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On permuting n-$(f, g)-$derivations of lattices

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Abstract

In this paper, we establish a generalization of derivation on a lattice; namely, permuting n-(f, g)-derivation of a lattice. Moreover, we introduce the concept of trace of permuting n-(f, g)-derivation of a lattice. Furthermore, we discuss some related properties.

1 Introduction

The notion of derivations, introduced from the analytic theory, is helpful for studying algebraic structures and properties in algebraic systems. In fact, the notion of derivations in ring theory is quite old and plays a significant role in algebraic geometry. First, Posner [2] introduced the notion of a derivation on a prime ring R as a function d from R into itself satisfying the following two conditions:

$d(xy) = d(x)y + xd(y)$ and $d(x + y) = d(x) + d(y)$ for all $x, y \in R$.

Based on this, a number of research articles have appeared on derivations in the theory of rings; for example, Albas [1], Bell and Mason [5], Bell and Kappe [6], Wang et al. [7]. Later on, Szasz [4] extended the notion of derivation to the lattice structures based on the meet ($\wedge$) and the join ($\vee$) operations. A derivation on a given lattice L with respect to meet ($\wedge$) and

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the join ($\vee$) operations is a function $d : L \to L$ satisfying the following two conditions:
$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$ and $d(x \vee y) = d(x) \vee d(y)$ for all $x, y \in L$. After that, Ferrari [9] and Xin et al. [15] investigated some properties of this notion of derivation on lattices and gave some characterizations of them. Thereafter, many researchers studied derivations and generalizations of derivations on lattices, such as f-derivation, symmetric bi-derivations, symmetric bi(f, g)derivations, permuting triderivations and permuting tri(f, g)derivations, and discussed some related properties; for example, Ceven and Ozturk [18], Harmaitree and Leerawat [13], Leerawat and Harmaitree [14], Balogun [3], Ceven [16], Ozturk et al. [10], Kim and Lee [8] and Asci et al. [11]. Recently, Ceven [17] introduced the notion of n-derivation, as a generalization of derivations on a lattice and the concept of (n, m)-derivation-homomorphism on lattices. Motivated and inspired by the above results, we establish a generalization of derivation on a lattice; namely, permuting n-(f, g)-derivation of a lattice. Moreover, we introduce the concept of trace of permuting n-(f, g)-derivation of a lattice. Furthermore, we discuss some related properties.

2 Preliminaries

First, we give some basic definitions and some results used throughout this paper. Details and proofs can be found in [12], [13] and [15].

Definition 2.1.[12] A lattice $(L,\wedge,\vee)$ is a nonempty set L with two binary operation $\wedge$ and $\vee$ on L which satisfy the following conditions for all $x, y, z \in L$:

(1) $x \wedge x = x$, $x \vee x = x$
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$
(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$.

In what follows, we denote by L a lattice $(L,\wedge,\vee)$, unless otherwise specified.

Lemma 2.2. Let L be a lattice. Then, for any $x, y \in L$, $x \wedge y = x$ if and only if $x \vee y = y$.

Proof. Let $x, y \in L$ and assume $x \wedge y = x$. Then $x \vee y = (x \wedge y) \vee y = y$. Conversely, let $x \vee y = y$. So $x \wedge y = x \wedge (x \vee y) = x$.

Definition 2.3. Let $(L,\wedge,\vee)$ be a lattice. A binary operation $\leqslant$ is defined by $x \leqslant y$ if and only if $x \wedge y = x$.

Lemma 2.4.[15] Let L be a lattice. Consider the binary operation $\leqslant$ as in Definition 2.3. Then $(L,\leqslant)$ is a poset and, for any $x, y \in L$, $x \wedge y$ is the greatest lower bound of $\{x, y\}$ (or $\inf\{x, y\}$) and $x \vee y$ is the least upper bound of $\{x, y\}$ (or $\sup\{x, y\}$).

Definition 2.5.[12] A poset $(L,\leqslant)$ is an ordered lattice if and only if for, every pair x, y of elements of L, both the $\sup\{x, y\}$ and the $\inf\{x, y\}$ exist.

Theorem 2.6.[12]

(1) Let $(L,\leqslant)$ be an ordered lattice. If we define $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$, then $(L, \wedge, \vee)$ is a lattice.

