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All intra-regular and relationship between some regular submonoids of Relhyp((m),(n))

Pornpimol Kunama, Sorasak Leeratanavalee

Department of Mathematics
Faculty of Science
Chiang Mai University
Chiang Mai 50200, Thailand

email: pornpimol5331@gmail.com, sorasak.l@cmu.ac.th

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Abstract

An algebraic system is a structure consisting of a nonempty set together with a sequence of operations and a sequence of relations on it. A relational hypersubstitution for algebraic systems is a mapping which maps any operation to a term and maps any relation to a relational term preserving its arities. The set of all such relational hypersubstitutions for algebraic systems forms a monoid. In this paper, we characterize the set of all intra-regular elements of the monoid of all relational hypersubstitutions of type $((m),(n))$ and show that the set of all intra-regular elements, set of all left (right) regular elements and the set of all completely regular elements of $Relhyp((m),(n))$ are the same.

1 Introduction

First, we recall the definition of some special elements in a semigroup. Let a be an element of a semigroup S. Then a is called *regular* if $a = axa$ for some $x \in S$, a is called a *completely regular* of S if a is regular such that $a = axa$ and $ax = xa$ for some $x \in S$, and a is called an *intra-regular* if and only if

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$a = xa^2y$ for some $x, y \in S$. We see that a completely regular element is an intra-regular element, but an intra-regular element need not be a completely regular element. So the set of all completely regular elements of S is a subset of the set of all intra-regular elements of S. In general, we know that the set of all intra-regular elements of the monoid is not its submonoid. In 2008, Denecke and Phusanga [5] introduced the concept of a hypersubstitution for algebraic systems of type (τ, τ') which is a mapping that assigns an operation to a term and assigns a relation to a formula which preserves the arities. The set of all hypersubstitutions for algebraic systems of type (τ, τ') is denoted by $Hyp(\tau, \tau')$. Denecke and Phusanga defined an associative operation $\circ_h$ on this set and proved that $(Hyp(\tau, \tau^{'}), \circ_h, \sigma_{id})$ forms a monoid where σ_{id} is an identity hypersubstitution for algebraic systems (for additional details see [5, 9, 10]). The concept of algebraic systems was first introduced by Mal'cev [8].

Definition 1.1. Let I and J be indexed sets. An algebraic system of type $(\tau, \tau^{'})$ is a triplet $(A, (f_i^A)_{i\in I}, (\gamma_j^A)_{j\in J})$ consisting of a nonempty set A, a sequence $(f_i^A)_{i\in I}$ of operations defined on A and a sequence $(\gamma_j^A)_{j\in J}$ of relations on A, where $\tau = (m_i)_{i\in I}$ is a sequence of the arity of each operation f_i^A and $\tau^{'} = (n_j)_{j\in J}$ is a sequence of the arity of each relation γ_j^A. The pair $(\tau, \tau^{'})$ is called the type of an algebraic system.

In 2018, Lekkoksung and Phusanga [7] improved this concept and introduced the concept of a relational hypersubstitution for algebraic systems of type (τ, τ'). The new concept generalizes the notion of a hypersubstitution (for universal algebras) of type τ in a canonical way. Since any hypersubstitution (for universal algebras) of type τ assigns an m_i-ary operation symbol f_i to an m_i-ary term for $i \in I$, it seems quite natural that any hypersubstitution for algebraic systems of type (τ, τ') assigns an m_i-ary operation f_i^A to an m_i-ary term for $i \in I$ and assigns an n_j-ary relation γ_j^A to an n_j-ary relational term for $j \in J$. Such mappings are called the relational hypersubstitutions for algebraic systems of type (τ, τ'). The set of all relational hypersubstitutions for algebraic systems together with a binary operation defined on this set and the identity element forms a monoid. In 2020, Daengsaen and Leeratanavalee [3] characterized all completely regular elements of the monoid of all relational hypersubstitutions for algebraic systems of type $(\tau, \tau') = ((m), (n))$. In this paper, we determine all intra-regular elements of this monoid and show that the set of all intra-regular elements and the set of all completely regular elements of the monoid of all relational hypersubstitutions for algebraic systems of type $(\tau, \tau') = ((m), (n))$ are the same.

2 The monoid of relational hypersubstitutions for algebraic systems

In this section, we introduce the concept of the monoid of all relational hypersubstitutions for algebraic systems of type $((m),(n))$. To define the concept of relational hypersubstitution for algebraic systems, we first introduce the concept of terms and relational terms.

Let $(f_i)_{i\in I}$ be a set of m_i-ary operation symbols indexed by the set I, where $m_i \in \mathbb{N}^+$. The set $X := \{x_1, \ldots, x_n, \ldots\}$ is a countably infinite set of symbols called variables. For each $m \geq 1$, let $X_m := \{x_1, \ldots, x_m\}$. We call the sequence $\tau := (m_i)_{i\in I}$ of arities of f_i, the type. An *m-ary term of type* τ is defined inductively in the following steps:

(i) Every variable $x_k \in X_m$ is an m-ary term of type τ.

(ii) If $t_1, \ldots, t_{m_i}$ are m-ary terms of type τ and f_i is an m_i-ary operation symbol, then $f_i(t_1, \ldots, t_{m_i})$ is an m-ary term of type τ.

Let $W_\tau(X_m)$ be the set of all m-ary terms of type τ which contains $x_1, \ldots, x_m$ and is closed under finite application of (ii) and let $W_\tau(X) := \bigcup_{m\in\mathbb{N}^+} W_\tau(X_m)$ be the set of all terms of type τ.

An $n-$ary relational term of type $(\tau, \tau^{'})$ is defined as follows:

Definition 2.1. ([10]) Let J be an indexed set. If $j \in J$ and $t_1, ..., t_{n_j}$ are $n-$ary terms of type τ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, ..., t_{n_j})$ is an $n-$ary relational term of type $(\tau, \tau^{'})$.

