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Hyperidentities in Graph Varieties Generated by Some Linear Term Equations

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Abstract

Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type (2,0). We say that a graph G satisfies a term equation $s \approx t$ if the corresponding graph algebra $A(G)$ satisfies $s \approx t$. The set of all graphs satisfying a set of term equations is called a graph variety. A term equation $s \approx t$ which is satisfied by all graphs in a graph variety is called an identity of that graph variety. An identity $s \approx t$ of a graph variety $\mathcal{V}$ is a hyperidentity of $\mathcal{V}$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $A(G)$ of the appropriate arity, the resulting identities hold in $A(G)$ for all G in $\mathcal{V}$.

1 Introduction

An identity $s \approx t$ of terms s and t of any type τ is called a *hyperidentity* of an algebra $A(G)$ if whenever the operation symbols occurring in s and t are replaced by any term operations of $A(G)$ of the appropriate arity, the resulting identity holds in $A(G)$. Hyperidentities can be defined more precisely by

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using the concept of a hypersubstitution which was introduced by Denecke, Lau, Pöschel and Schweigert [7].

We fix a type $\tau = (n_i)_{i\in I},\ n_i > 0$ for all $i \in I$ and operation symbols $(f_i)_{i\in I}$, where f_i is n_i-ary. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X and let $Alg(\tau)$ be the class of all algebras of type τ. Then, a mapping

$$\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every n_i-ary operation symbol f_i an n_i-ary term will be called a *hypersubstitution* of type τ (for short, a hypersubstitution). We denote the extension of the hypersubstitution σ by a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X).$$

The term $\hat{\sigma}[t]$ is defined inductively by

(i) $\hat{\sigma}[x] = x$ for any variable x in the alphabet X, and

(ii) $\hat{\sigma}[f_i(t_1, ..., t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$.

Here $\sigma(f_i)^{W_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra with the universe $W_\tau(X)$.

Shallon [2] invented graph algebras (see also [4]), to obtain examples of nonfinitely based finite algebras. To recall this concept, let $G = (V, E)$ be a (directed) graph with the vertex set V and the set of edges $E \subseteq V \times V$. Define the *graph algebra* $A(G)$ corresponding to G with the underlying set $V \cup \{\infty\}$, where ∞ is a symbol outside V and with two basic operations; namely, a nullary operation pointing to ∞ and a binary one, denoted by juxtaposition, such that for $u, v \in V \cup \{\infty\}$

$$uv = \begin{cases} u, & \text{if} \quad (u, v) \in E, \\ \infty, & \text{otherwise.} \end{cases}$$

Pöschel and Wessel [9] investigated graph varieties for finite undirected graphs in order to get graph theoretic results (structure theorems) from universal algebra via graph algebras. In [11], these investigations were extended to arbitrary (finite) directed graphs where the authors ask for a graph theoretic characterization of graph varieties; i.e., of classes of graphs which can be defined by identities for their corresponding graph algebras. The answer is a theorem of **Birkhoff-type** which uses graph theoretic closure operations. *A class of finite directed graphs is equational* (i.e., a graph variety) *if and only if it is closed with respect to finite restricted pointed subproducts and isomorphic copies.*

Let $s \approx t$ be a term equation. In [13], [14], [15], [16], Poomsa-ard et al. characterized the graph variety $\mathcal{V} = Mod_g(\{s \approx t\})$ in various kind of terms s and t. Moreover, they characterized identities and hyperidentities in these graph varieties. In [1], Manyuen, Jampachon and Poomsa-ard characterizedthe graph variety generated by some linear term equations.

In this paper, we characterize graph varieties generated by other linear term equations. In addition, we characterize identities and hyperidentities in these graph varieties.

2 Terms, Identities and Graph varieties.

We start with the basic definitions, some propositions and some theorems which will be needed in later sections. Throughout this paper, by a graph we mean a finite directed graph without multiple edges.

Definition 2.1. *Let $X = \{x_1, x_2, x_3 \ldots\}$ be a countable set of variables. We define terms over X in the language of graph algebras by the following recursion:*
(i) Every $x \in X$ is a term.
(ii) ∞ is a term.
(iii) If t_1 and t_2 are terms, then t_1t_2 is a term.
The set of all terms over X is denoted by $W_\tau(X)$.

Definition 2.2. *Let $G = (V, E)$ be a graph. Let $h : X \to V \cup \{\infty\}$ be a map called an assignment. Extend h to a map $\bar{h} : W_\tau(X) \to V \cup \{\infty\}$ by the rule $\bar{h}(t) = h(t)$ if $t = x \in X$, $\bar{h}(t) = \bar{h}(t_1)\bar{h}(t_2)$ if $t = t_1t_2$, where the product take in $A(G)$. Then $\bar{h}$ is called the valuation of the term t in the graph G respect to the assignment h.*

Definition 2.3. *A term equation is an ordered pair (s, t) of terms $s, t \in W_\tau(X)$, usually written as $s \approx t$. Let $A(G)$ be a graph algebra corresponding to $G = (V, E)$. We say that $A(G)$ satisfies $s \approx t$, and we write $A(G) \models s \approx t$, if $\bar{h}(s) = \bar{h}(t)$ for every assignment $h : X \to V \cup \{\infty\}$. In this case, we also say that G satisfies $s \approx t$ and we write $G \models s \approx t$.*

The above notation extends to an arbitrary class $\mathcal{G}$ of graphs and to any set Σ of terms equations as follows:
$G \models \Sigma$ if $G \models s \approx t$ for all $s \approx t \in \Sigma$,
$\mathcal{G} \models s \approx t$ if $G \models s \approx t$ for all $G \in \mathcal{G}$,
$\mathcal{G} \models \Sigma$ if $G \models \Sigma$ for all $G \in \mathcal{G}$,

$Id\mathcal{G} = \{s \approx t | s, t \in W_\tau(X), \mathcal{G} \models s \approx t\}$,
$Mod_g\Sigma = \{G | G$ is a graph and $G \models \Sigma\}$,
$V_g(\mathcal{G}) = Mod_g Id\mathcal{G}$.
$V_g(\mathcal{G})$ is called the *graph variety generated by* $\mathcal{G}$ and $\mathcal{G}$ is called the *graph variety* if $V_g(\mathcal{G}) = \mathcal{G}$. $\mathcal{G}$ is called *equational* if there exists a set Σ of term equations such that $\mathcal{G} = Mod_g\Sigma$. Obviously, $V_g(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G}$ is an equational class.

Definition 2.4. *The leftmost [rightmost] variable occurring in a term t is denoted by $L(t)$ [$R(t)$]. We say that a term t is trivial if ∞ occurs in t.*

To a nontrivial term t, we associate a directed graph $G = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by
(i) $E(t) = \emptyset$ if $t = x$ for some $x \in X$,
(ii) $E(t) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$, where t_1, t_2 are terms.
We associate the empty graph $\emptyset$ to every trivial term.