(2) Let $(L, \wedge, \vee)$ be a lattice. If we define $x \leqslant y$ if and only if $x \wedge y = x$ (or $x \leqslant y$ if and only if $x \vee y = y$), then $(L,\leqslant)$ is a lattice ordered set.
One can verify that Theorem 2.6 yields a one-to-one relationship between ordered lattice and lattices. Therefore, we shall use the term lattice for both concepts.

Theorem 2.7.[12]

(1) Every ordered set is a lattice.

(2) In a lattice $(L,\leqslant)$, for all $x, y \in L$, the following statements are equivalent:
(a) $x \leqslant y$; (b) $\sup\{x, y\} = y$; (c) $\inf\{x, y\} = x$.

Definition 2.8.[12] If a lattice L contains a least (or greatest) element with respect to $\leqslant$, then this uniquely determined element is called the zero element, denoted by 0 (or 1).

Lemma 2.9.[12] Let L be a lattice. If $y \leqslant z$, then $x \wedge y \leqslant x \wedge z$ and $x \vee y \leqslant x \vee z$ for all $x, y, z \in L$.

Definition 2.10.[12] A nonempty subset S of a lattice L is called a sublattice of L if S is a lattice with respect to the restriction of $\wedge$ and $\vee$ of L onto S.

Definition 2.11.[12] A lattice L is called modular if, for any $x, y, z \in L$, if $x \leqslant z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Definition 2.12.[12] A lattice L is called distributive if the identity (1) or (2) holds for all $x, y, z \in L$:

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

In any lattice, the conditions (1) and (2) are equivalent.

Corollary 2.13.[12] Every distributive lattice is a modular lattice.

Definition 2.14.[12] Let $f : L \to M$ be a function from a lattice L to a lattice M.

(1) f is called a join-homomorphism if $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$.

(2) f is called a meet-homomorphism if $f(x \wedge y) = f(x) \vee f(y)$ for all $x, y \in L$.

(3) f is called a lattice-homomorphism if f is both a join-homomorphism and a meet- homomorphism.

(4) f is called an order-preserving if $x \leqslant y$ implies $f(x) \leqslant f(y)$ for all $x, y \in L$.

3 Permuting n-(f, g)-derivations of lattices

From now on, let L denote a lattice and $f, g : L \to L$ be functions. Let n be a fixed positive integer and L^n denote $L \times L \times \cdots \times L$ (n terms).

Definition 3.1. A mapping $D : L^n \to L$ is said to be permuting if the relation

$$D(x_1, x_2, \ldots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$$

holds for all $x_i \in L$ and for every permutation $\pi \in S_n$, where S_n is the permutation group on $\{1, 2, \ldots, n\}$.

Definition 3.2. Let $D : L^n \to L$ be a permuting mapping. A mapping $d : L \to L$ defined by $d(x) = D(x, x, \ldots, x)$ for all $x \in L$ is called the trace of D.

Definition 3.3. A mapping $D : L^n \to L$ is called an n-join-homomorphism of L if D satisfies the following conditions:

$$\begin{aligned}
D(x_1 \vee y, x_2, \ldots, x_n) &= D(x_1, x_2, \ldots, x_n) \vee D(y, x_2, \ldots, x_n) \\
D(x_1, x_2 \vee y, \ldots, x_n) &= D(x_1, x_2, \ldots, x_n) \vee D(x_1, y, \ldots, x_n) \\
&\vdots \\
D(x_1, x_2, \ldots, x_n \vee y) &= D(x_1, x_2, \ldots, x_n) \vee D(x_1, x_2, \ldots, y)
\end{aligned}$$

for all $x_1, x_2, \ldots, x_n, y \in L$.

.

Definition 3.4. A mapping $D : L^n \to L$ is called an $n-(f,g)-$derivation of L if D is an n-join-homomorphism of L and satisfies the following conditions:

$$
\begin{aligned}
D(x_1 \wedge y, x_2, \ldots, x_n) &= (D(x_1, x_2, \ldots, x_n) \wedge f(y)) \vee (g(x_1) \wedge D(y, x_2, \ldots, x_n)) \\
D(x_1, x_2 \wedge y, \ldots, x_n) &= (D(x_1, x_2, \ldots, x_n) \wedge f(y)) \vee (g(x_2) \wedge D(x_1, y, \ldots, x_n)) \\
&\vdots \\
D(x_1, x_2, \ldots, x_n \wedge y) &= (D(x_1, x_2, \ldots, x_n) \wedge f(y)) \vee (g(x_n) \wedge D(x_1, x_2, \ldots, y))
\end{aligned}
$$

for all $x_1, x_2, \ldots, x_n, y \in L$.

