ideal of R, the inclusion $\iota : L \longrightarrow R$ and $h : L \longrightarrow M'$ any homomorphism with $\ker h \supseteq \mathrm{ann}(m')$ of some $m' \in M'$ and $\beta : M \longrightarrow M'$ an epimorphism from an injective module M. By assumption, M' is ps-injective. This implies that there is a homomorphism $g : R \longrightarrow M'$ extending h. But R is projective. As a result, there is a homomorphism $f : R \longrightarrow M$ lifting g.

$$\begin{array}{ccc} R & \overset{\iota}{\hookleftarrow} & L \\ {\scriptstyle f}\,\downarrow & \overset{g}{\searrow} & \downarrow h \\ M & \underset{\beta}{\longrightarrow} & M' \end{array}$$

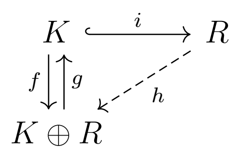
Consequently, $\beta f \iota = g\iota = h$ and L are projective by Lemma 3.1. □

A ring R is called *Von Neumann regular* [2] if any principal left ideal of R is a direct-summand. In a Von Neumann regular ring, all R-modules are ps-injective. The converse is also true.

Theorem 3.3. *For a ring R, the following statements are equivalent:*
(1) R is Von Neumann regular,
(2) Every R-module is ps-injective,
(3) Every R-module generated by two elements is ps-injective.

Proof. (1) $\Rightarrow$ (2). Every R-module is absolutely self pure modules [2] and hence is ps-injective.

(2) $\Rightarrow$ (3). Obvious.
(3) $\Rightarrow$ (1). Consider the following diagram:



where K is a principal left ideal of R. Hence $K \oplus R$ is generated by two elements. By Corollary 2.6 K is ps-injective. □

Corollary 3.4. *For a ring R, the following statements are equivalent:*
(1) R is Von Neumann regular,
(2) Every R-module is absolutely self pure,
(3) Every R-module generated by two elements is absolutely self pure.

Recall that a ring R is a *right SF-ring* [10] if every simple right R-module is flat.

Theorem 3.5. *Suppose that R is a commutative ring. Then R is Von Neumann regular if and only if any ps-injectve R-module is flat.*

Proof. ($\Rightarrow$) Trivial.
($\Leftarrow$) We will prove that R is Von Neumann regular. Every simple R-module is quasi-injective ([6], p 237). Hence every simple R-module is ps-injective and every ps-injective module is flat, by assumption. So R is an SF-ring. Therefore, R is Von Neumann regular ([3], Theorem 3.16). □

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