In [3], Kiss, Pöschel and Pöhle characterized identities in the class of all graph algebras and the result provided a useful tool for checking whether a graph satisfies a term equation as the following propositions show:

Proposition 2.5. *Let $s \approx t$ be a term equation and let $\mathcal{G}$ be the class of all graphs. Then $\mathcal{G} \models s \approx t$ if and only if s and t are trivial terms or $G(s) = G(t)$ and $L(s) = L(t)$.*

Proposition 2.6. *Let $G = (V, E)$ be a graph and let $h : X \to V \cup \{\infty\}$ be an evaluation of variables. Consider the canonical extension $\bar{h}$ of h to the set of all terms. Then the following hold:*
If t is a trivial term, then $\bar{h}(t) = \infty$. Otherwise, if $h : G(t) \to G$ is a homomorphism of graphs, then $\bar{h}(t) = \bar{h}(L(t))$, and if h is not a homomorphism of graphs, then $\bar{h}(t) = \infty$.

In [9], Pöschel and Wessel gave another result that provides a useful tool for checking whether a graph satisfies a term equation as the following proposition shows:

Proposition 2.7. *Let s and t be nontrivial terms such that $V(s) = V(t)$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if G has the following property:*
a mapping $h : V(s) \to V$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G.

In [12], Changphas, Pibaljommee and Deneke gave the definition of a linear term:

Definition 2.8. *A linear term over the alphabet*

$$X = \{x_1, x_2, x_3, \ldots\}$$

is defined inductively as follows:

(i) *every variable* $x_i, i = 1, 2, 3, \ldots$ *and* ∞ *are linear terms,*

(ii) *if* t_1 *and* t_2 *are linear terms and* $V(t_1) \cap V(t_2) = \emptyset$*, then* $t_1 t_2$ *is a linear term.*

$W_\tau^{lin}(X)$ *is the set of all linear terms which can be obtained from* (i) *and* (ii) *in finitely many steps.*

Definition 2.9. *Let* G *be a graph and* $a, b \in V(G)$*. Then the length of longest walk in* G *is denoted by* $\mathcal{L}_w(G)$ *and the length of the longest walk from* a *to* b *in* G *is denoted by* $\mathcal{L}_a^b(G)$*.*

In [1], Manyuen, Jampachon and Poomsa-ard proved the following:

Proposition 2.10. *Let* G *be a graph. Then* G *is a graph corresponding to some terms if and only if* G *is a rooted graph.*

Theorem 2.11. *Let* G *be a graph. Then* G *corresponds to a linear term if and only if*

(i) *the underlying graph of* G *is a tree,*

(ii) *if* $(y_1, y_2) \in E(G)$*, then* $(y_2, y_1) \notin E(G)$*.*

Theorem 2.12. *Let* G *be a graph. Let* s *and* t *be linear terms in which* $L(s) \neq L(t)$ *or* $|V(s)| \neq |V(t)|$*. Then* $G \models s \approx t$ *if an only if* $\mathcal{L}_w(G) < min\{\mathcal{L}_w(G(s)),$
$\mathcal{L}_w(G(t))\}$*.*

Corollary 2.13. *Let* G *be a graph. Let* s *and* t *be linear terms. If* $\mathcal{L}_w(G) < min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\}$*, then* $G \models s \approx t$*.*

Lemma 2.14. *Let* G *be a graph. Let* s *and* t *be linear terms with* $G(s) \neq G(t)$*,* $L(s) = L(t)$*,* $Vs) = V(t)$ *and* $|V(s)| > 1$*. If* $\mathcal{L}_w(G(s)) = 1$ *or* $\mathcal{L}_w(G(t)) = 1$*, then* $G \models s \approx t$ *if and only if each component* G' *of* G *such that* $|V(G')| > 1$ *satisfies the following conditions:*

(i) *For* $b \in V(G')$*,* $(b, b) \in E(G)$ *if and only if there exists* $(a, b) \in E(G')$ *for some* $a \in V(G'), a \neq b$*,* *(ii) for* $a, b, c \in V(G')$ *if* $(a, b) \in E(G')$ *and* $(b, c) \in E(G')$*, then* $(a, c) \in E(G')$*,*

(iii) for $a, b, c \in V(G')$ *if* $(a, b) \in E(G')$ *and* $(a, c) \in E(G')$*, then* $(b, c) \in E(G')$ *and* $(c, b) \in E(G')$*.*

Theorem 2.15. *Let* Σ_l *be the set of all linear term equations such that for every* $s \approx t \in \Sigma_l$*,* $L(s) = L(t)$ *and* $V(s) = V(t)$*. Let* $s' \approx t' \in \Sigma_l$*. If* $\mathcal{L}_w(G(s')) = 1$*,* $G(s') \neq G(t')$ *and* $|V(s')| > 2$*, then* $Mod_g(\Sigma_l) = Mod_g(\{s' \approx t'\})$*.*

Using the methods of proof of Theorems 2.12 and 2.15, we can prove the following theorems:

Theorem 2.16. *Let* $\Sigma_{l'}$ *be the set of all linear term equations such that for every* $s' \approx t' \in \Sigma_{l'}$*,* $L(s') \neq L(t')$ *or* $V(s') \neq V(t')$*. Let* $\Sigma' \subseteq \Sigma_{l'}$ *and let* G *be a graph. Then* $G \in Mod_g(\Sigma') = \mathcal{K}_{\Sigma'}$ *if and only if* $\mathcal{L}_w(G) < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$*,* $s' \approx t' \in \Sigma'$*.*

Proof. Suppose that $G \in Mod_g(\Sigma')$. We have $G \in Mod_g(\{s' \approx t'\})$, for all $s' \approx t' \in \Sigma'$. By Theorem 2.12, we get $\mathcal{L}_w(G) < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$. Conversely, suppose that $\mathcal{L}_w(G) < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$. By Theorem 2.12, we get $G \models s' \approx t'$ for all $s' \approx t' \in \Sigma'$. We have $G \in Mod_g(\{s' \approx t'\})$ for all $s' \approx t' \in \Sigma'$. Hence $G \in Mod_g(\Sigma') = \mathcal{K}_{\Sigma'}$. □

Theorem 2.17. *Let* Σ_l *be the set of all linear term equations such that for every* $s \approx t \in \Sigma_l$*,* $L(s) = L(t)$ *and* $V(s) = V(t)$*. Let* $\Sigma \subseteq \Sigma_l$ *and* $s' \approx t' \in \Sigma$*. If* $\mathcal{L}_w(G(s')) = 1$*,* $G(s') \neq G(t')$*, then* $\mathcal{K}_\Sigma = Mod_g(\Sigma) = Mod_g(\{s' \approx t'\})$*.*