Definition 3.5. A permuting mapping $D : L^n \to L$ is called a permuting $n-(f,g)-$derivation of L if D satisfies the following conditions:

$$
\begin{aligned}
&D(x_1 \vee y, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n) \vee D(y, x_2, \ldots, x_n) \quad \text{and} \\
&D(x_1 \wedge y, x_2, \ldots, x_n) = (D(x_1, x_2, \ldots, x_n) \wedge f(y)) \vee (g(x_1) \wedge D(y, x_2, \ldots, x_n))
\end{aligned}
$$

for all $x_1, x_2, \ldots, x_n, y \in L$.

Example 3.6. Let L={0,a,1}. Define operations $\wedge$ and $\vee$ on L as follows:

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

and

$\vee$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

Then it can be easily verified that (L,$\wedge$,$\vee$) is lattice.

Consider functions $f, g : L \to L$. Define a function $D : L^3 \to L$ by $D(x, y, z) = min\{x, y, z\}$ for all $x, y, z \in L$,

$$
f(x) = \begin{cases} 0 & \text{if } x = 0 \\ a & \text{if } x = a, 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } x = 0, a \\ 1 & \text{if } x = 1. \end{cases}
$$

Then it can be easily verified that D is a permuting 3-(f,g)-derivation on L.

Example 3.7. Let L be the lattice as in Example 3.6.

Define a function $D : L^3 \to L$, and function $f, g : L \to L$ by

$D(x, y, z) = max\{x, y, z\}$ for all $x, y, z \in L$

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ a & \text{if } x = a, 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ a & \text{if } x = a \\ 0 & \text{if } x = 1 \end{cases}$$

D is not a permuting 3-(f, g)-derivation since
$1 = max\{a, 1, 0\} = D(a, 1, 0) = D(1 \wedge a, 1, 0) \neq (D(1, 1, 0) \wedge f(a)) \vee (g(1) \wedge D(a, 1, 0)) = (max\{1, 1, 0\} \wedge a) \vee (0 \wedge max\{a, 1, 0\}) = (1 \wedge a) \vee (0 \wedge 1) = a \vee 0 = a$.

Proposition 3.8. Let D be an $n-(f, g)-$derivation of L with trace d. Then $d(x) \leqslant f(x) \vee g(x)$ for all $x \in L$.

Proof. Let $x \in L$. Then

$$\begin{aligned} d(x) &= D(x, x, \ldots, x) \\ &= D(x \wedge x, x, \ldots, x) \\ &= (D(x, x, \ldots, x) \wedge f(x)) \vee (g(x) \wedge D(x, x, \ldots, x)) \\ &= (d(x) \wedge f(x)) \vee (g(x) \wedge d(x)). \end{aligned}$$

Since $d(x) \wedge f(x) \leq f(x)$ and $g(x) \wedge d(x) \leq g(x)$,
we get $d(x) \leqslant f(x) \vee g(x)$. $\square$

Proposition 3.9. Let D be an $n - (f, g)-$derivation of L. Then

$$\begin{aligned} D(x_1, x_2, \ldots, x_n) &\leqslant f(x_1) \vee g(x_1) \\ D(x_1, x_2, \ldots, x_n) &\leqslant f(x_2) \vee g(x_2) \\ &\vdots \\ D(x_1, x_2, \ldots, x_n) &\leqslant f(x_n) \vee g(x_n) \end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in L$.

Proof. Let $x_1, x_2, \ldots, x_n \in L$. Then

$$\begin{aligned} D(x_1, x_2, \ldots, x_n) &= D(x_1 \wedge x_1, x_2, \ldots, x_n) \\ &= (D(x_1, x_2, \ldots, x_n) \wedge f(x_1)) \vee (g(x_1) \wedge D(x_1, x_2, \ldots, x_n)). \end{aligned}$$

Since $D(x_1, x_2, \ldots, x_n) \wedge f(x_1) \leq f(x_1)$ and
$g(x_1) \wedge D(x_1, x_2, \ldots, x_n) \leq g(x_1)$, we get $D(x_1, x_2, \ldots, x_n) \leq f(x_1) \vee g(x_1)$.

