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Constructing Supereulerian Digraphs by Control Function of Arc-Connectivity and Matching Number

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Abstract

A digraph D is supereulerian if D has a spanning connected eulerian subdigraph. In this paper, we construct a supereulerian digraph controlling its matching number, vertices, arcs and arc-connectivity.

1 Introduction

We consider finite and simple digraphs. Usually, we use G to denote a graph and D to denote a digraph. Undefined terms and notations will follow [6]. In particular, $\alpha'(D)$ denotes the matching number of a digraph D and $\lambda(D)$ denotes the arc-strong connectivity of a digraph D. Throughout this paper, we use (u, v) to denote an arc oriented from u to v in a digraph and use $[u, v]$ to denote either (u, v) or (v, u) such that $[u, v]$ represents one arc in $A(D)$. When $[u, v] \in A(D)$, we say that u and v are adjacent.

If D is a digraph, then we often use $G(D)$ to denote the underlying undirected graph of D, the graph obtained from D by erasing all orientations on the arcs of D. The matching number of a digraph D is defined as

$$\alpha'(D) = \alpha'(G(D)).$$

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For a digraph D with $X, Y \subseteq V(D)$, define

$$(X,Y)_D = \{(x,y) \in A(D) \ : \ x \in X, y \in Y\}.$$

When $Y = V(D) - X$, we define

$$\partial_D^+(X) = (X, V(D) - X)_D \text{ and } \partial_D^-(X) = (V(D) - X, X)_D.$$

For a vertex $v \in V(D)$, $d_D^+(v) = |\partial_D^+(\{v\})|$ and $d_D^-(v) = |\partial_D^-(\{v\})|$ are the out-degree and the in-degree of v in D, respectively. For any vertex $v \in V(D)$, define

$$\partial_D(v) = \partial_D^+(v) \cup \partial_D^-(v), d_D(v) = d_D^+(v) + d_D^-(v) \text{ and } d_D^*(v) = min\{d_D^+(v), d_D^-(v)\}.$$

By the definition of $\lambda(D)$ in [6], we note that for any integer $k \geq 0$ and a digraph D,

$$\lambda(D) \geq k \text{ if and only if for any nonempty proper subset } X \subset V(D), |\partial_D^+(X)| \geq k. \tag{1.1}$$

In 1977, motivated by the Chinese Postman Problem, Boesch, Suffel, and Tindell [7] proposed the supereulerian problem, which seeks to characterize graphs that have spanning Eulerian subgraphs, and they [7] indicated that this problem would be very difficult. Later, in 1979, Pulleyblank [13] proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been considerable research on this topic. In 1992, Catlin [8] presented the first survey on supereulerian graphs. Later, in 1995, Chen et al. [9] gave an update specifically on the reduction method associated with the supereulerian problem. A recent survey on supereulerian graphs was given in [12].

It is natural to consider the supereulerian problem in digraphs. A strong digraph D is **eulerian** if for any $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. A strong-connected digraph D is **supereulerian** if D contains a spanning eulerian subdigraph. The main problem is to determine supereulerian digraphs. Several efforts have been made in this direction. The earlier studies were done by Gutin ([10, 11]). More studies can be found in ([1, 2, 3, 4]), among others.

In [2], Algefari and Lai discussed supereulerianicity of a digraph D when $\lambda(D) \geq \alpha'(D)$ by proving the following result:

Theorem 1.1. *(Algefari and Lai [2]) Let D be a digraph. If $\lambda(D) \geq \alpha'(D)$, then D is supereulerian.*

It is natural to consider whether we can use $\lambda(D) < \alpha'(D)$ to replace $\lambda(D) \geq \alpha'(D)$ in predicting supereulerianicity of a digraph.

These motivate the current research. A natural question is:
If we subdue D to some features would it warrant supereulerianicity when $d^*_D < \alpha'(D)$, where $d^*_D = \min\{d^*_D(v) : v \in V(D)\}$, which is slightly stronger than $\lambda(D) < \alpha'(D)$?

The main problem in this paper is to discuss connectivity and supereulerianicity of digraphs. In section 3, we will discuss connectivity of a featured digraph, which is our first main result, and supereulerianicity of this featured digraph, which is our second main result.

In the next section, we present some preliminaries and define the digraph which will be under discussion as well as prove an associated result. This will be utilized in Section 3 to justify the main results. In Section 4, we shall display an example of nonsupereulerian digraph to show that our second main result is sharp.