Proof. Since s' and t' are linear terms with $\mathcal{L}_w(G(s')) = 1$ and $G(s') \neq G(t')$, we have $|V(s')| > 2$. Since $s' \approx t' \in \Sigma$, we have $Mod_g(\Sigma) \subseteq Mod_g(\{s' \approx t'\})$. To show that $Mod_g(\{s' \approx t'\}) \subseteq Mod_g(\Sigma)$, let $G = (V, E)$ be a graph. Suppose that $G \in Mod_g(\{s' \approx t'\})$ (i.e., $G \models s' \approx t'$) and $s \approx t \in \Sigma$. We want to show that $G \models s \approx t$. If $G(s) = G(t)$, then by Proposition 2.5 we get $G \models s \approx t$. Suppose that $G(s) \neq G(t)$. Then we get $|V(s)| > 2$. If $\mathcal{L}_w(G(s)) = 1$ or $\mathcal{L}_w(G(t)) = 1$, then by Lemma 2.14, we have $G \models s \approx t$. Assume that $\mathcal{L}_w(G(s)) \neq 1$ and $\mathcal{L}_w(G(t)) \neq 1$. Since $\mathcal{L}_w(G(s')) = 1$ and $|V(s')| > 1$. Then, by Lemma 2.14, we have $G \models s' \approx s$ and $G \models s' \approx t$. Let $h : V(s) \to V \cup \{\infty\}$. Suppose that h is a homomorphism from $G(s)$ into G. Since $G \models s' \approx s$, h is a homomorphism from $G(s')$ into G. Since $G \models s' \approx t$, h is a homomorphism from $G(t)$ into G. Similarly, if h is a homomorphism from $G(t)$ into G, then we can prove that h is a homomorphism from $G(s)$ into G. By Proposition 2.7, $G \models s \approx t$. Hence, $G \in Mod_g(\Sigma)$. Therefore, $\mathcal{K}_\Sigma = Mod_g(\Sigma) = Mod_g(\{s' \approx t'\})$. □

Theorem 2.18. *Let G be a graph, Σ_l be the set of all linear term equations such that for every $s \approx t \in \Sigma_l$, $|V(s)| = |V(t)|$, $L(s) = L(t)$ and $\Sigma_{l'}$ be the set of all linear term equations such that for every $s \approx t \in \Sigma_{l'}$, $|V(s)| \neq |V(t)|$ or $L(s) \neq L(t)$. Let $\Sigma \subseteq \Sigma_l$, $\Sigma' \subseteq \Sigma_{l'}$ and $s' \approx t' \in \Sigma$. If $\mathcal{L}_w(G(s')) = 1$ and $G(s') \neq G(t')$, then $\mathcal{K}_{\Sigma \cup \Sigma'} = Mod_g(\Sigma \cup \Sigma') = \{G \in Mod_g(\{s' \approx t'\} | \mathcal{L}_w(G) < min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\}$ for all $s \approx t \in \Sigma'\}$.*

Proof. This is immediate from Theorem 2.12, Lemma 2.14 and Theorem 2.16. □

Next, we will characterize identities of each of the graph varieties generated by linear term equations. Clearly, by Proposition 2.5, if both of s and t are trivial or $G(s) = G(t)$ and $L(s) = L(t)$, then $s \approx t$ is an identity of all graph varieties. So, we consider only the case s and t are nontrivial and $G(s) \neq G(t)$ or $L(s) \neq L(t)$.

3 Identities in graph varieties generated by linear term equations.

Let Σ_l be the set of all linear term equations such that for every $s \approx t \in \Sigma_l$, $|V(s)| = |V(t)|$ and $L(s) = L(t)$. Let Σ is a subset of Σ_l which contains $s \approx t$ such that $\mathcal{L}_w(G(s)) = 1$ and $G(s) \neq G(t)$. Then, by Lemma 2.14 and Theorem 2.16, we can characterize all identities in $Mod_g(\Sigma)$ as follows:

Theorem 3.1. *Let s and t be nontrivial terms. Then $s \approx t$ is an identity in $Mod_g(\Sigma)$ if and only if the following conditions hold:*

(i) $|V(s)| = |V(t)|$ and $L(s) = L(t)$,

(ii) for any $x \in V(s)$, there exists $y \in V(s)$ such that $(y, x) \in E(s)$ if and only if there exists $z \in V(s)$ such that $(z, x) \in E(t)$,

(iii) for any $x, y \in V(s), x \neq y$, there exists a path from x to y in $G(s)$ or there exists $z \in V(s)$ such that there exist a path from z to x and a path from z to y in $G(s)$ if and only if there exists a path from x to y in $G(t)$ or there exists $w \in V(s)$ such that there exist a path from w to x and a path from w to y in $G(t)$.

Proof. $(\to)$ Suppose that $s \approx t$ is an identity of $Mod_g(\Sigma)$.

(i) Suppose that $|V(s)| \neq |V(t)|$. We can assume that $y \in V(s)$ but $y \notin V(t)$. Let $G = (V, E)$ be a complete graph. By Lemma 2.14, we get $G \in Mod_g(\Sigma)$. Let $h : V(s) \cup V(t) \to V \cup \{\infty\}$ be a function such that $h(x) \in V$ for all $x \in V(s) \cup V(t), x \neq y$ and $h(y) = \infty$. We see that h is a homomorphism from $G(t)$ into G but h is not a homomorphism from $G(s)$ into G. By Proposition 2.6, we get $\bar{h}(s) = \infty \neq h(L(t)) = \bar{h}(t)$. Hence, $G \nvDash s \approx t$. Let $G = (V, E)$ be a complete graph such that $V = V(s)$. Let $h : V(s) \to V$ be an identity map. We have $\bar{h}(s) = L(s) = L(t) = \bar{h}(t)$.

(ii) Let $x \in V(s)$ and suppose that there exists $y \in V(s)$ such that $(y, x) \in E(s)$ but there exists no $z \in V(s)$ such that $(z, x) \in E(t)$. We see that $x = L(t)$ because if $x \neq L(t)$, then there exists a path from $L(t)$ to x. Let $G = (V, E)$ be a graph such that $V = \{a, b\}$ and $E = \{(a, b), (b, b)\}$. By Lemma 2.14, we have $G \in Mod_g(\Sigma)$. Let $h : V(s) \to V$ such that $h(x) = h(L(s)) = a$ and $h(y) = b$ for all $y \in V(s), y \neq x$. We get h is a homomorphism from $G(t)$ into G but h is not a homomorphism from $G(s)$ into G). Hence $\bar{h}(t) = h(L(t)) = a \neq \infty = \bar{h}(s)$. So, $G \nvDash s \approx t$.