Let $\gamma\mathcal{F}_{(\tau,\tau')}(X_n)$ be the set of all $n-$ary relational terms of type $(\tau, \tau^{'})$ and let $\gamma\mathcal{F}_{(\tau,\tau')}(X) := \cup_{n\in\mathbb{N}}\gamma\mathcal{F}_{(\tau,\tau')}(X_n)$ be the set of all relational terms of type $(\tau, \tau^{'})$.

A relational hypersubstitution for algebraic systems of type
$(\tau, \tau^{'}) = ((m_i)_{i\in I}, (n_j)_{j\in J})$ is a mapping

$$\sigma : \{f_i : i \in I\} \cup \{\gamma_j : j \in J\} \to W_\tau(X_{m_i}) \cup \gamma\mathcal{F}_{(\tau,\tau')}(X_{n_j})$$

with $\sigma(f_i) \in W_\tau(X_{m_i})$ and $\sigma(\gamma_j) \in \gamma\mathcal{F}_{(\tau,\tau')}(X_{n_j})$. The set of all relational hypersubstitutions for algebraic systems of type $(\tau, \tau^{'})$ is denoted by $Relhyp(\tau, \tau^{'})$.

Definition 2.2. ([10]) For any $m, n \in \mathbb{N}^+$ and $t, t_1, \ldots, t_{n_j} \in W_\tau(X_m)$, $s_1, \ldots, s_m \in W_\tau(X_n)$ and $F = \gamma_j(t_1, ..., t_{n_j}) \in \gamma\mathcal{F}_{(\tau,\tau')}(X_m)$, a superposition partial operation

$$R_n^m : (W_\tau(X_m) \cup \gamma\mathcal{F}_{(\tau,\tau')}(X_m)) \times (W_\tau(X_n))^m \multimap W_\tau(X_n) \cup \gamma\mathcal{F}_{(\tau,\tau')}(X_n)$$

is defined by the following steps:

(i) $R_n^m(t, s_1, \ldots, s_m) := S_n^m(t, s_1, \ldots, s_m)$,

(ii) $R_n^m(F, s_1, \ldots, s_m) := \gamma_j(S_n^m(t_1, s_1, \ldots, s_m), \ldots, S_n^m(t_{n_j}, s_1, \ldots, s_m))$.

Every relational hypersubstitution for algebraic systems σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \cup \gamma\mathcal{F}_{(\tau,\tau')}(X) \to W_\tau(X) \cup \gamma\mathcal{F}_{(\tau,\tau')}(X)$ defined by the following steps:

(i) $\widehat{\sigma}[x_i] := x_i$ for $i \in \mathbb{N}$,

(ii) $\widehat{\sigma}[f_i(t_1 \ldots, t_{m_i})] := S_m^{m_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{m_i}])$,
where $i \in I$ and $t_1, \ldots, t_{m_i} \in W_\tau(X_m)$; i.e., any occurrence of the variable x_k in $\sigma(f_i)$ is replaced by the term $\widehat{\sigma}[t_k], 1 \leq k \leq m_i$,

(iii) $\widehat{\sigma}[\gamma_j(s_1 \ldots, s_{n_j})] := R_n^{n_j}(\sigma(\gamma_j), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_{n_j}])$, where $j \in J$ and $s_1, \ldots, s_{n_j} \in W_\tau(X_n)$; i.e., any occurrence of the variable x_k in $\sigma(\gamma_j)$ is replaced by the term $\widehat{\sigma}[s_k], 1 \leq k \leq n_j$.

We define a binary operation $\circ_r$ on $Relhyp(\tau, \tau^{'})$ by $\sigma \circ_r \alpha := \hat{\sigma} \circ \alpha$, where $\circ$ is the usual composition of mappings and $\sigma, \alpha \in Relhyp(\tau, \tau^{'})$. Let σ_{id} be the relational hypersubstitution which maps each m_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{m_i})$ and maps each n_j-ary relation symbol γ_j to the relational term $\gamma_j(x_1, \ldots, x_{n_j})$. Lekkoksung and Phusanga [7] proved that $(Relhyp(\tau, \tau^{'}), \circ_r, \sigma_{id})$ is a monoid.

In 2015, Wongpinit and Leeratanavalee [11] introduced the concept of the $i - most$ of terms as follows:

Definition 2.3. ([11]) For a type $\tau = (m)$ with an m-ary operation symbol f, $t \in W_{(m)}(X)$ and $1 \leq i \leq m$. An $i - most(t)$ is defined inductively by the following steps.

(i) If t is a variable, then $i - most(t) = t$.

(ii) If $t = f(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n \in W_{(m)}(X)$, then $i - most(t) := i - most(t_i)$.

Example 1. *Let* $\tau = (3)$ *be a type,* $t = f(f(x_3, x_1, x_2), x_2, f(x_2, x_1, x_1))$*. Then* $1-most(t) = 1-most(f(x_3, x_1, x_2)) = x_3, 2-most(t) = 2-most(x_2) = x_2$ *and* $3-most(t) = 3-most(f(x_2, x_1, x_1)) = x_1$*.*

Let $(\tau, \tau^{'}) = ((m), (n))$ be a type with an $m-$ary operation symbol f and an $n-$ary relation symbol γ. Let $t \in W_{(m)}(X_m)$ and $F \in \gamma F_{((m),(n))}(X_n)$. We use the following notations:

$\sigma_{t,F}$:= the relational hypersubstitution of type $(\tau, \tau') = ((m), (n))$ with maps f to a term $t \in W_{(m)}(X_m)$ and maps γ to a relational term $F \in \gamma F_{((m),(n))}(X_n)$,

$var(t)$:= the set of all variables occurring in the term t,

$var(F)$:= the set of all variables occurring in the relational term F,

$vb^t(x)$:= the total number of x-variable occurring in the term t,

$vb^F(x)$:= the total number of x-variable occurring in the relational term F.

Example 2. *Let* $(\tau, \tau') = ((3), (2))$ *where* $t = f(x_3, f(x_1, x_2, x_1), f(x_3, x_1, x_2)) \in W_{(3)}(X_3)$ *and* $F = \gamma(f(x_2, f(x_1, x_2, x_1), x_1), x_1) \in \gamma F_{((3),(2))}(X_2)$*. Then*

$var(t) = \{x_1, x_2, x_3\}$ *and* $vb^t(x_1) = 3, vb^t(x_2) = 2, vb^t(x_3) = 2$

$var(F) = \{x_1, x_2\}$ *and* $vb^F(x_1) = 4, vb^F(x_2) = 2$.