Similarly, $D(x_1, x_2, \ldots, x_n) \leqslant f(x_i) \vee g(x_i)$ for all $i = 2, 3, \ldots, n$. $\square$

Corollary 3.10. Let D be an $n-(f,g)-$derivation of L. If $g(x) \leqslant f(x)$ for all $x \in L$, then

$$\begin{aligned} D(x_1, x_2, \ldots, x_n) &\leqslant f(x_1) \\ D(x_1, x_2, \ldots, x_n) &\leqslant f(x_2) \\ &\vdots \\ D(x_1, x_2, \ldots, x_n) &\leqslant f(x_n) \end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in L$.

Proposition 3.11. Let D be an $n-(f,g)-$derivation of L. If L has a least element 0 such that $f(0) = 0 = g(0)$, then

$$\begin{aligned} D(0, x_2, \ldots, x_n) &= 0 \\ D(x_1, 0, \ldots, x_n) &= 0 \\ &\vdots \\ D(x_1, x_2, \ldots, 0) &= 0 \end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in L$.

Proof. Let $x_2, \ldots, x_n \in L$. Then
By Proposition 3.9, we have $D(0, x_2, \ldots, x_n) \leqslant f(0) \vee g(0) = 0$.
Since 0 is the least element, $D(0, x_2, \ldots, x_n) = 0$.
Similarly, $D(x_1, 0, \ldots, x_n) = 0, \ldots, D(x_1, x_2, \ldots, 0) = 0$. □

Proposition 3.12. Let D be a permuting $n-(f,g)-$derivation of L. If $g(x) \leq f(x)$ for all $x \in L$, then, for all $x_1, x_2, \ldots, x_n, y \in L$,

$$\begin{aligned} D(x_1, x_2, \ldots, x_n) \wedge D(y, x_2, \ldots, x_n) &\leqslant D(x_1 \wedge y, x_2, \ldots, x_n) \\ &\leqslant D(x_1, x_2, \ldots, x_n) \vee D(y, x_2, \ldots, x_n). \end{aligned}$$

Proof. Let $x_1, x_2, \ldots, x_n, y \in L$. Then
$D(x_1 \wedge y, x_2, \ldots, x_n) = (D(x_1, x_2, \ldots, x_n) \wedge f(y)) \vee (g(x_1) \wedge D(y, x_2, \ldots, x_n))$,
which implies $D(x_1, x_2, \ldots, x_n) \wedge f(y) \leqslant D(x_1 \wedge y, x_2, \ldots, x_n)$.
Since $D(y, x_2, \ldots, x_n) \leqslant f(y)$, and
$D(x_1, x_2, \ldots, x_n) \wedge D(y, x_2, \ldots, x_n) \leqslant D(x_1, x_2, \ldots, x_n) \wedge f(y)$,
we have $D(x_1, x_2, \ldots, x_n) \wedge D(y, x_2, \ldots, x_n) \leqslant D(x_1 \wedge y, x_2, \ldots, x_n)$.
Since $D(x_1, x_2, \ldots, x_n) \wedge f(y) \leqslant D(x_1, x_2, \ldots, x_n)$ and
$g(x_1) \wedge D(y, x_2, \ldots, x_n) \leqslant D(y, x_2, \ldots, x_n)$, we get

$(D(x_1, x_2, \ldots, x_n) \wedge f(y)) \vee (g(x_1) \wedge D(y, x_2, \ldots, x_n)) \leqslant D(x_1, x_2, \ldots, x_n) \vee D(y, x_2, \ldots, x_n)$.
Hence $D(x_1 \wedge y, x_2, \ldots, x_n)) \leqslant D(x_1, x_2, \ldots, x_n) \vee D(y, x_2, \ldots, x_n)$. □

Proposition 3.13. Let D be a permuting $n-(f,g)-$derivation of L. If $g(x) \leqslant f(x)$ for all $x \in L$, then $D(x_1 \wedge y, x_2, \ldots, x_n) \leqslant f(x_1) \vee f(y)$ for all $x_1, x_2, \ldots, x_n, y \in L$.