(iii) Suppose that there exist $x, y \in V(s), x \neq y$ and there exists a path from x to y in $G(s)$ or there exists $z \in V(s)$ such that there exist a path from z to x and a path from z to y in $G(s)$ but there exists no path from x to y and for all $w \in V(s)$ there exist no path from w to x and a path from w to y. We see that $y = L(s)$, because if $y \neq L(s)$, then there exists a path from $L(s)$ to x and a path from $L(s)$ to y. Let $G = (V, E)$ be a graph such that $V = \{a, b\}$ and $E = \{(a, b), (b, b)\}$. By Lemma 2.14, we have $G \in Mod_g(\Sigma)$. Let $h : V(s) \to V$ such that $h(y) = h(L(s)) = a$ and $h(u) = b$ for all $u \in V(s), y \neq u$. We get h is a homomorphism from $G(t)$ into G but h is not a homomorphism from $G(s)$ into G. Hence $\bar{h}(t) = h(L(t)) = a \neq \infty = \bar{h}(s)$. So, $G \nvDash s \approx t$.

$(\leftarrow)$ Suppose that s and t are nontrivial terms satisfying (i), (ii) and (iii). Let $G \in Mod_g(\Sigma)$ and let $h : V(s) \to V \cup \{\infty\}$ be a function. Suppose that h is a homomorphism from $G(s)$ into G and let $(x, y) \in E(t)$. If $x = y$ (i.e., $(x, x) \in Et)$), then by (ii) there exists $z \in V(s)$ such that there exists a path $z, z_1, z_2, \ldots, x$ from z to x in $G(s)$. By assumption, we get $h(z), h(z_1), h(z_2), \ldots, h(x)$ a path from $h(z)$ to $h(x)$ in G. By Lemma 2.14 (ii), we have $(h(x), h(x)) \in E(G)$. Suppose that $x \neq y$. By (iii), there exists a path from x to y or there exists $z \in V(s)$ such that there exist a path from z to x and a path from z to y. Then, by Lemma 2.14 (ii), (iii), we get $(h(x), h(y)) \in E(G)$. Hence, h is a homomorphism from $G(t)$ into G. Similarly, if h is a homomorphism from $G(t)$ into G, we can prove that h is a homomorphism from $G(s)$ into G. By Proposition 2.5, we get $G \models s \approx t$. $\square$

By Theorem 2.12 and Corollary 2.13, we have the following theorem:

Theorem 3.2. *Let $\Sigma_{l'}$ be the set of all linear term equations $s' \approx t'$ such that $|V(s')| \neq |V(t')|$ or $L(s') \neq L(t')$ and $\Sigma' \subseteq \Sigma_{l'}$. Let s and t be non trivial terms. Then, $s \approx t$ is an identity of $Mod_g(\Sigma')$ if and only if $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} \geq min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$ or if $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma$, then $G(s) = G(t)$ and $L(s) = L(t)$.*

Proof. ($\rightarrow$) Suppose that $s \approx t$ is an identity of $Mod_g(\Sigma')$. Clearly, $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} \geq min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$ or $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$. We want to prove that if $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$, then $G(s) = G(t)$ and $L(s) = L(t)$.

Assume that $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$ and $G(s) \neq G(t)$. Suppose that $\mathcal{L}_w(G(s)) < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$. By Theorem 2.12, we have $G(s) \in Mod_g(\Sigma')$. Let $G = (V, E)$ be a graph such that $V = V(s)$, $E = E(s)$ and let $h : V(s) \cup V(t) \rightarrow V \cup \{\infty\}$ such that $h(x) = x$ if $x \in V(s)$ and $h(y) = \infty$ if $y \notin V(s)$. We see that h is a homomorphism from $G(s)$ into G but it is not a homomorphism from $G(t)$ into G. We get $\bar{h}(s) = L(s) \neq \infty = \bar{h}(t)$ which contradicts $G \models s \approx t$. Hence, $G(s) = G(t)$. Let $h : V(s) \rightarrow V$ be an identity map. We have h is a homomorphism from $G(s)$ into G and h is a homomorphism from $G(t)$ into G. We get $L(s) = \bar{h}(s) = \bar{h}(t) = L(t)$.

($\leftarrow$) Let G be a graph and let $G \in Mod_g(\Sigma')$. Suppose that $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} \geq min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$. By Theorem 2.12, we get $\mathcal{L}_w(G) < min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(s))\}$. By Corollary 2.13, we have $G \models s \approx t$. Suppose that $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$. Then, we get $G(s) = G(t)$ and $L(s) = L(t)$. By Proposition 2.5, we have $G \models s \approx t$. $\square$

By Theorem 2.16 and Corollary 2.13, we have the following theorem:

Theorem 3.3. *Let G be a graph, Σ_l be the set of all linear term equations such that for every $s \approx t \in \Sigma_l$, $|V(s)| = |V(t)|$, $L(s) = L(t)$ and $\Sigma_{l'}$ be the set of all linear term equations such that for every $s \approx t \in \Sigma_{l'}$, $|V(s)| \neq |V(t)|$ or $L(s) \neq L(t)$. Let $\Sigma \subseteq \Sigma_l$, $\Sigma' \subseteq \Sigma_{l'}$ and Σ contain the linear term equation $s' \approx t'$ such that $\mathcal{L}_w(G(s')) = 1$ and $G(s') \neq G(t')$. Then, for any non trivial terms s and t, the term equation $s \approx t$ is an identity of $Mod_g(\Sigma \cup \Sigma')$ if and* *only if* $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} \geq min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma^{'}$ *or if* $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G$ $(t'))\}$ $s' \approx t' \in \Sigma^{'}$, *then*

(i) $|V(s)| = |V(t)|$ *and* $L(s) = L(t)$,

(ii) for any $x \in V(s)$, *there exists* $y \in V(s)$ *such that* $(y, x) \in E(s)$ *if and only if there exists* $z \in V(s)$ *such that* $(z, x) \in E(t)$,

(iii) for any $x, y \in V(s), x \neq y$, *there exists a path from* x *to* y *in* $G(s)$ *or there exists* $z \in V(s)$ *such that there exist a path from* z *to* x *and a path from* z *to* y *in* $G(s)$ *if and only if there exists a path from* x *to* y *in* $G(t)$ *or there exists* $w \in V(s)$ *such that there exist a path from* w *to* x *and a path from* w *to* y *in* $G(t)$.