2 Preliminaries

Throughout the rest of this section, D always denotes a simple digraph under discussion.

When D is understood from the context, we often use $d(v)$ for $d_D(v)$. If $X \subseteq V(D)$, then $D[X]$ denotes the subdigraph induced by X. We say that r is a partner of $[u, v]$ if $\{u, v\} \subseteq N(r)$. Let M be a matching in D. We use $V(M)$ to denote the vertices of M.

Now, we construct a digraph with some features.

Definition 2.1. *Let D be a digraph with:*
(1) a maximum matching $M = \{[u_1, v_1], ..., [u_m, v_m], [\overline{u_1}, \overline{v_1}], ..., [\overline{u_{k-m}}, \overline{v_{k-m}}]\}$, $1 \leq m \leq k-1$, $k \geq 2$.
(2) $C = \{u_1, u_2, ..., u_m\}$.
(3) $M_C = \{[u_1, v_1], ..., [u_m, v_m]\}$.
(4) $\overline{M_C} = \{[\overline{u_1}, \overline{v_1}], ..., [\overline{u_{k-m}}, \overline{v_{k-m}}]\}$.
(5) $R = \{r_l : r_l$ *is a partner of at least one arc of* $\overline{M_C},\ 1 \leq l \leq h\}$, $h \geq 1$.
(6) $X = \{x : x \in V(D) - V(M) \cup R\}$, $|X| \geq 2$.
(7) $Y = \{v_i : v_i \in V(M_C) - C,\ 1 \leq i \leq m\}$.
(8) $|V(D)| = n \geq 2k + h + 2$.
Then we call D an m-central matching digraph if:

(1) $\forall\ v \in X \cup Y$, *then* $C \subseteq N(v)$.
(2) $\forall\ [\overline{u_i}, \overline{v_i}] \in \overline{M_C}$, $\exists\ r_j \in R$ *such that* r_j *is a partner of* $[\overline{u_i}, \overline{v_i}]$.

Note that there might be more than one member in $\{[\overline{u_1}, \overline{v_1}], ..., [\overline{u_{k-m}}, \overline{v_{k-m}}]\}$ share the same partner in R. In addition, as we will prove in the next lemma, any member of the previous arc set has only one partner in R.
We call the matching $M_C = \{[u_1, v_1], [u_2, v_2], ..., [u_m, v_m]\}$ in the previous definition a central matching.
We call the matching $\overline{M_C} = \{[\overline{u_1}, \overline{v_1}], ..., [\overline{u_{k-m}}, \overline{v_{k-m}}]\}$ in the previous definition a complementary matching.
We call the matching subset of $\overline{M_C}$, which contains all arcs that share the same partner $r_l \in R$, $1 \leq l \leq h$, partner matching of r_l and denote it as $\overline{M_{C_{r_l}}}$.
So we write $\overline{M_C} = \overline{M_{C_{r_1}}} \cup \overline{M_{C_{r_2}}} \cup ... \cup \overline{M_{C_{r_h}}}$.

We will use the following lemma in the proof of the first main theorem.

Lemma 2.2. *Let D be an m-central matching digraph with $\alpha'(D) = k$, $n \geq 2k + h + 2$ and $|C| = m \geq 1$.*
(1) If $d^*(v) \geq m$, $\forall v \in Y \cup X$, then $d^+(v) = d^-(v) = m$ and $N(v) = C$.
(2) $\forall\ [\overline{u_i}, \overline{v_i}]$ and $[\overline{u_j}, \overline{v_j}]$ with different partners where $1 \leq i, j \leq k - m$ then $((\{\overline{u_i}, \overline{v_i}\}, \{\overline{u_j}, \overline{v_j}\})_D \cup (\{\overline{u_j}, \overline{v_j}\}, \{\overline{u_i}, \overline{v_i}\})_D) \cap A(D) = \phi$.
(3) $\forall\ [\overline{u_s}, \overline{v_s}] \in \overline{M_C}$, $1 \leq s \leq k - m$ $\exists$ only one partner.
(4) $\forall\ r_i \in R$, then $(r_i, V(\overline{M_{C_{r_j}}}))_D \cup (V(\overline{M_{C_{r_j}}}), r_i)_D = \phi\ \forall\ i \neq j$.
(5) $\forall\ r_i, r_j \in R$, then $\{(r_i, r_j), (r_j, r_i)\} \cap A(D) = \phi$ where $i \neq j$.