Proof. Let $x_1, x_2, \ldots, x_n, y \in L$. Then
$D(x_1 \wedge y, x_2, \ldots, x_n) = (D(x_1, x_2, \ldots, x_n) \wedge f(y)) \vee (g(x_1) \wedge D(y, x_2, \ldots, x_n))$.
Since $D(x_1, x_2, \ldots, x_n) \wedge f(y) \leqslant f(y)$ and
$g(x_1) \wedge D(y, x_2, \ldots, x_n) \leqslant f(x_1) \wedge D(y, x_2, \ldots, x_n) \leqslant f(x_1)$, we get
$(D(x_1, x_2, \ldots, x_n) \wedge f(y)) \vee (g(x_1) \wedge D(y, x_2, \ldots, x_n)) \leqslant f(x_1) \vee f(y)$.
Hence $D(x_1 \wedge y, x_2, \ldots, x_n) \leqslant f(x_1) \vee f(y)$. □

Proposition 3.14. Let D be a permuting $n-(f,g)-$derivation of L. If f, g are order-preserving, then

(1) $D(x_1 \wedge y, x_2, \ldots, x_n) \leqslant f(x_1) \vee g(y)$
(2) $D(x_1 \wedge y, x_2, \ldots, x_n) \leqslant f(y) \vee g(x_1)$
(3) $D(x_1 \wedge y, x_2, \ldots, x_n) \leqslant f(x_1) \vee g(x_1)$
(4) $D(x_1 \wedge y, x_2, \ldots, x_n) \leqslant f(y) \vee g(y)$

for all $x_1, x_2, \ldots, x_n, y \in L$.

Proof. Let $x_1, x_2, \ldots, x_n, y \in L$.

(1) By Proposition 3.9, we get $D(x_1 \wedge y, x_2, \ldots, x_n) \leqslant f(x_1 \wedge y) \vee g(x_1 \wedge y)$.
Since $x_1 \wedge y \leqslant x_1, x_1 \wedge y \leqslant y$ and f, g are order-preserving, we have
$f(x_1 \wedge y) \leqslant f(x_1)$ and $g(x_1 \wedge y) \leqslant g(y)$. Hence
$f(x_1 \wedge y) \vee g(x_1 \wedge y) \leqslant f(x)_1 \vee g(y)$.

(2)-(4) Similar. □

Proposition 3.15. Let D be a permuting $n-(f,g)-$derivation of L. If L has a greatest element 1 such that $f(1) = 1 = g(1)$, then the following conditions hold for all $x_1, x_2, \ldots, x_n \in L$.

(1) If $f(x_1) \leqslant D(1, x_2, \ldots, x_n)$ and $g(x_1) \leqslant D(1, x_2, \ldots, x_n)$, then $D(x_1, x_2, \ldots, x_n) = f(x_1) \vee g(x_1)$.
(2) If $D(1, x_2, \ldots, x_n) \leqslant f(x_1)$ or $D(1, x_2, \ldots, x_n) \leqslant g(x_1)$ then $D(1, x_2, \ldots, x_n) \leqslant D(x_1, x_2, \ldots, x_n)$.

Proof.

(1) Let $x_1, x_2, \ldots, x_n \in L$ and assume that $f(x_1) \leqslant D(1, x_2, \ldots, x_n)$ and $g(x_1) \leqslant D(1, x_2, \ldots, x_n)$. Then

$$\begin{aligned} D(x_1, x_2, \ldots, x_n) &= D(x_1 \wedge 1, x_2, \ldots, x_n) \\ &= (D(x_1, x_2, \ldots, x_n) \wedge f(1)) \vee (g(x_1) \wedge D(1, x_2, \ldots, x_n)) \\ &= (D(x_1, x_2, \ldots, x_n) \wedge 1) \vee (g(x_1) \wedge D(1, x_2, \ldots, x_n) \\ &= D(x_1, x_2, \ldots, x_n) \vee g(x_1). \end{aligned}$$

Hence $\quad g(x_1) \leqslant D(x_1, x_2, \ldots, x_n)$. (3.15.1)