2.1 Sequence of relational terms

In this section, we construct some tools which are used to characterize all intra-regular elements in $Relhyp((m), (n))$. These tools are called the sequence of a relational term and the depth of a relational term, respectively

Definition 2.4. ([1]) Let $t \in W_{(m)}(X_m)\backslash X$ where $t = f(t_1, \ldots, t_m)$ for some $t_1, \ldots, t_m \in W_{(m)}(X_m)$. Let $x_j^{(l)}$ be variable x_j occurring in the l^{th} component of t (from the left). For $i_l = 1, \ldots, m$, let $\pi_{i_l} : W_{(m)}(X_m)\backslash X \to W_{(m)}(X_m)$ be defined by $\pi_{i_l}(t) = \pi_{i_l}(f(t_1, \ldots, t_m)) = t_{i_l}$. The sequence and the depth of $x_j^{(l)}$ in t are denoted by $seq^t(x_j^{(l)})$ and $depth^t(x_j^{(l)})$, respectively. If $x_j^{(l)} = \pi_{i_k} \circ \cdots \circ \pi_{i_1}(t)$ for some $k \in \mathbb{N}$, then we define

$$seq^t(x_j^{(l)}) = (i_1, \ldots, i_k) \text{ and } depth^t(x_j^{(l)}) = k.$$

Definition 2.5. Let $F \in \gamma F_{((m),(n))}(X_n)$ where $F = \gamma(s_1, \ldots, s_n)$ for some $s_1, \ldots, s_n \in W_{(n)}(X_n)$. Let $x_j^{(l)}$ be variable x_j occurring in the l^{th} component of F (from the left). For $i_l = 1, \ldots, n$, let $\varphi_{i_l} : \gamma F_{((m),(n))}(X_n) \to W_{(n)}(X_n)$ be defined by $\varphi_{i_l}(F) = \varphi_{i_l}(\gamma(s_1, \ldots, s_n)) = s_{i_l}$. For $i_k \in \{1, \ldots, m\}$, let $\phi_{i_k} : W_{(n)}(X_n)\backslash X \to W_{(n)}(X_n)$ be defined by $\phi_{i_k}(f(t_1, \ldots, t_m)) = t_{i_k}$. The sequence and the depth of $x_j^{(l)}$ in F are denoted by $seq^F(x_j^{(l)})$ and $depth^F(x_j^{(l)})$, respectively. If $x_j^{(l)} = \phi_{i_k} \circ \cdots \circ \phi_{i_2} \circ \varphi_{i_1}(F)$ for some $k \in \mathbb{N}$, then we define

$$seq^F(x_j^{(l)}) = (i_1, \ldots, i_k) \text{ and } depth^F(x_j^{(l)}) = k.$$

Example 3. *Let* $(\tau, \tau') = ((3),(2))$ *and let* $t \in W_{(3)}(X_3)$, $F \in \gamma F_{((3),(2))}(X_2)$ *where* $F = \gamma(f(x_2, f(x_1, x_2, x_1), x_1), x_1)$ *and*
$t = f(x_2, f(x_3, f(x_1, f(x_2, x_3, x_1), x_2), x_1), f(f(x_3, x_2, f(x_2, x_3, x_1)), x_1, x_3))$.
Then
$seq^t(x_1^{(1)}) = (2,2,1)$ *and* $depth^t(x_1^{(1)}) = 3$
$seq^t(x_1^{(2)}) = (2,2,2,3)$ *and* $depth^t(x_1^{(2)}) = 4$
$seq^t(x_1^{(3)}) = (2,3)$ *and* $depth^t(x_1^{(3)}) = 2$
$seq^t(x_1^{(4)}) = (3,1,3,3)$ *and* $depth^t(x_1^{(4)}) = 4$
$seq^t(x_1^{(5)}) = (3,2)$ *and* $depth^t(x_1^{(5)}) = 2$
$seq^F(x_1^{(1)}) = (1,2,1)$ *and* $depth^t(x_1^{(1)}) = 3$
$seq^F(x_1^{(2)}) = (1,2,3)$ *and* $depth^t(x_1^{(2)}) = 3$
$seq^F(x_1^{(3)}) = (1,3)$ *and* $depth^t(x_1^{(3)}) = 2$
$seq^F(x_1^{(4)}) = (2)$ *and* $depth^t(x_1^{(4)}) = 1$

Definition 2.6. ([2]) Let $t \in W_{(m)}(X_m)\backslash X$ and $k \in \mathbb{N}^+$. Let $x_i^{(j)}$ be variable x_i occurring in the j^{th} component of t (from the left). Let $var(t)_{X_m}^{d(k)}$ denote the set of all distinct variables $x_i \in var(t)$ such that $depth^t(x_j^{(l)}) = k$; i.e.,

$$var(t)_{X_m}^{d(k)} = \{x_i \in var(t) \mid depth^t(x_j^{(l)}) = k \; \exists\, l \in \mathbb{N}^+\}$$

Definition 2.7. Let $F \in \gamma F_{((m),(n))}(X_n)$ and $k \in \mathbb{N}^+$. Let $x_i^{(j)}$ be variable x_i occurring in the j^{th} component of t (from the left). Let $var(F)_{X_n}^{d(k)}$ denote the set of all distinct variables $x_i \in var(F)$ such that $depth^F(x_j^{(l)}) = k$; i.e.,

$$var(F)_{X_n}^{d(k)} = \{x_i \in var(F) \mid depth^F(x_j^{(l)}) = k \; \exists\, l \in \mathbb{N}^+\}$$