Proof. $(\rightarrow)$ Suppose that $s \approx t$ is an identity of $Mod_g(\Sigma \cup \Sigma^{'})$. Clearly, $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} \geq min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma^{'}$ or $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$ $s' \approx t' \in \Sigma^{'}$. We want to prove that if $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$ $s' \approx t' \in \Sigma^{'}$, then (i), (ii) and (iii) hold.

(i) Suppose that $V(s) \neq V(t)$. If $1 = min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma^{'}$, then let $s = x$ and $t = y, x \neq y$, $G = (V, E)$ such that $V = \{a, b\}$, $E = \emptyset$. By Lemma 2.14, we see that $G \in Mod_g(\Sigma \cup \Sigma^{'})$. Let $h : V(s) \cup V(t) \rightarrow V \cup \{\infty\}$ such that $h(x) = a$ and $h(y) = b$. We get $\bar{h}(s) = a \neq b = \bar{h}(t)$, which contradicts $G \models s \approx t$. Hence, $V(s) = V(t) = \{x\}$. So, $L(s) = x = L(t)$. Suppose that $1 < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma^{'}$ and $V(s) \neq V(t)$. We can assume that there exists $y \in V(s)$ but $y \notin V(t)$. Let $G = (V, E)$, where $V = \{a\}, E = \{(a, a)\}$. We have $G \in Mod_g(\Sigma \cup \Sigma^{'})$. Let $h : V(s) \cup V(t) \rightarrow V \cup \{\infty\}$ such that $h(x) = a$ for all $x \in V(s) \cup V(t), x \neq y$ and $h(y) = \infty$. We get $\bar{h}(s) = a \neq \infty = \bar{h}(t)$ which contradicts $G \models s \approx t$. Hence $V(s) = V(t)$. Suppose that $L(s) \neq L(t)$. Let $L(s) = x, L(t) = y, x \neq y$ and let $G = (V, E)$ such that $V = \{a, b\}$ and $E = \{(a, a)\}$. We see that $G \in Mod_g(\Sigma \cup \Sigma^{'})$. Let $h : V(s) \rightarrow V \cup \{\infty\}$ such that $h(z) = a$ for all $z \in V(s), z \neq y$, $h(y) = b$. We get $\bar{h}(s) = a \neq \infty = \bar{h}(t)$ which contradicts $G \models s \approx t$. Hence $L(s) = L(t)$.

(ii) Suppose that there exists $x \in V(s)$ and there exists $y \in V(s)$ such that $(y, x) \in E$ but that there exists no $z \in V(s)$ such that $(z, x) \in E(t)$. We see that $x = L(t)$. Assume that $n = min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma^{'}$. If $n = 1$ or $n = 2$, then the graphs in $Mod_g(\Sigma \cup \Sigma^{'})$ are the graphs which each component has only one vertex and terms s and t are x or xx. Let $t = x$, $s = xx$, $G = (V, E)$ such that $V = \{a\}, E = \emptyset$ and $h : V(s) \rightarrow V \cup \{\infty\}$ such that $h(x) = a$. We have $\bar{h}(s) = \infty \neq a = \bar{h}(t)$ which is contradiction to $G \models s \approx t$. Suppose that $n > 2$. We have $G = (V, E)$ with $V = \{a, b\}$ and

$E = \{(a,b),(b,b)\}$ belongs to $Mod_g(\Sigma\cup\Sigma^{'})$. Let $h : V(s) \to V \cup \{\infty\}$ such that $h(x) = a$ and $h(y) = b$ for all $y \in V(s), y \neq x$. We have $\bar{h}(s) = \infty \neq a = \bar{h}(t)$ which contradicts $G \models s \approx t$.

(iii) Suppose that there exist $x, y \in V(s)$ such that there exists a path from x to y in $G(s)$ or there exists $z \in V(s)$ such that there exist a path from z to x and a path from z to y in $G(s)$ but there is no path from x to y in $G(t)$ and there is no $w \in V(s)$ such that there is a path from w to x and a path from w to y in $G(t)$. We see that $y = L(t)$. Assume that $n = min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$. For $n = 1$ or $n = 2$, we have the graphs in $Mod_g(\Sigma\cup\Sigma^{'})$ are the graphs in which each component has only one vertex and terms s and t are x or xx. Let $t = x$, $s = xx$, $G = (V,E)$ such that $V = \{a\}, E = \emptyset$ and $h : V(s) \to V \cup \{\infty\}$ such that $h(x) = a$. We have $\bar{h}(s) = \infty \neq a = \bar{h}(t)$ which is contradiction to $G \models s \approx t$. Suppose that $n > 2$. We have $G = (V,E)$ with $V = \{a,b\}$ and $E = \{(a,b),(b,b)\}$ belongs to $Mod_g(\Sigma\cup\Sigma^{'})$. Let $h : V(s) \to V \cup \{\infty\}$ such that $h(y) = a$ and $h(x) = b$ for all $x \in V(s), y \neq x$. We have $\bar{h}(s) = \infty \neq a = \bar{h}(t)$ which contradicts $G \models s \approx t$.

$(\leftarrow)$ Let s and t be nontrivial terms, $G = (V,E)$ be a graph and $G \in Mod_g(\Sigma\cup\Sigma^{'})$. Suppose that $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} \geq min\{\mathcal{L}_w(G(s')), \mathcal{L}_w (G(t'))\}$, $s' \approx t' \in \Sigma^{'}$ or if $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w (G(t'))\}$ $s' \approx t' \in \Sigma^{'}$, then the conditions (i), (ii) and (iii) hold. Suppose that $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} \geq min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma$. By Theorem 2.12, we get $\mathcal{L}_w(G) < min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(s))\}$. Then, by Corollary 2.13, we have $G \models s \approx t$. Suppose that $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma$. Let $h : V(s) \to V \cup \{\infty\}$. Suppose that h is a homomorphism from $G(s)$ into G and let $(x,y) \in E(t)$. If $x = y$ (i.e., $(x,x) \in E(t)$), then by assumption there exists $z \in V(s)$ such that $(z,x) \in E(s)$. Hence, $(h(z), h(x)) \in E$. By the property of G, we get $(h(x), h(x)) \in E$. Suppose that $x \neq y$. By assumption, there exists a path from x to y in $G(s)$ or there exists $z \in V(s)$ such that there exist a path from z to x and a path from z to y in $G(s)$. Suppose that there exists a path from x to y in $G(s)$. Since h is a homomorphism, by the property of G, we get $(h(x), h(y)) \in E$. Suppose that there exists $z \in V(s)$ such that there exist a path from z to x and a path from z to y in $G(s)$. Since h is a homomorphism from $G(s)$ into G, there exists a path from $h(z)$ to $h(x)$ and a path from $h(z)$ to $h(y)$ in G. By the property of G, we get $(h(x), h(y)) \in E$. Hence, h is a homomorphism from $G(t)$ into G. Similarly, if h is a homomorphism from $G(t)$ into G, we can prove that it is a homomorphism from $G(s)$ into G, too. By Proposition 2.7, we get $G \models s \approx t$. □

4 Hyperidentities in graph varieties generated by linear term equations

Let $\mathcal{K}$ be any graph variety. Now, we precisely formulate the concept of a graph hypersubstitution for graph algebras.