Proof. (1). (a) Let $v = v_i \in Y$, $i \in \{1, 2, ..., m\}$ where $[u_i, v_i] \in M_C$.
If $d^{+(-)}(v_i) > m$, then
$\exists\ y \in V(D) \setminus \{C \cup \{v_i\}\}$ such that $y \in N^{+(-)}(v_i)$.

(i). $y \in X \cup R$.
Since $|V(D)| \geq 2k+h+2$ and C is central then $\exists$ a vertex $x \in X$, $x \neq y$ such that $\{[x, u_i], [v_i, y]\} \cup M - [u_i, v_i]$ is greater matching than M (contradicts that M is a maximum matching).

(ii). $y = v_j \in Y$, $j \in \{1, 2, ..., m\} \setminus \{i\}$ (we do not need this case when $m = 1$).
Since $|V(D)| \geq 2k + h + 2$ and C is central, $\exists$ two vertices $\{x, \overline{x}\} \subseteq X$ such that $\{[x, u_i], [v_i, v_j], [u_j, \overline{x}]\} \cup M \setminus \{[u_i, v_i], [u_j, v_j]\}$ is greater matching than

M (contradicts that M is a maximum matching).

(iii). $y = \overline{y_s} \in V([\overline{u_s}, \overline{v_s}]), 1 \leq s \leq k - m$.
Since $|V(D)| \geq 2k + h + 2$ and C is central then $\exists\ x \in X$ and $r_l \in R$, $1 \leq l \leq h$, where r_l is the partner of $[\overline{u_s}, \overline{v_s}]$ such that $\{[v_i, \overline{y_s}], [V([\overline{u_s}, \overline{v_s}]) - \overline{y_s}, r_l], [u_i, x]\} \cup M \setminus \{[u_i, v_i], [\overline{u_s}, \overline{v_s}]\}$ is greater matching than M (contradicts that M is a maximum matching).

(b) Let $v = x \in X$.
If $d^{+(-)}(x) > m$, then $\exists$ a vertex $y \in V(D) - C$ such that $y \in N^{+(-)}(x)$.

(i). $y \in X \cup R$, then $M \cup \{[x, y]\}$ is a greater matching.

(ii). $y = v_i \in Y$, $i \in \{1, 2, ..., m\}$. Since $|V(D)| \geq 2k + h + 2$ and C is central $\exists$ a vertex $\overline{x} \in X$, $\overline{x} \neq x$ such that $\{[x, v_i], [u_i, \overline{x}]\} \cup \{M - [u_i, v_i]\}$ is greater matching than M (contradicts that M is a maximum matching).

(iii). $y = \overline{y_s} \in V([\overline{u_s}, \overline{v_s}])$, $1 \leq s \leq k - m$, then $\{[x, \overline{y_s}], [V([\overline{u_s}, \overline{v_s}]) - \overline{y_s}, r_l]\} \cup (M - [\overline{u_s}, \overline{v_s}])$ is a greater matching, where r_l is the partner of $[\overline{u_s}, \overline{v_s}]$.

(2). If $[e, d] \in A(D)$, where $e \in \{\overline{u_i}, \overline{v_i}\}$ and $d \in \{\overline{u_j}, \overline{v_j}\}$, then $\{[e, d], [V([\overline{u_i}, \overline{v_i}]) - e, r_s]$,
$[V([\overline{u_j}, \overline{v_j}]) - d, r_t]\} \cup \{M - \{[\overline{u_i}, \overline{v_i}], [\overline{u_j}, \overline{v_j}]\}\}$ is a greater matching, where r_s and r_t are the partners of $[\overline{u_i}, \overline{v_i}]$ and $[\overline{u_j}, \overline{v_j}]$, respectively.

(3). Let r_1 and r_2 be two partners of $[\overline{u_s}, \overline{v_s}]$. Then $\{[\overline{u_s}, r_1], [\overline{v_s}, r_2]\} \cup (M - [\overline{u_s}, \overline{v_s}])$ is a greater matching.

(4). Let $[\overline{u_s}, \overline{v_s}] \in \overline{M_{c_{r_j}}}$ and $[r_i, \overline{y_s}] \in A(D)$, where $\overline{y_s} \in \{\overline{u_s}, \overline{v_s}\}$. Then $\{[r_i, \overline{y_s}]$,
$[r_j, V([\overline{u_s}, \overline{v_s}]) - \overline{y_s}]\} \cup (M - [\overline{u_s}, \overline{v_s}])$ is a greater matching.