Example 4. *Let* $(\tau, \tau') = ((3),(2))$ *and* $t \in W_{(3)}(X_3)$, $F \in \gamma F_{((3),(2))}(X_2)$ *where* $t = f(x_2, f(x_3, f(x_1, f(x_2, x_3, x_1), x_2), x_1), f(f(x_3, x_2, f(x_2, x_3, x_1), x_1)x_3))$ *and* $F = \gamma(f(x_2, f(x_1, x_2, x_1), x_1), x_1)$. *Then*
$var(t)_{X_3}^{d(1)} = \{x_2\}$, $var(t)_{X_3}^{d(2)} = \{x_1, x_3\}$, $var(t)_{X_3}^{d(3)} = \{x_1, x_2, x_3\}$, $var(t)_{X_3}^{d(4)} = \{x_1, x_2, x_3\}$, $var(t)_{X_3}^{d(n)} = \emptyset$ *if* $n \geq 5$ *and* $var(F)_{X_2}^{d(1)} = \{x_1\}$, $var(F)_{X_2}^{d(2)} = \{x_1, x_2\}$, $var(F)_{X_2}^{d(3)} = \{x_1, x_2\}$, $var(F)_{X_2}^{d(n)} = \emptyset$ *if* $n \geq 4$.

3 All Intra-regular Elements in $Relhyp((m),(n))$

Let $\sigma_{t,F} \in Relhyp((m),(n))$. We use the following notations:

$CR(R_x) := \{\sigma_{t,F} \mid t = x_i \in X_m$ and $F = \gamma(s_1, ..., s_n)$ with $var(F) = \{x_{b_1}, ..., x_{b_l}\} \subseteq X_n$ such that $i - most(s_{b_j}) = x_{\phi(b_j)}$ for all $j = 1, ..., l$ where ϕ is a bijective on $\{b_1, ..., b_l\}\}$;

$CR(R_t) := \{\sigma_{t,F} \mid t = f(t_1, \ldots, t_m)$ and $F = \gamma(s_1, ..., s_n)$ with $var(t) = \{x_{a_1}, ..., x_{a_k}\}$ and $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $t_{a_i} = x_{\pi a_i}$ and $s_{b_j} = x_{\phi(b_j)}$ for all $i = 1, ..., k, j = 1, ..., l$ where π is a bijective on $\{a_1, ..., a_k\}$ and ϕ is a bijective on $\{b_1, ..., b_l\}\}$.

In 2020, Daengsaen and Leeratanavalee [3] showed that $CR(Relhyp((m),(n))) := CR(R_x) \cup CR(R_t)$ is the set of all completely regular elements in $Relhyp((m),(n))$. In general, $CR(Relhyp((m),(n)))$ is a subset of the set all intra-regular elements in $Relhyp((m),(n))$. In this section, we show that every $\sigma_{t,F} \in Relhyp((m),(n)) \backslash CR(Relhyp((m),(n)))$ is not intra-regular. So $CR(Relhyp((m),(n)))$ is the set of all intra-regular elements in $Relhyp((m),(n))$.

From now on, we construct some tools which will be used in this section.

Theorem 3.1. ([1]) Let $t, s \in W_{(m)}(X_m) \backslash X$ and $x_i \in var(t)$. Let $x_i^{(j)}$ be variable x_i occurring in the j^{th} component of t (from the left) such that $seq^t(x_i^{(j)}) = (i_1, \ldots, i_n)$ for some $i_1, \ldots, i_n \in \{1, \ldots, m\}$. Then $x_i^{(j)} \in var(\hat{\sigma}_s[t])$ if and only if $x_{i_k} \in var(s)$ for all $1 \leq k \leq n$; i.e.,

$$seq^{\hat{\sigma_s}[t]}(x_i^{(j,h)}) = (a_{i_1}, \ldots, a_{i_n})$$

where (a_{i_k}) is a sequence of $k_1, \ldots, k_z$ such that $(k_1, \ldots, k_z) \in seq^s(x_{i_k})$ for all $k \in \{1, \ldots, n\}$. Moreover,

$$depth^{\hat{\sigma_s}[t]}(x_i^{(j,h)}) = depth^s(x_{i_1}^{(l_1)}) + \cdots + depth^s(x_{i_n}^{(l_n)}),$$

where $x_{i_k}^{(l_k)}$ be variable x_{i_k} occurring in the l_k^{th} component of s (from the left) for all $k \in \{1, \ldots, n\}$.

Theorem 3.2. *Let $F, H \in \gamma F_{((m),(n))}(X_n)$ and $x_i \in var(F)$. Let $x_i^{(j)}$ be variable x_i occurring in the j^{th} component of F (from the left) such that $seq^F(x_i^{(j)}) = (i_1, \ldots, i_m)$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$. Then $x_i^{(j)} \in var(\hat{\sigma}_{u,H}[F])$ if and only if $x_{i_k} \in var(H) \cup var(u)$ for all $1 \leq k \leq m$; i.e.,*

$$seq^{\hat{\sigma}_{u,H}[F]}(x_i^{(j,h)}) = (a_{i_1}, \ldots, a_{i_m})$$

where (a_{i_k}) *is a sequence of* $p_1,\ldots,p_g$ *such that* $(p_1,\ldots,p_g) \in seq^H(x_{i_k})$ *or* (a_{i_l}) *is a sequence of* $q_1,\ldots,q_r$ *such that* $(q_1,\ldots,q_r) \in seq^u(x_{i_l})$ *for all* $k,l \in \{1,\ldots,n\}$*. Moreover,*

$$depth^{\hat{\sigma}_{u,H}[F]}(x_i^{(j,h)}) = depth^H(x_{i_1}^{(l_1)}) + \cdots + depth^u(x_{i_m}^{(l_m)}),$$

for some $l_1,\ldots,l_m \in \mathbb{N}$.