Definition 4.1. *Let $X_2 = \{x_1, x_2\}$ and let f be the operation symbol corresponding to the binary operation of a graph algebra.*
A mapping $\sigma : \{f, \infty\} \to W_\tau(X_2)$ is called *graph hypersubstitution* if $\sigma(\infty) = \infty$ and $\sigma(f) = s \in W_\tau(X_2)$. The graph hypersubstitution with $\sigma(f) = s$ is denoted by σ_s.

Definition 4.2. An identity $s \approx t$ is a $\mathcal{K}$ *graph hyperidentity* iff for all graph hypersubstitutions σ, the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities in $\mathcal{K}$.

If we want to check that an identity $s \approx t$ is a hyperidentity in $\mathcal{K}$, we can restrict our consideration to a (small) subset of $Hyp\mathcal{G}$ - the set of all graph hypersubstitutions. In [5], the following relation between hypersubstitutions was defined:

Definition 4.3. Two graph hypersubstitutions σ_1, σ_2 are called $\mathcal{K}$-*equivalent* iff $\sigma_1(f) \approx \sigma_2(f)$ is an identity in $\mathcal{K}$. In this case we write $\sigma_1 \sim_{\mathcal{K}} \sigma_2$.

The following lemma was proven in [6].

Lemma 4.4. *If $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{K}$ and $\sigma_1 \sim_{\mathcal{K}} \sigma_2$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{K}$.*

As a result, it is enough to consider the quotient set $Hyp\mathcal{G}/\sim_{\mathcal{K}}$.

In [10], it was shown that any non-trivial term t over the class of graph algebras has a uniquely determined normal form term $NF(t)$ and there is an algorithm to construct the normal form term to a given term t. It is easy to show that $G(NF(t)) = G(t), L(NF(t)) = L(t)$.

Definition 4.5. *[8]* The graph hypersubstitution $\sigma_{NF(t)}$ is called a *normal form graph hypersubstitution*. Here, $NF(t)$ is the normal form of the binary term t.

Since for any binary term t the rooted graphs of t and $NF(t)$ are the same, we have $t \approx NF(t) \in Id\mathcal{K}$. Then, for any graph hypersubstitution σ_t with $\sigma_t(f) = t \in W_\tau(X_2)$, one obtains $\sigma_t \sim_{\mathcal{K}} \sigma_{NF(t)}$.

In [8], all rooted graphs with at most two vertices were considered. As a result, we form the corresponding binary terms and use the algorithm to construct normal form terms. The result is given in the following table:

normal form term	graph hypers	normal form term	graph hypers
x_1x_2	σ_0	x_1	σ_1
x_2	σ_2	x_1x_1	σ_3
x_2x_2	σ_4	x_2x_1	σ_5
$(x_1x_1)x_2$	σ_6	$(x_2x_1)x_2$	σ_7
$x_1(x_2x_2)$	σ_8	$x_2(x_1x_1)$	σ_9
$(x_1x_1)(x_2x_2)$	σ_{10}	$(x_2(x_1x_1))x_2$	σ_{11}
$x_1(x_2x_1)$	σ_{12}	$x_2(x_1x_2)$	σ_{13}
$(x_1x_1)(x_2x_1)$	σ_{14}	$(x_2(x_1x_2))x_2$	σ_{15}
$x_1((x_2x_1)x_2)$	σ_{16}	$x_2((x_1x_1)x_2)$	σ_{17}
$(x_1x_1)((x_2x_1)x_2)$	σ_{18}	$(x_2((x_1x_1)x_2))x_2$	σ_{19}

Clearly, if s and t are trivial terms, then $s \approx t$ is a hyperidentity in $\mathcal{K}$ if and only if $L(s) = L(t)$, $R(s) = R(t)$ and $s \approx t$ is a hyperidentity in $\mathcal{K}$ if s and t are variables. To find all hypeidentities in $\mathcal{K}$, we consider the case in which s and t are non-trivial terms and are not variables.

In [8], the following lemma was proven:

Lemma 4.6. *For each non-trivial term s which is not a variable, the following hold*

(i) $E(\hat{\sigma}_6[s]) = E(s) \cup \{(u,u)|(u,v) \in E(s)\}$,
(ii) $E(\hat{\sigma}_8[s]) = E(s) \cup \{(v,v)|(u,v) \in Es)\}$,
(iii) $E(\hat{\sigma}_{12}[s]) = E(s) \cup \{(v,u)|(u,v) \in E(s)\}$.

Similarly, we can prove that
$E(\hat{\sigma}_{10}[s]) = E(s) \cup \{(u,u),(v,v)|(u,v) \in E(s)\}$,
$E(\hat{\sigma}_{14}[s]) = E(s) \cup \{(u,u),(v,u)|(u,v) \in E(s)\}$,
$E(\hat{\sigma}_{16}[s]) = E(s) \cup \{(v,v),(v,u)|(u,v) \in E(s)\}$,
$E(\hat{\sigma}_{18}[s]) = E(s) \cup \{(u,u),(v,v),(v,u)|(u,v) \in E(s)\}$.

We define the product of two normal form graph hypersubstitutions in $\mathcal{K}$ as follows:

Definition 4.7. The *product* $\sigma_{1N} \circ_N \sigma_{2N}$ *of two normal form graph hypersubstitutions* is defined by:
$(\sigma_{1N} \circ_N \sigma_{2N})(f) = NF(\hat{\sigma}_{1N}[\sigma_{2N}(f)])$.

Let $\mathcal{K}_\Sigma = Mod_g(\Sigma)$. By Theorem 3.1, we have the following relations:

(i) $\sigma_O \sim_{\mathcal{K}_\Sigma} \sigma_8$ (ii) $\sigma_5 \sim_{\mathcal{K}_\Sigma} \sigma_9$ (iii) $\sigma_6 \sim_{\mathcal{K}_\Sigma} \sigma_{10}$ (iv) $\sigma_7 \sim_{\mathcal{K}_\Sigma} \sigma_{11}$

(v) $\sigma_{12} \sim_{\mathcal{K}_\Sigma} \sigma_{14} \sim_{\mathcal{K}_\Sigma} \sigma_{16} \sim_{\mathcal{K}_\Sigma} \sigma_{18}$ (vi) $\sigma_{13} \sim_{\mathcal{K}_\Sigma} \sigma_{15} \sim_{\mathcal{K}_\Sigma} \sigma_{17} \sim_{\mathcal{K}_\Sigma} \sigma_{19}$.