(5). If $[r_i, r_j] \in A(D)$, then $M \cup \{[r_i, r_j]\}$ is greater matching than M. $\square$

3 Main Theorems

Our main results will be proved in this section. First, we will prove the main result of connectivity.

Definition 3.1. *we say $\overline{M_C}$ is an independent partner matching if for any two arcs $[\overline{u_i}, \overline{v_i}], [\overline{u_j}, \overline{v_j}]$ in the same partner matching $\overline{M_{C_{r_l}}}$, $1 \leq l \leq h$ then we have $((\{\overline{u_i}, \overline{v_i}\}, \{\overline{u_j}, \overline{v_j}\})_D \cup (\{\overline{u_j}, \overline{v_j}\}, \{\overline{u_i}, \overline{v_i}\})_D) \cap A(D) = \phi$.*

Definition 3.2. *Let $M_{C_{r_l}} = D[V(\overline{M_{C_{r_l}}}) \cup \{r_l\}]$ (induced subdigraph).*

Theorem 3.3. *Let D be an m-central matching digraph with $\alpha'(D) = k$, $|R| = h$, $n \geq 2k + h + 2$, $m \geq 1$, $d^*(v) \geq m$, $\forall\ v \in X \cup Y$, $d^*(\overline{v}) = d \geq 3$ $\forall$ $\overline{v} \in V(\overline{M_C})$ and $d^*(r_l) = s \geq 1$, $1 \leq l \leq h$. If*

$$\overline{M_C} \text{ is an independent partner matching,} \tag{3.2}$$

then $\lambda(D) \geq min\{m, d, s\}$.

Proof. Suppose that $\overline{M_C}$ is an independent partner matching. Let $A \cup B = V(D)$ and $A \cap B = \phi$, where A and B are nonempty sets.

(I) $(X \cup Y \cup C) \cap A \neq \phi$ and $(X \cup Y \cup C) \cap B \neq \phi$.
(1). $|C \cap A| = m \Rightarrow \exists\ v \in (X \cup Y) \cap B$. Then by Lemma 2.2 $\Rightarrow$ $|\partial_D^+(A)| \geq (m)(1)$.
(2). $1 \leq |C \cap A| = l < m$, where $|C| = m$. (we do not need this case when $m = 1$).
(a) $1 \leq |(X \cup Y) \cap B| = f \leq m + |X|$, where $|Y| = m$.
Then by Lemma 2.2 $\Rightarrow |\partial_D^+(A)| \geq (l)(f) + (m + |X| - f)(m - l) \geq f + (m + |X| - f) \geq m + |X| \geq m$.
(b) $|(X \cup Y) \cap B| = 0 \Rightarrow X \cup Y \subseteq A$. Since $C \cap B \neq \phi$ and $|Y| = m$ then by Lemma 2.2 $|\partial_D^+(A)| \geq m + |X| > m$.
(3). $|C \cap A| = 0 \Rightarrow \exists\ v \in (Y \cup X) \cap A$. Then by Lemma 2.2 $\Rightarrow |\partial_D^+(A)| \geq m$.

(II) $(C \cup Y \cup X) \subseteq A$.
There exists a partner $M_{C_{r_l}}$, $1 \leq l \leq h$ such that $|V(M_{C_{r_l}}) \cap B| \geq 1$.
(1). $|V(M_{C_{r_l}}) \cap B| = 1$.
(i). If $V(M_{C_{r_l}}) \cap B \neq \{r_l\}$, then by Lemma 2.2 and since $d^*(\overline{v}) = d$, $\forall$ $\overline{v} \in V(\overline{M_C})$ we have $|\partial_D^+(A)| \geq d$.
(ii). If $V(M_{C_{r_l}}) \cap B = \{r_l\}$, then by Lemma 2.2 and since $d^*(r_l) = s$, $1 \leq l \leq h$ we have $|\partial_D^+(A)| \geq s$.
(2). $|V(M_{C_{r_l}}) \cap B| \geq 2$.
(i). The partner vertex r_l of $M_{C_{r_l}}$ is contained in B. In this case, let $\{r_l, z_1, z_2, ..., z_t\} \subseteq V(M_{C_{r_i}}) \cap B$, where $t \geq 1$.
Since $d^*(\overline{v}) = d$, $\forall$ $\overline{v} \in V(\overline{M_C})$, and by Lemma 2.2 and (3.2) we have:

(a) When $t < d$ and $s > t$, then
$|\partial_D^+(A)| \geq (t)(d-2) + (s-t) \geq t + (s-t) \geq s$, where $d \geq 3$.
(b) When $t < d$ and $s \leq t$, then
$|\partial_D^+(A)| \geq (t)(d-2) \geq t \geq s$, where $d \geq 3$.
(c) When $t \geq d$, then
$|\partial_D^+(A)| \geq (t)(d-2) \geq t \geq d$, where $d \geq 3$.
(ii). The partner vertex r_l of $M_{C_{r_l}}$ is not contained in B. In this case, let $\{z_1, z_2, ..., z_t\} \subseteq V(M_{C_{r_l}}) \cap B$, where $t \geq 2$.
Since $d^*(\overline{v}) = d$, $\forall\, \overline{v} \in V(\overline{M_C})$, and by Lemma 2.2 and (3.2) we have:
$|\partial_D^+(A)| \geq (t)(d-1) \geq 2(d-1) \geq d+1$, where $d \geq 3$.

(III) $(C \cup Y \cup X) \subseteq B$.
Using the same argument of (II) with $\partial_D^-(B)$, we are done. □
Now, we will prove the main result of supereulerianicity.

Definition 3.4. *If $U \subseteq V(D)$, then we call a collection of closed ditrails $R_1, R_2, ..., R_t$ of the induced subdigraph $D[U]$ a cover of U if $\bigcup_{i=1}^{t} V(R_i) = U$ and $A(R_i) \cap A(R_j) = \phi$, whenever $i \neq j$.*

Theorem 3.5. *Let D be a strong m-central matching digraph with $\alpha'(D) = k$, $m \geq 1$ and*
$$|X| + |Y| \geq |\overline{M_C}|. \tag{3.3}$$
If
$$d^*(v) \geq m, \forall v \in X \cup Y,$$
$$d^*(v) \geq 3, \forall v \in V(\overline{M_C}), \text{ and} \tag{3.4}$$
$$\overline{M_C} \text{ is an independent partner matching,}$$
then D is supereulerian.

Proof.
(A) First, we will prove that for each $r_l \in R$, $1 \leq l \leq h$ there exists a cover that contains r_l and some vertices of $\overline{M_{C_{r_l}}}$. Then we will cover the remaining vertices of $\overline{M_{C_{r_l}}}$, where all these covers are arc disjoint.
Let $\{s_1, s_2, ..., s_h\} \subseteq X \cup Y$, $\{r_l, \overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}\} \subseteq V(M_{C_{r_l}})$ and $[\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}] \in \overline{M_{C_{r_l}}}$, $1 \leq l \leq h$, where $h = |R|$.

Case (1): $\{(\overline{v_{r_{l_1}}}, r_l), (r_l, \overline{u_{r_{l_1}}})\} \subset A(D)$.
By Lemma 2.2 and (3.4) then $\exists\, u_j \in \{u_1, u_2, ..., u_m\} = C$ such that $(\overline{u_{r_{l_1}}}, u_j) \in A(D)$

and $u_i \in \{u_1, u_2, ..., u_m\}$ such that $(u_i, \overline{v_{r_{l_1}}}) \in A(D)$.
Thus, by Lemma 2.2, $C_{l_1} = \{(u_j, s_l), (s_l, u_i), (u_i, \overline{v_{r_{l_1}}}), (\overline{v_{r_{l_1}}}, r_l), (r_l, \overline{u_{r_{l_1}}}), (\overline{u_{r_{l_1}}}, u_j)\} \subseteq A(D)$ is a cover of $\{\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}, r_l\}$.

Case (2): $\{(\overline{u_{r_{l_1}}}, r_l), (r_l, \overline{v_{r_{l_1}}})\} \subset A(D)$.
By Lemma 2.2 and (3.4), $\exists\ u_j \in \{u_1, u_2, ..., u_m\} = C$ such that $(u_j, \overline{u_{r_{l_1}}}) \in A(D)$
and $u_i \in \{u_1, u_2, ..., u_m\}$ such that $(\overline{v_{r_{l_1}}}, u_i) \in A(D)$.
Thus, by Lemma 2.2 $C_{l_1} = \{(u_j, \overline{u_{r_{l_1}}}), (\overline{u_{r_{l_1}}}, r_l), (r_l, \overline{v_{r_{l_1}}}), (\overline{v_{r_{l_1}}}, u_i), (u_i, s_l), (s_l, u_j)\} \subseteq A(D)$
is a cover of $\{\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}, r_l\}$.