Proof. Let $F = \gamma(s_1,\ldots,s_n)$ and $seq^F(x_i^{(j)}) = (i_1,\ldots,i_m)$. Let us proceed by mathematical induction on m. If $(i_1) \in seq^F(x_i)$, then $\phi_{i_1}(F) = s_{i_1} = x_i$, where $s_{i_1} \in \{s_1,\ldots,s_n\}$. Hence $\hat{\sigma}_{u,H}[s_{i_1}] = x_i$. Consider

$$(\sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) = \hat{\sigma}_{u,H}[F] = R_n^n(H, \hat{\sigma}_{u,H}[s_1],\ldots,\hat{\sigma}_{u,H}[s_n]).$$

By $x_{i_1} \in var(H) \cup var(u), x_i = \hat{\sigma}_{u,H}[F] \in var(\hat{\sigma}_{u,H}[F])$ and there is $(a_{i_1}) \in seq^{\hat{\sigma}_{u,H}[F]}(x_i)$ where (a_{i_1}) is a sequence of natural number $p_1,\ldots,p_g$ such that $(p_1\ldots,p_g) \in seq^H(x_{i_1})$. Let m be a natural number and assume that for $G \in \gamma F_{((m),(n))}(X_n), x_i \in var(G)$ and $(l_1,\ldots,l_d) \in seq^G(x_i)$. Then $x_i \in var(\hat{\sigma}_{u,H}[G])$ and $(a_{l_1},\ldots,a_{l_d}) \in seq^{\hat{\sigma}_{u,H}[G]}(x_i)$ where (a_{l_b}) is a sequence of $j_1,\ldots,j_{g^*}$ such that $(j_1,\ldots,j_{g^*}) \in seq^H(x_{l_b})$ or is a sequence of $z_1,\ldots,z_{r^*}$ such that $(z_1,\ldots,z_{r^*}) \in seq^u(x_{l_b})$ for all $j,z \in \{1,\ldots,y\}$, for all natural numbers $y < m$. If $(i_1,\ldots,i_m) \in seq^F(x_i)$, then $x_i = \pi_{i_m} \circ \cdots \circ \pi_{i_2} \circ \phi_{i_1}(F) = \pi_{i_m} \circ \cdots \circ \pi_{i_2}(s_{i_1})$; i.e. $x_i \in var(s_{i_1})$ and $(i_2,\ldots,i_m) \in seq^{s_{i_1}}(x_i)$. By the assumption, we get $x_i \in var(\hat{\sigma}_{u,H}[s_{i_1}])$ and there is $(a_{i_2},\ldots,a_{i_m}) \in seq^{\hat{\sigma}_{u,H}[s_{i_1}]}(x_i)$ where (a_{i_k}) is a sequence of $p_1,\ldots,p_g$ such that $seq^H(x_{i_k}) = (p_1,\ldots,p_g)$ or (a_{i_l}) is a sequence of $q_1,\ldots,q_r$ such that $seq^u(x_{i_l}) = (q_1,\ldots,q_r)$. Since $x_{i_l} \in var(H) \cup var(u)$ and $seq^{\hat{\sigma}_{u,H}[F]}(\hat{\sigma}_{u,H}[s_{i_1}])$. Hence $x_i \in var(\hat{\sigma}_{u,H}[F])$ and there is $(a_{i_1},\ldots,a_{i_m}) \in seq^{\hat{\sigma}_{u,H}[F]}(x_i)$ where (a_{i_j}) is a sequence of $p_1,\ldots,p_g$ such that $(p_1,\ldots,p_g) \in seq^H(x_{i_j})$ or (a_{i_j}) is a sequence of $q_1,\ldots,q_r$ such that $(q_1,\ldots,q_r) \in seq^u(x_{i_j})$.

Conversely, assume $x_i^{(j,h)} \in var(\hat{\sigma}_{u,H}[F])$ and $seq^{\hat{\sigma}_{u,H}[F]}(x_i^{(j,h)}) = (a_{i_1},\ldots,a_{i_m})$, where (a_{i_l}) is a sequence of $p_1,\ldots,p_q$ such that $(p_1,\ldots,p_q) = seq^H(x_{i_k}^{(l_k)})$ or (a_{i_l}) is a sequence of $q_1,\ldots,q_r$ such that $(q_1,\ldots,q_r) = seq^u(x_{i_k}^{(l_k)})$ for some $l_k \in \mathbb{N}$ and $k \in \{1,\ldots,m\}$. Then $vb^{\hat{\sigma}_{u,H}[F]}(x_i^{(j)}) = vb^H(x_{i_1}) \times \cdots \times vb^u(x_{i_m})$. Suppose that $x_{i_k} \notin var(H) \cup var(u)$ for some $1 \leq k \leq m$. Then $vb^H(x_{i_z}) = 0$ or $vb^u(x_{i_z}) = 0$; i.e., $vb^{\hat{\sigma}_{u,H}[F]}(x_i^{(j)}) = 0$, which contradicts the assumption. Consequently, $x_{i_1},\ldots,x_{i_m} \in var(H) \cup var(u)$. □

Lemma 3.3. ([2]) Let $t,s \in W_{(m)}(X_m)\backslash(X)$. Then $\left|var(\hat{\sigma}_t[s])_{X_m}^{d(1)}\right| \leq \left|var(t)_{X_m}^{d(1)}\right|$.

Lemma 3.4. *Let* $F,H \in \gamma F_{((m),(n))}(X_n)$*. Then* $\left|var(\hat{\sigma}_{t,F}[H])_{X_n}^{d(1)}\right| \leq \left|var(F)_{X_n}^{d(1)}\right|$.

Proof. Let $F, H \in \gamma F_{((m),(n))}(X_n)$, where $F = \gamma(s_1, \ldots, s_n), H = \gamma(h_1, \ldots, h_n)$. If $var(F)^{d(1)}_{X_n} = \emptyset$, then $var(\hat{\sigma}_{t,F}[H])^{d(1)}_{X_n} = \emptyset$. So $|var(F)^{d(1)}_{X_n}| = 0 = |var(\hat{\sigma}_{t,F}[H])^{d(1)}_{X_n}|$. If $var(F)^{d(1)}_{X_n} \neq \emptyset$, then there are $s_{i_1}, \ldots, s_{i_m} \in \{s_1, \ldots, s_n\}$ such that $s_{i_1} = x_{j_1}, \ldots, s_{i_m} = x_{j_m}$ where $var(F)^{d(1)}_{X_n} = \{x_{j_1}, \ldots, x_{j_m}\}$. Then $\left|var(F)^{d(1)}_{X_n}\right| = m$. Consider $(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = \hat{\sigma}_{t,F}[H] = \gamma(w_1, \ldots, w_n)$, where $w_i = R^n_n(s_i, \hat{\sigma}_{t,F}[h_1], \ldots, \hat{\sigma}_{t,F}[h_n])$ for all $i = \{1, \ldots, n\}$. Then $var(\hat{\sigma}_{t,F}[H])^{d(1)}_{X_n} \subset \{w_{i_1}, \ldots, w_{i_m}\}$. So $\left|var(\hat{\sigma}_{t,F}[H])^{d(1)}_{X_n}\right| \leq m$. Therefore, $\left|var(\hat{\sigma}_{t,F}[H])^{d(1)}_{X_n}\right| \leq \left|var(F)^{d(1)}_{X_n}\right|$. $\square$