Similarly, by the assumption and $1 \wedge x_1 = x_1$, we get

$$\begin{aligned} D(x_1, x_2, \ldots, x_n) &= D(1 \wedge x_1, x_2, \ldots, x_n) \\ &= (D(1, x_2, \ldots, x_n) \wedge f(x_1)) \vee (g(1) \wedge D(x_1, x_2, \ldots, x_n)) \\ &= f(x_1) \vee D(x_1, x_2, \ldots, x_n) \end{aligned}$$

Hence $\quad f(x_1) \leqslant D(x_1, x_2, \ldots, x_n)$. (3.15.2)

From (3.15.1) and (3.15.2), we have $f(x_1) \vee g(x_1) \leqslant D(x_1, x_2, \ldots, x_n)$.
By Proposition 3.9, we have $D(x_1, x_2, \ldots, x_n) \leqslant f(x_1) \vee g(x_1)$.
Combining the above two inequalities, we have
$f(x_1) \vee g(x_1) \leqslant D(x_1, x_2, \ldots, x_n) \leqslant f(x_1) \vee g(x_1)$,
this implies $D(x_1, x_2, \ldots, x_n) = f(x_1) \vee g(x_1)$.

(2) Let $x_1, x_2, \ldots, x_n \in L$ and assume that $D(1, x_2, , x_n) \leqslant f(x_1)$. Then

$$\begin{aligned} D(x_1, x_2, \ldots, x_n) &= D(1 \wedge x_1, x_2, \ldots, x_n) \\ &= (D(1, x_2, \ldots, x_n) \wedge f(x_1)) \vee (g(1) \wedge D(x_1, x_2, \ldots, x_n) \\ &= D(1, x_2, \ldots, x_n) \vee D(x_1, x_2, \ldots, x_n). \end{aligned}$$

We have $D(1, x_2, \ldots, x_n) \leqslant D(x_1, x_2, \ldots, x_n)$.
Similarly, if $D(1, x_2, \ldots, x_n) \leqslant g(x_1)$ *for all* $x_1, x_2, \ldots, x_n \in L$, then $D(1, x_2, \ldots, x_n) \leqslant D(x_1, x_2, \ldots, x_n)$. □

Now, we state and prove our main results. In order to fix the framework needed to state our main results, we recall the following notions. For simplicity, we denote $D(x^{(n-k)}, y^{(k)})$ by $D(\underbrace{x, x, \ldots, x}_{n-k \text{ copies}}, \underbrace{y, y, \ldots, y}_{k \text{ copies}})$, where $k = 1, 2, 3, \ldots, n-1$, $x, y \in L$, and D is a permuting n-(f,g)-derivation of L.

Theorem 3.16. Let D be a permuting $n-(f,g)-$derivation of L and let d be the trace of D. Then, for all $x, y \in L$,

$$d(x \vee y) = d(x) \vee d(y) \vee [D(x^{(n-1)}), y) \vee D(x^{(n-2)}), y^{(2)}) \vee ... \vee D(x, y^{(n-1)})].$$

Proof. Since D is a permuting $n-(f,g)-$derivation of L, we have

$$\begin{aligned} d(x \vee y) &= D(x \vee y, x \vee y, \ldots, x \vee y) \\ &= D(x, (x \vee y)^{(n-1)})) \vee D(y, (x \vee y)^{(n-1)}) \\ &= D(x, x, (x \vee y)^{(n-2)}) \vee D(x, y, (x \vee y)^{(n-2)}) \vee D(y, y, (x \vee y)^{(n-2)})) \\ &= D(x, x, x, (x \vee y)^{(n-3)}) \vee D(x, x, y, (x \vee y)^{(n-3)}) \vee D(x, y, y, (x \vee y)^{(n-3)}) \\ &\vee D(y, y, y, (x \vee y)^{(n-3)}) \\ &\quad \vdots \\ &= D(x, x, \ldots, x) \vee [D(x, y^{(n-1)}) \vee D(x^{(2)}, y^{(n-2)}) \vee \cdots \vee D(x^{(n-1)}, y)] \\ &\vee D(y, y, \ldots, y) \\ &= d(x) \vee d(y) \vee [D(x, y^{(n-1)}) \vee D(x^{(2)}, y^{(n-2)}) \vee \cdots \vee D(x^{(n-1)}, y)]. \end{aligned}$$