Case (3): $\{(r_l, \overline{u_{r_{l_1}}}), (r_l, \overline{v_{r_{l_1}}})\} \cap A(D) = \phi$ and let by symmetry $[\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}] = (\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}) \in A(D)$.
By definition of a partner $\exists\ (\overline{v_{r_{l_1}}}, r_l) \in A(D)$.
Since D is strong and by Lemma 2.2 $\exists\ y_l \in \{u_1, u_2, ..., u_m\} \cup (V(\overline{M_{C_{r_l}}}) - \{\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}\})$, such that $(r_l, y_l) \in A(D)$.
By Lemma 2.2 and (3.4) $\exists\ u_j \in \{u_1, u_2, ..., u_m\}$ such that $(u_j, \overline{u_{r_{l_1}}}) \in A(D)$.

(i). If $y_l \in \{u_1, u_2, ..., u_m\}$, then
$C_{l_1} = \{(u_j, \overline{u_{r_{l_1}}}), (\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}), (\overline{v_{r_{l_1}}}, r_l), (r_l, y_l), (y_l, s_l), (s_l, u_j)\}$ is a cover of $\{\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}, r_l\}$.

(ii). If $y_l \in V(M_{C_{r_l}}) - \{\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}, r_l\}$, where $[y_l, \overline{y_l}] \in \overline{M_{C_{r_l}}}$.
For convenience, let $[y_l, \overline{y_l}] = [\overline{u_{r_{l_2}}}, \overline{v_{r_{l_2}}}]$ and $z_{l_2} \in X \cup Y - \{s_1, s_2, ..., s_h\}$. By Lemma 2.2 and (3.4) $\exists\ \{u_{y_l}, u_i, u_{\overline{y_l}}\} \subseteq \{u_1, u_2, ..., u_m\}$ such that $\{(y_l, u_{y_l}), (\overline{y_l}, u_{\overline{y_l}}), (u_i, \overline{y_l})\} \subseteq A(D)$.
Thus, $C_{l_1} = \{(u_{y_l}, s_l), (s_l, u_j), (u_j, \overline{u_{r_{l_1}}}), (\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}), (\overline{v_{r_{l_1}}}, r_l), (r_l, y_l), (y_l, u_{y_l})\}$
and $C_{l_2} = \{(\overline{y_l}, u_{\overline{y_l}}), (u_{\overline{y_l}}, z_{l_2}), (z_{l_2}, u_i), (u_i, \overline{y_l})\}$ is a cover of $\{\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}, r_l, \overline{u_{r_{l_2}}}, \overline{v_{r_{l_2}}}\}$.

Case (4): $\{(\overline{u_{r_{l_1}}}, r_l), (\overline{v_{r_{l_1}}}, r_l)\} \cap A(D) = \phi$ and let by symmetry $[\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}] = (\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}) \in A(D)$.
By deffinition of a partner $\exists\ (r_l, \overline{u_{r_{l_1}}}) \in A(D)$.
Since D is strong and by Lemma 2.2 $\exists\ y_l \in \{u_1, u_2, ..., u_m\} \cup (V(\overline{M_{C_{r_l}}}) - \{\overline{u_{r_l}}, \overline{v_{r_l}}\})$, such that $(y_l, r_l) \in A(D)$.
By Lemma 2.2 and (3.4) $\exists\ u_j \in \{u_1, u_2, ..., u_m\}$ such that $(\overline{v_{r_{l_1}}}, u_j) \in A(D)$.

(i). If $y_l \in \{u_1, u_2, ..., u_m\}$, then
$C_{l_1} = \{(\overline{v_{r_{l_1}}}, u_j), (u_j, s_l), (s_l, y_l), (y_l, r_l), (r_l, \overline{u_{r_{l_1}}}), (\overline{u_{r_{l_1}}}, v_{r_l})\}$ is a cover of $\{\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}, r_l\}$.