Theorem 3.5. ([2]) Let $t, s \in W_{(m)}(X_m)\backslash X$ and $x_i \in var(t)$. Let $x_i^{(j)}$ be a variable x_i occurring in the j^{th} component of t (from the left) such that $seq^t(x_i^{(j)}) = (i_1, \ldots, i_n)$. Let $x_i^{(j)} \in var(\sigma_{s,H}[t])$ and $x_i^{(j,h)}$ be a variable $x_i^{(j)}$ occurring in the h^{th} component of $\hat{\sigma}_s[t]$ (from the left). If there is $i_k \in \{i_1, \ldots, i_m\}$ such that $x_{i_k} \notin var(s)^{d(1)}_{X_m}$, then $depth^{\hat{\sigma}_s[t]}(x_i^{(j,h)}) > depth^t(x_i^{(j)})$.

Theorem 3.6. *Let $F, H \in \gamma F_{((m),(n))}(X_n)$ and $x_i \in var(F)$. Let $x_i^{(j)}$ be a variable x_i occurring in the j^{th} component of F (from the left) such that $seq^F(x_i^{(j)}) = (i_1, \ldots, i_m)$. Let $x_i^{(j)} \in var(\hat{\sigma}_{s,H}[F])$ and $x_i^{(j,h)}$ be a variable $x_i^{(j)}$ occurring in the h^{th}component of $\hat{\sigma}_{u,H}[F]$ (from the left). If there is $i_k \in \{i_1, \ldots, i_m\}$ such that $x_{i_k} \notin var(H)^{d(1)}_{X_n}$ or $x_{i_k} \notin var(u)^{d(1)}_{X_m}$, then $depth^{\hat{\sigma}_{u,H}[F]}(x_i^{(j,h)}) \geq depth^F(x_i^{(j)})$.*

Proof. We have $depth^F(x_i^{(j)}) = m$. Also $x_{i_1}, \ldots, x_{i_k}, \ldots, x_{i_m} \in var(H) \cup var(u)$ and

$$depth^{\hat{\sigma}_{u,H}[F]}(x_i^{(j,h)}) = depth^H(x_{i_1}^{(q_1)}) + \cdots + depth^u(x_{i_m}^{(q_m)}),$$

where $x_{i_p}^{(q_p)}$ is a variable x_{i_p} occurring in the q_p^{th} component of H (from the left) or is a variable $x_{(i_p)}$ occurring in the q_p^{th} component of u (from the left). Since $x_{i_k} \notin var(H)^{d(1)}_{X_n}$ or $x_{i_k} \notin var(u)^{d(1)}_{X_m}$, we have $depth^H(x_{i_k}^{(q_k)}) \geq 1$ or $depth^u(x_{i_k}^{(q_k)}) \geq 1$. Hence $depth^{\hat{\sigma}_{u,H}[F]}(x_i^{(j,h)}) \geq m$. $\square$

Corollary 3.7. ([2]) Let $t \in W_{(m)}(X_m)\backslash X$ and $x_i \in var(t)\backslash var(t)^{d(1)}_{X_m}$. For each $x_i^{(j)} \in var(t_l)$ for some $l \in \{1, \ldots, m\}$ such that $x_i \notin var(t)^{d(1)}_{X_m}$ where $x_i^{(j)}$ is a variable x_i occurring in the j^{th} component of t (from the left), if $x_i^{(j)} \in var(\hat{\sigma}_t[t])$ and denote $x_i^{(j,h)}$ be a variable $x_i^{(j)}$ occurring in the h^{th}component of $\hat{\sigma}_t[t]$ (from the left), then $depth^{\hat{\sigma}_t[t]}(x_i^{(j,h)}) > depth^t(x_i^{(j)})$.

Corollary 3.8. *Let $F \in \gamma F_{((m),(n))}(X_n)$ and $x_i \in var(F) \backslash var(F)^{d(1)}_{X_n}$. For each $x_i^{(j)} \in var(s_l)$ for some $l \in \{1, \ldots, n\}$ such that $x_i \notin var(F)^{d(1)}_{X_n}$ where $x_i^{(j)}$ be a variable x_i occurring in the j^{th} component of F (from the left), if $x_i^{(j)} \in var(\hat{\sigma}_{t,F}[F])$ and denote $x_i^{(j,h)}$ be a variable $x_i^{(j)}$ occurring in the h^{th} component of $\hat{\sigma}_{t,F}[F]$ (from the left), then $depth^{\hat{\sigma}_{t,F}[F]}(x_i^{(j,h)}) \geq depth^{F}(x_i^{(j)})$.*

Theorem 3.9. *Let $t = x_i$ and $F = \gamma(s_1, \ldots, s_n) \in \gamma F_{((m),(n))}(X_n)$ with $var(F) = \{x_{b_1}, \ldots, x_{b_l}\}$. Then $\sigma_{t,F}$ is intra-regular if and only if $\sigma_{t,F} \in CR(R_x)$.*