Let $M_{\mathcal{K}_\Sigma}$ be the set of all normal form graph hypersubstitutions in $M_{\mathcal{K}_\Sigma}$. Then,

$$M_{\mathcal{K}_\Sigma} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_{12}, \sigma_{13}\}.$$

The concept of a proper hypersubstitution of a class of algebras was introduced in [6].

Definition 4.8. A hypersubstitution σ is called *proper with respect to a class* $\mathcal{K}$ *of algebras* if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}$ for all $s \approx t \in Id\mathcal{K}$.

A graph hypersubstitution with the property that $\sigma(f)$ contains both variables x_1 and x_2 is called *regular*. It is easy to check that the set of all regular graph hypersubstitutions forms a groupoid M_{reg}.

We want to find all proper graph hypersubstitutions with respect to $M_{\mathcal{K}_\Sigma}$. We have,

Theorem 4.9. $\{\sigma_0, \sigma_6, \sigma_{12}\}$ *is the set of all proper graph hypersubstitutions with respect to* $\mathcal{K}_\Sigma$.

Proof. Clearly, σ_0 is a proper graph hypersubstitution. Consider σ_6 and σ_{12}. If $s \approx t \in Id\mathcal{K}_\Sigma$ and s and t are trivial terms, then $\hat{\sigma}_6[s]$, $\hat{\sigma}_{12}[s]$, $\hat{\sigma}_6[t]$ and $\hat{\sigma}_{12}[t]$ are also trivial terms and thus $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_\Sigma$ and $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{K}_\Sigma$.

If $s \approx t \in Id\mathcal{K}_\Sigma$ and s and t are variables, then $\hat{\sigma}_6[s]$, $\hat{\sigma}_{12}[s]$, $\hat{\sigma}_6[t]$ and $\hat{\sigma}_{12}[t]$ are also variables and thus $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_\Sigma$ and $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{K}_\Sigma$.

Now, assume that s and t are nontrivial terms, not variables and $s \approx t \in Id\mathcal{K}_\Sigma$. Then (i) and (ii) of Theorem 3.1 hold.

By Lemma 4.6, for σ_6, we have:
if $(x, y) \in E(s)$, then $(x, x) \in E(\hat{\sigma}_6[s])$. We want to show that there exists $z \in V(s)$ such that $(z, x) \in E(\hat{\sigma}_6[t])$. If $(x, y) \in E(t)$, then $(x, x) \in E(\hat{\sigma}_6[t])$. If $(x, y) \notin E(t)$, then, by Theorem 3.1(iii), there exists $z' \in V(s)$ such that there exists a path from z' to x; that is, there exists $z \in V(s)$ such that $(z, x) \in E(\hat{\sigma}_6[t])$. Since $(x, y) \in E(t)$, if $(x, x) \in E(\hat{\sigma}_6[t])$, then there exists $w \in V(s)$ such that $(w, x) \in E(\hat{\sigma}_6[s])$. Therefore, if $s \approx t \in Id\mathcal{K}_\Sigma$, then $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{K}_\Sigma$.

For σ_{12}, by Lemma 4.6, if $(x, y) \in E(s), x \neq y$, then $(y, x) \in E(\hat{\sigma}_{12}[s])$. We want to show that there exists a path from y to x in $G(\hat{\sigma}_{12}[t])$ or there exists $z \in V(s)$ such that there exists a path from z to x and a path from z

to y in $G(\hat{\sigma}_{12}[t])$. Since $(x,y) \in E(s), x \neq y$, by Theorem 3.1, there exists a path from x to y in $G(t)$ or there exists $w \in V(s)$ such that there exist a path from w to x and a path from w to y in $G(t)$. For both cases, there exists a path from y to x in $G(\hat{\sigma}_{12}[t])$ or there exists $z \in V(s)$ such that there exist a path from z to x and a path from z to y in $G(\hat{\sigma}_{12}[t])$. Since $(x',y') \in E(t), x' \neq y'$, if $(y',x') \in E(\hat{\sigma}_{12}[t])$, there exists a path from y' to x' in $G(\hat{\sigma}_{12}[s])$ or there exists $z' \in V(s)$ such that there exist a path from z' to x' and a path from z' to y' in $G(\hat{\sigma}_{12}[s])$. Hence, $\hat{\sigma}_{12}[s] \approx \hat{\sigma}_{12}[t] \in Id\mathcal{K}_\Sigma$.

For each $\sigma \notin \{\sigma_0, \sigma_6, \sigma_{12}\}$ we get an identity $s \approx t \in Id\mathcal{K}_\Sigma$ such that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{K}_\Sigma$. Clearly, if s and t are trivial terms with different leftmost and different rightmost, then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \notin Id\mathcal{K}_\Sigma$, $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \notin Id\mathcal{K}_\Sigma$ $\hat{\sigma}_3[s] \approx \hat{\sigma}_3[t] \notin Id\mathcal{K}_\Sigma$. and $\hat{\sigma}_4[s] \approx \hat{\sigma}_4[t] \notin Id\mathcal{K}_\Sigma$.

Now, let $s = (xy)(zx)$ and let $t = (x((zx)y))$. By Theorem 3.1, $s \approx t \in Id\mathcal{K}_\Sigma$. Since $L(\hat{\sigma}_5[s]) = L(\hat{\sigma}_7[s]) = L(\hat{\sigma}_{13}[s]) = x$ and $L(\hat{\sigma}_5[t]) = L(\hat{\sigma}_7[t]) = L(\hat{\sigma}_{13}[t]) = y$, again by Theorem 3.1, we get $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \notin Id\mathcal{K}_\Sigma$, $\hat{\sigma}_7[s] \approx t \notin Id\mathcal{K}_\Sigma$ and $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \notin Id\mathcal{K}_\Sigma$. □

By Theorem 3.1, we can characterize all hyperidentities in $\mathcal{K}_\Sigma$ as the following theorem shows:

Theorem 4.10. *Let s and t be nontrivial terms which are not variables. An identity $s \approx t$ in $\mathcal{K}_\Sigma$ is a hyperidentity in $\mathcal{K}_\Sigma$ if and only if $R(s) = R(t)$ and $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_\Sigma$.*

Proof. Since $s \approx t$ is a hyperidentity in $\mathcal{K}_\Sigma$, $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_\Sigma$ and $R(s) = \hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] = R(t) \in Id\mathcal{K}_\Sigma$. Hence, $R(s) = R(t)$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_\Sigma$ and that $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_\Sigma$ and $R(s) = R(t)$, too. Then (i), (ii) and (iii) of Theorem 3.1 hold. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions from $M_{\mathcal{K}_\Sigma}$.