(ii). If $y_l \in V(M_{C_{r_l}}) - \{\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}, r_l\}$, where $[y_l, \overline{y_l}] \in \overline{M_{C_{r_l}}}$.
For convenience, let $[y_l, \overline{y_l}] = [\overline{u_{r_{l_2}}}, \overline{v_{r_{l_2}}}]$ and $z_{l_2} \in X \cup Y - \{s_1, s_2, ..., s_h\}$. By Lemma 2.2 and (3.4) $\exists \{u_{y_l}, u_i, u_{\overline{y_l}}\} \in \{u_1, u_2, ..., u_m\}$ such that $\{(u_i, \overline{y_l}), (\overline{y_l}, u_{\overline{y_l}})(u_{y_l}, y_l)\} \subseteq A(D)$.
Thus, $C_{l_1} = \{(u_j, s_l), (s_l, u_{y_l}), (u_{y_l}, y_l), (y_l, r_l), (r_l, \overline{u_{r_{l_1}}}), (\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}), (\overline{v_{r_{l_1}}}, u_j)\}$ and $C_{l_2} = \{(\overline{y_l}, u_{\overline{y_l}}), (u_{\overline{y_l}}, z_{l_2}), (z_{l_2}, u_i), (u_i, \overline{y_l})\}$ is a cover of $\{\overline{u_{r_{l_1}}}, \overline{v_{r_{l_1}}}, r_l, \overline{u_{r_{l_2}}}, \overline{v_{r_{l_2}}}\}$.

Now, we will cover the remaining vertices of $V(\bar{M}_C)$ which are not covered yet, where such case happens only for the partner matching that has more than one arc of $\overline{M_C}$.
Since $|X|+|Y| \geq |\overline{M_C}|$, let $z_{l_i} \in \bigcup_{1 \leq l \leq h} \{z_{l_2}, ..., z_{l_{|\overline{M_{C_{r_l}}}|}}\} \subseteq X \cup Y - \{s_1, s_2, ..., s_h\}$.

(a) If case(1), case (2), case (3)(i) or case (4)(i) is satisfied and $|\overline{M_{C_{r_l}}}| \geq 2$, then:
let $[\overline{u_{l_i}}, \overline{v_{l_i}}] \in \overline{M_{C_{r_l}}} - \{[\overline{u_{l_1}}, \overline{v_{l_1}}]\}$, $1 \leq l \leq h$, $2 \leq i \leq |\overline{M_{C_{r_l}}}|$.
Let by symmetry $[\overline{u_{l_i}}, \overline{v_{l_i}}] = (\overline{u_{l_i}}, \overline{v_{l_i}})$.
By Lemma 2.2 and (3.4) $\exists \{u_{i_0}, u_{j_0}\} \subset \{u_1, u_2, ..., u_m\}$, it is possible that $u_{i_0} = u_{j_0}$, such that
$C_{l_i} = \{(\overline{u_{l_i}}, \overline{v_{l_i}}), (\overline{v_{l_i}}, u_{i_0}), (u_{i_0}, z_{l_i}), (z_{l_i}, u_{j_0}), (u_{j_0}, \overline{u_{l_i}})\}$, $2 \leq i \leq |\overline{M_{C_{r_l}}}|$, $1 \leq l \leq h$ is a cover of $\{\overline{u_{l_i}}, \overline{v_{l_i}}\}$.

(b) If case (3)(ii) or case (4)(ii) is satisfied and $|\overline{M_{C_{r_l}}}| \geq 3$, then:
let $[\overline{u_{l_i}}, \overline{v_{l_i}}] \in \overline{M_{C_{r_l}}} - \{[\overline{u_{l_1}}, \overline{v_{l_1}}], [\overline{u_{l_2}}, \overline{v_{l_2}}]\}$, $1 \leq l \leq h$, $3 \leq i \leq |\overline{M_{C_{r_l}}}|$.
By symmetry, let $[\overline{u_{l_i}}, \overline{v_{l_i}}] = (\overline{u_{l_i}}, \overline{v_{l_i}})$.
By Lemma 2.2 and (3.4) $\exists \{u_{i_0}, u_{j_0}\} \subset \{u_1, u_2, ..., u_m\}$, it is possible that $u_{i_0} = u_{j_0}$, such that
$C_{l_i} = \{(\overline{u_{l_i}}, \overline{v_{l_i}}), (\overline{v_{l_i}}, u_{i_0}), (u_{i_0}, z_{l_i}), (z_{l_i}, u_{j_0}), (u_{j_0}, \overline{u_{l_i}})\}$, $3 \leq i \leq |\overline{M_{C_{r_l}}}|$, $1 \leq l \leq h$ is a cover of $\{\overline{u_{l_i}}, \overline{v_{l_i}}\}$.