Proof. Let $\sigma_{t,F}$ be an intra-regular. There exist $\sigma_{u,H}, \sigma_{v,G} \in Relhyp((m)(n))$ such that $\sigma_{t,F} = \sigma_{u,H} \circ_r \sigma^2_{t,F} \circ_r \sigma_{v,G}$ where $u, v \in W_{(m)}(X_m)$ and $H = \gamma(h_1, \ldots, h_n), G = \gamma(g_1, \ldots, g_n) \in Relhyp((m)(n))$. The proof is by contradiction. Assuming that $\sigma_{t,F} \notin CR(R_x)$, there exists $x_{b_j} \in var(F)$ such that $i - most(s_{b_k}) \neq x_{\phi(b_j)}$ for all $k = \{1, \ldots, l\}$, where ϕ is bijective on $\{b_1, ..., b_l\}$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{x_i,F}[\sigma_{x_i,F}(\gamma)] \\ &= \widehat{\sigma}_{x_i,F}[F] \\ &= R^n_n(F, i - most(s_1), \ldots, i - most(s_n)).\end{aligned}$$

Since we know that every variable $x_{b_k} \in var(F)$ must be replaced by $i - most(s_{b_k})$ but $x_{\phi(b_j)} \notin \{i - most(s_{b_k}) : k = 1, \ldots, l\}$, $x_{b_j} \notin var((\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma))$. This implies that $|var((\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma))| < |var(F)|$. Consider

$$\begin{aligned}(\sigma^2_{t,F} \circ_r \sigma_{v,G})(\gamma) &= \widehat{\sigma}^2_{t,F}[\sigma_{v,G}(\gamma)] \\ &= \widehat{\sigma}^2_{x_i,F}[G] \\ &= R^n_n(\sigma^2_{x_i,F}, i - most(g_1), \ldots, i - most(g_n)).\end{aligned}$$

Since every variable $x_{b_k} \in \sigma^2_{t,F}(\gamma)$ is replaced by $i - most(g_{b_k})$, $|var((\sigma^2_{t,F} \circ_r \sigma_{v,G})(\gamma))| = |\{i - most(g_{b_k}) : x_{b_k} \in var(\sigma^2_{t,F})(\gamma)\}| \leq |var(\sigma^2_{t,F})(\gamma)| < |var(F)|$. Since $F = \sigma_{t,F}(\gamma) = (\sigma_{u,H} \circ_r \sigma^2_{t,F} \circ_r \sigma_{v,G})(\gamma)$ and $var(F) = var((\sigma_{u,H} \circ_r \sigma^2_{t,F} \circ_r \sigma_{v,G})(\gamma)) \subseteq var((\sigma^2_{t,F} \circ_r \sigma_{v,G})(\gamma))$, $|var(F)| \leq |var((\sigma^2_{t,F} \circ_r \sigma_{v,G})(\gamma))| < |var(F)|$, this is contradiction. Conversely, let $\sigma_{t,F} \in CR(R_x)$. Then $\sigma_{t,F}$ is completely regular. It follows that $\sigma_{t,F}$ is intra-regular. □

Theorem 3.10. *Let $t = f(t_1, \ldots, t_m) \in W_{(m)}(X_m)$ with $var(t) = \{x_{a_1}, \ldots, x_{a_k}\}$ and $F = \gamma(s_1, \ldots, s_n) \in \gamma F_{((m),(n))}(X_n)$ with $var(F) = \{x_{b_1}, \ldots, x_{b_l}\}$. Then $\sigma_{t,F}$ is intra-regular if and only if $\sigma_{t,F} \in CR(R_t)$.*

Proof. Let $\sigma_{t,F}$ be an intra-regular. There exist $\sigma_{u,H}, \sigma_{v,G} \in Relhyp((m)(n))$ such that

$$\sigma_{t,F} = \sigma_{u,H} \circ_r \sigma^2_{t,F} \circ_r \sigma_{v,G}, \tag{3.1}$$

where $u = f(u_1, \ldots, u_m), v = f(v_1, \ldots, v_m) \in W_{(m)}(X_m)$ and $H = \gamma(h_1, \ldots, h_n)$, $G = \gamma(g_1, \ldots, g_n) \in \gamma F_{((m)(n))}(X_n)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{t,F})(f) = f(w_1, \ldots, w_m) \text{ where } w_i = S^m_m(t_i, \hat{\sigma}_{t,F}[t_1], \ldots, \hat{\sigma}_{t,F}[t_m]),$$

$$(\sigma_{w,F} \circ_r \sigma_{v,G})(f) = f(\tilde{w_1}, \ldots, \tilde{w_m}) \text{ where } \tilde{w_i} = S^m_m(w_i, \hat{\sigma}_{w,F}[v_1], \ldots, \hat{\sigma}_{w,F}[v_m]),$$

$$(\sigma_{u,H} \circ_r \sigma_{\tilde{w},F})(f) = f(z_1, \ldots, z_m) \text{ where } z_i = S^m_m(u_i, \hat{\sigma}_{u,H}[\tilde{w_1}], \ldots, \hat{\sigma}_{u,H}[\tilde{w_m}])$$

for all $i \in \{1, \ldots, m\}$ and

$$(\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma) = f(q_1, \ldots, q_n) \text{ where } q_j = R^n_n(s_j, \hat{\sigma}_{t,F}[s_1], \ldots, \hat{\sigma}_{t,F}[s_n]),$$

$$(\sigma_{t,Q} \circ_r \sigma_{v,G})(\gamma) = \gamma(\tilde{q_1}, \ldots, \tilde{q_n}) \text{ where } \tilde{q_j} = R^n_n(q_j, \hat{\sigma}_{t,Q}[g_1], \ldots, \hat{\sigma}_{t,Q}[g_n]),$$

$$(\sigma_{u,H} \circ_r \sigma_{t,\tilde{Q}})(\gamma) = \gamma(z'_1, \ldots, z'_n) \text{ where } z'_j = R^n_n(h_j, \hat{\sigma}_{u,H}[\tilde{q_1}], \ldots, \hat{\sigma}_{u,H}[\tilde{q_n}])$$

for all $j \in \{1, \ldots, n\}$.
Assuming that $\sigma_{t,F} \notin CR(R_t)$, we consider four cases:

(1) $t_{a'_1} = x_{a_1}, \ldots, t_{a'_k} = x_{a_k}$ such that $x_{a'_i} \notin var(t)^{d(1)}_{X_m}$,