This completes the proof. □

Theorem 3.17. Let L be a distributive lattice and D be a permuting $n-(f,g)-$derivation of L with the trace d. Then, for all $x, y \in L$,

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (g(x) \wedge d(y))$$

$$\vee [(g(x) \wedge f(y)) \wedge [D(x^{(n-1)}, y) \vee D(x^{(n-2)}, y^{(2)}) \vee \cdots \vee D(x, y^{(n-1)})]],$$

Proof. Since D is a permuting $n-(f,g)-$derivation of L, we have

$$\begin{aligned}
d(x\wedge y) &= D(x\wedge y, x\wedge y, \ldots, x\wedge y)\\
&= [D(x,(x\wedge y)^{(n-1)})\wedge f(y)]\vee[g(x)\wedge D(y,(x\wedge y)^{(n-1)})]\\
&= [[D(x,x,(x\wedge y)^{(n-2)}))\wedge f(y)]\vee[g(x)\wedge D(x,y,(x\wedge y)^{(n-2)})]\wedge f(y)]\\
&\quad\vee[g(x)\wedge[[D(y,x,(x\wedge y)^{(n-2)})\wedge f(y)]\vee[g(x)\wedge D(y,y,(x\wedge y)^{(n-2)})]]]\\
&= [D(x,x,(x\wedge y)^{(n-2)})\wedge f(y)]\vee[g(x)\wedge f(y)\wedge D(x,y,(x\wedge y)^{(n-2)})]\\
&\quad\vee[g(x)\wedge D(y,y,(x\wedge y)^{(n-2)})]\\
&= [D(x,x,x,(x\wedge y)^{(n-3))})\wedge f(y)]\vee[g(x)\wedge f(y)\wedge D(x,x,y,(x\wedge y)^{(n-3)})]\\
&\quad\vee[(g(x)\wedge f(y)\wedge D(x,y,y,(x\wedge y)^{(n-3)}))]\\
&\quad\vee[g(x)\wedge D(y,y,y,(x\wedge y)^{(n-3))})]\\
&\qquad\qquad\vdots\\
&= [D(x,x,x,\ldots,x)\wedge f(y)]\vee[g(x)\wedge f(y)\wedge D(x^{(n-1)},y)]\\
&\quad\vee[g(x)\wedge f(y)\wedge D(x^{(n-2)},y^{(2)})]\\
&\quad\vee[g(x)\wedge f(y)\wedge D(x^{(n-3)},y^{(3)})]\\
&\qquad\qquad\vdots\\
&\quad\vee[g(x)\wedge f(y)\wedge D(x,y^{(n-1)})]\\
&\quad\vee[g(x)\wedge D(y,y,y,\ldots,y)]\\
&= (d(x)\wedge f(y))\vee(g(x)\wedge d(y))\\
&\quad\vee[(g(x)\wedge f(y))\wedge[D(x^{(n-1)},y)\vee D(x^{(n-2)},y^{(2)})\vee\cdots\vee D(x,y^{(n-1)})]].
\end{aligned}$$

This completes the proof. □

The following results are immediate from Theorems 3.16 and 3.17.

Corollary 3.18. Let L be a distributive lattice and D be a permuting $n-(f,g)-$derivation of L with the trace d. Then for all $x,y\in L$,

(1) $d(x)\vee d(y)\leqslant d(x\vee y)$.

(2) $g(x)\wedge f(y)\wedge[D(x^{(n-1)},y)\vee D(x^{(n-2)},y^{(2)})\vee\cdots\vee D(x,y^{(n-1)})]\leqslant d(x\wedge y)$.

(3) $g(x)\wedge d(y)\leqslant d(x\wedge y)$.

(4) $d(x)\wedge f(y)\leqslant d(x\wedge y)$.

Theorem 3.19. Let L be a distributive lattice and D be a permuting $n-(f,g)-$derivation of L with the trace d. Then d is order-preserving.

Proof. Let $x, y \in L$ be such that $x \leqslant y$. Then,
by Corollary 3.18, $d(x) \leqslant d(x) \vee d(y) \leqslant d(x \vee y) = d(y)$.
This implies $d(x) \leqslant d(y)$. So d is order-preserving. □

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