If $\sigma \in \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, then by $L(s) = L(t)$ and $R(s) = R(t)$ we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_\Sigma$. By assumption, we have $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_\Sigma$.

For $\sigma \in \{\sigma_0, \sigma_6, \sigma_{12}\}$ by σ is proper, we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_\Sigma$. Since $\sigma_6 \circ_N \sigma_5 = \sigma_7$ and $\sigma_{12} \circ_N \sigma_5 = \sigma_{13}$. Then, since $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_\Sigma$ and σ_6, σ_{12} are proper, we get $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_\Sigma$ and $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_\Sigma$. Hence, $s \approx t$ is a hyperidentity in $\mathcal{K}_\Sigma$. □

Next, we want to show that if $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$, then σ is a proper graph hypersubstitutions with respect to $M_{\mathcal{K}_{\Sigma'}}$.

Theorem 4.11. *Let $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$. Then, σ is a proper graph hypersubstitution with respect to $\mathcal{K}_{\Sigma'}$.*

Proof. Clearly, σ_0 is a proper graph hypersubstitution. Consider $\sigma \in \{\sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$. If $s \approx t \in Id\mathcal{K}_{\Sigma'}$ and s, t are trivial terms, then $\hat{\sigma}[s]$ and $\hat{\sigma}[t]$ are also trivial terms and thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{\Sigma'}$. If $s \approx t \in Id\mathcal{K}_{\Sigma'}$ and s, t are variables, then $\hat{\sigma}[s]$ and $\hat{\sigma}[t]$ are also variables and thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{\Sigma'}$.

Now, assume that s and t are nontrivial terms, not variables and $s \approx t \in Id\mathcal{K}_{\Sigma'}$. Then, the conditions of Theorem 3.2 hold.

Suppose that $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} \geq min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$. Since $\mathcal{L}_w(G(\hat{\sigma}[s])) \geq \mathcal{L}_w(G(s))$ and $\mathcal{L}_w(G(\hat{\sigma}[t])) \geq \mathcal{L}_w(G(t))$, we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{\Sigma'}$.

If $min\{\mathcal{L}_w(G(s)), \mathcal{L}_w(G(t))\} < min\{\mathcal{L}_w(G(s')), \mathcal{L}_w(G(t'))\}$, $s' \approx t' \in \Sigma'$, then by assumption we get $G(s) = G(t)$ and $L(s) = L(t)$. Moreover, we have $G(\hat{\sigma}[s]) = G(\hat{\sigma}[t])$ and $L(\hat{\sigma}[s]) = L(\hat{\sigma}[t])$. Hence, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{\Sigma'}$. □

By Theorem 3.2, we can characterize all hyperidentities in $\mathcal{K}_{\Sigma'}$ as the following theorem shows:

Theorem 4.12. *An identity $s \approx t$ in $\mathcal{K}_{\Sigma'}$ where s and t are non-trivial terms and are not variables is a hyperidentity in $\mathcal{K}_{\Sigma'}$ if and only if $L(s) = L(t)$, $R(s) = R(t)$ and $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_{\Sigma'}$.*

Proof. Since $s \approx t$ is a hyperidentity in $\mathcal{K}_{\Sigma'}$, $L(s) = \hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] = L(t)$ and $R(s) = \hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] = R(t)$ are identities in $\mathcal{K}_{\Sigma'}$. We get $L(s) = L(t)$ and $R(s) = R(t)$. Moreover, we get $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_{\Sigma'}$. Conversely, assume that $s \approx t$ is an identity in $\mathcal{K}_{\Sigma'}$ and that $L(s) = L(t)$, $R(s) = R(t)$ and $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_{\Sigma'}$, too. Then the conditions of Theorem 3.2 hold. We have to prove that $s \approx t$ is closed under all graph hypersubstitutions.

If $\sigma \in \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, then by $L(s) = L(t)$ and $R(s) = R(t)$ we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{\Sigma'}$. By assumption, we have $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_{\Sigma'}$.

For $\sigma \in \{\sigma_0, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}, \sigma_{14}, \sigma_{16}, \sigma_{18}\}$ by σ is proper, we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}_{\Sigma'}$. Since $\sigma_6 \circ_N \sigma_5 = \sigma_7$, $\sigma_8 \circ_N \sigma_5 = \sigma_9$, $\sigma_{10} \circ_N \sigma_5 = \sigma_{11}$, $\sigma_{12} \circ_N \sigma_5 = \sigma_{13}$, $\sigma_{14} \circ_N \sigma_5 = \sigma_{15}$, $\sigma_{16} \circ_N \sigma_5 = \sigma_{17}$ and $\sigma_{18} \circ_N \sigma_5 = \sigma_{19}$. Then by $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_{\Sigma'}$ and σ_6, σ_8, σ_{10}, σ_{12}, σ_{14}, σ_{16}, σ_{18} are proper we get $\hat{\sigma}_7[s] \approx \hat{\sigma}_7[t] \in Id\mathcal{K}_{\Sigma'}$, $\hat{\sigma}_9[s] \approx \hat{\sigma}_9[t] \in Id\mathcal{K}_{\Sigma'}$, $\hat{\sigma}_{11}[s] \approx \hat{\sigma}_{11}[t] \in Id\mathcal{K}_{\Sigma'}$, $\hat{\sigma}_{13}[s] \approx \hat{\sigma}_{13}[t] \in Id\mathcal{K}_{\Sigma'}$, $\hat{\sigma}_{15}[s] \approx \hat{\sigma}_{15}[t] \in Id\mathcal{K}_{\Sigma'}$, $\hat{\sigma}_{17}[s] \approx \hat{\sigma}_{17}[t] \in Id\mathcal{K}_{\Sigma'}$ and $\hat{\sigma}_{19}[s] \approx \hat{\sigma}_{19}[t] \in Id\mathcal{K}_{\Sigma'}$. Hence $s \approx t$ is a hyperidentity in $\mathcal{K}_{\Sigma'}$.

□

By Theorems 3.2, 4.10 and 4.12, we can characterize all hyperidentities in $\mathcal{K}_{\Sigma\cup\Sigma'}$ as the following theorem shows:

Theorem 4.13. *An identity* $s \approx t$ *in* $\mathcal{K}_{\Sigma \cup \Sigma'}$ *where* s *and* t *are nontrivial terms and are not variables is a hyperidentity in* $\mathcal{K}_{\Sigma \cup \Sigma'}$ *if and only if* $L(s) = L(t)$, $R(s) = R(t)$ *and* $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t] \in Id\mathcal{K}_{\Sigma \cup \Sigma'}$.

Proof. This follows from the proofs of Theorems 4.10 and 4.12. □

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