(2) $var(t) = \{x_{a_1}, \ldots, x_{a_k}\}$ such that $var(t)^{d(1)}_{X_m} \subset var(t)$,

(3) $s_{b'_1} = x_{b_1}, \ldots, s_{b'_l} = x_{b_l}$ such that $x_{b'_j} \notin var(F)^{d(1)}_{X_n}$,

(4) $var(F) = \{x_{b_1}, \ldots, x_{b_l}\}$ such that $var(F)^{d(1)}_{X_n} \subset var(F)$,

<u>Case 1</u>: We will show that $x_{a'_i} \notin var(\hat{\sigma}_{t,H}[t])^{d(1)}_{X_m}$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{t,F})(f) = f(w_1, \ldots, w_m) \text{ where } w_i = S^m_m(t_i, \hat{\sigma}_{t,H}[t_1], \ldots, \hat{\sigma}_{t,H}[t_m]).$$

If $x_{a'_i} \in var(\hat{\sigma}_{t,H}[t])^{d(1)}_{X_m}$, then $w_k = x_{a'_i}$. So there exists $l \in \{a'_1, \ldots, a'_k\}$ such that $t_l = x_{a'_i}$ and $x_{a'_i} \in var(t)^{d(1)}_{X_m}$ which contradicts to $x_{a'_i} \notin var(t)^{d(1)}_{X_m}$. Hence $var(\hat{\sigma}_{t,H}[t])^{d(1)}_{X_m} \subset var(t)^{d(1)}_{X_m}$ and so $|var(\hat{\sigma}_{t,H}[t])^{d(1)}_{X_m}| < |var(t)^{d(1)}_{X_m}|$. Next, we

will show that $\sigma_{t,F} \neq \sigma_{u,H} \circ_r \sigma^2_{t,F} \circ_r \sigma_{v,G}$. So $var(\sigma_{u,H} \circ_r \sigma^2_{t,F} \circ_r \sigma_{v,G})^{d(1)}_{X_m} \subseteq var(\sigma^2_{t,F} \circ_r \sigma_{v,G})^{d(1)}_{X_m}$ and we have that $|var(\sigma_{u,H} \circ_r \sigma^2_{t,F} \circ_r \sigma_{v,G})^{d(1)}_{X_m}| \leq |var(\sigma^2_{t,F} \circ_r \sigma_{v,G})^{d(1)}_{X_m}|$. By Lemma, we get $|var(\sigma^2_{t,F} \circ_r \sigma_{v,G})^{d(1)}_{X_m}| \leq |var(\sigma^2_{t,F})^{d(1)}_{X_m}|$. So $|var(\sigma_{u,H} \circ_r \sigma^2_{t,F} \circ_r \sigma_{v,G})^{d(1)}_{X_m}| \leq |var(t)^{d(1)}_{X_m}|$ which contradicts (1). Hence $\sigma_{t,F} \neq \sigma_{u,H} \circ_r \sigma^2_{t,F} \circ_r \sigma_{v,G}$.
Case 2: There exists at least one element of $var(t)$ which is not an element of the set $var(t)^{d(1)}_{X_m}$, say $x^{(l)}_{a_i}$ where $x^{(l)}_{a_i}$ is a variable x_{a_i} occurring in the l^{th} component of t (from the left). Let $x^{(l_v)}_{a_i}$ be a variable x_{a_i} occurring in the l^{th}_v component of v (from the left), $x^{(l_v,l_{\tilde{w}})}_{a_i}$ be a variable $x^{(l_v)}_{a_i}$ occurring in the $l^{th}_{\tilde{w}}$ component of $\tilde{w}$ (from the left) and $x^{(l_v,l_{\tilde{w}},l_z)}_{a_i}$ be a variable $x^{(l_v,l_{\tilde{w}})}_{a_i}$ occurring in the l^{th}_z component of z (from the left). There exist $x_{i_1}, \ldots, x_{i_r} \in var(w)$ such that

$$depth^{\tilde{w}}(x^{(l_v,l_{\tilde{w}})}_{a_i}) = depth^w(x^{(b_1)}_{d_1}) + \cdots + depth^w(x^{(b_r)}_{d_r}), \qquad (3.2)$$

where $x^{(b_k)}_{d_k}$ is a variable x_{d_k} occurring in the b^{th}_k component of w (from the left). Since $x^{(b_1)}_{d_1} \in var(w) \backslash var(w)^{d(1)}_{X_m}$ and $w = \hat{\sigma}_{t,F}[t]$, by Corollary 1, we have $depth^w(x^{(b_1)}_{d_1}) > depth^t(x^{(l)}_{a_i})$. By (2), $depth^{\tilde{w}}(x^{(l_v,l_{\tilde{w}})}_{a_i}) > depth^t(x^{(l)}_{a_i})$. So $depth^z(x^{(l_v,l_{\tilde{w}},l_z)}_{a_i}) \geq depth^z(x^{(l_v,l_{\tilde{w}})}_{a_i}) > depth^t(x^{(l)}_{a_i})$ which contradicts (1). Hence $\sigma_{t,F} \neq \sigma_{u,H} \circ_r \sigma^2_{t,F} \circ_r \sigma_{v,G}$. The proofs of cases 3 and 4 are similar to cases 1 and 2, respectively. Therefore, $\sigma_{t,F} \in CR(R_t)$. Conversely, since the set of all completely regular elements is a subset of the set of all intra-regular elements, $\sigma_{t,F}$ is intra-regular. □

The following results now follow easily.

Corollary 3.11. *$CR(Relhyp((m),(n)))$ is the set of all intra-regular elements in $Relhyp((m),(n))$.*

Theorem 3.12. *Let $\sigma_{t,F} \in Relhyp((m),(n))$. Then the following statement are equivalent:*

(i) $\sigma_{t,F}$ is completely regular in $Relhyp((m),(n))$;
(i) $\sigma_{t,F}$ is left regular in $Relhyp((m),(n))$;
(i) $\sigma_{t,F}$ is right regular in $Relhyp((m),(n))$;
(i) $\sigma_{t,F}$ is intra-regular in $Relhyp((m),(n))$.

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