(B) Let $T = \{t_1, ..., t_d\}$, $|T| \geq 0$ be the vertices of $X \cup Y$ that are not covered yet and let $G = \{u_{i_1}, ..., u_{i_p}\}$, $|G| \geq 0$ be the vertices of C that are not covered yet.
(1). If $|T| + |G| \geq 1$, then let $c \in C - G$ and $z \in X \cup Y - T$. Now let $C_H = D[T \cup G \cup \{c, z\}] - (A(D[G]) \cup \{(c, z), (z, c)\})$ (the induced digraph of $T \cup G \cup \{c, z\}$ minus the arcs of the induced digraph of G and the arcs (c, z) and (z, c)) is a cover of $T \cup G$ which is arc-disjoint from previous covers.
(2). If $|T| + |G| = 0$, then let $C_H = \phi$.

Thus we have the following spanning closed ditrail:

$C_H \cup (\bigcup_{1\leq l\leq h} (\bigcup_{1\leq i\leq|\overline{M_{C_{r_l}}}|} C_{l_i}))$.

And D is supereulerian. □

4 Sharpness of Supereulerianicity Theorem

The following lemma is a necessary condition for a digraph to be supereulerian. Alsatami et al. [5] proved this lemma for the opposite direction.

Lemma 4.1. *A digraph D is nonsupereulerian if there exist vertex-disjoint subdigraphs $\{B, B_1, ..., B_m\}$ of D, for some integer $m > 0$, satisfying each of the following:*
(i) $N^+(B_i) \subseteq V(B), \forall i \in \{1, 2, ..., m\}$.
(ii) $|\partial^+_D(B)| \leq m - 1$.

Proof. By contradiction we assume that both (i) and (ii) hold and D is supereulerian. Let S be a spanning closed ditrail in D. By (ii), $|\partial^+_D(B) \cap A(S)| \leq m - 1$. From the fact that S is eulerian, it follows that

$$|\partial^-_D(B) \cap A(S)| \leq m - 1. \tag{4.5}$$

By (4.5), there exists a B_j with $j \in \{1, 2, ..., m\}$ such that $\partial^+(B_j) \cap A(S) = \emptyset$, a contradiction with S is a spanning closed ditrail in D. □

The following example shows that that there exists an infinitely many nonsupereulerian m-central matching digraphs with $|X| + |Y| = |\overline{M_C}| - 1$.

Example. Let D be an m-central matching digraph with:
(1) $|C| = m \geq 1$, $|X| = d \geq 2$, $\overline{M_C} = \overline{M_{C_r}}$ (one partner) and $|\overline{M_{C_r}}| = m + d + 1$.
(2) $\forall\ [\overline{u_i}, \overline{v_i}] \in \overline{M_{C_r}}$, then $\{(\overline{u_i}, \overline{v_i}), (\overline{v_i}, \overline{u_i})\} \subset A(D)$, $1 \leq i \leq m + d + 1$.
(3) $\forall\ v \in V(\overline{M_{C_r}})$, $d^+(v) = d^-(v) = 3$ and $\{u, r\} \subset N^+(v)$ where $u \in C$ and r is the partner of $\overline{M_{C_r}}$.
(4) $N^+(r) = \{u\}$ and $N^+(u) = X \cup Y$.
(5) $B = D[\{u, r\}]$ and $B_i = D[\{\overline{u_i}, \overline{v_i}\}]$, $1 \leq i \leq m + d + 1$ (induced subdigraphs of D).
Then we have the following:
(i) By Theorem 3.3, D is strong.
(ii) $N^+(B_i) \subseteq \{u, r\} = V(B)$, $\forall\ i \in \{1, 2, ..., m + d + 1\}$.
(iii) $|\partial^+_D(B)| = m + d \leq (m + d + 1) - 1$.
Thus, by Lemma 4.1, D is not supereulerian.

5 Conclusion

Using Definition 2.1 and Theorems 3.3 and 3.5, we constructed infinitely many supereulerian m-central matching digraphs controlling its matching number, central matching, partners, vertices, arcs and arc-connectivity. We stated our control function whose images are classes of digraphs and each class has the same number of inputs:

$$f(k, m, h, n, a, \lambda)$$

where
(1) $k \geq 2$, (k is matching number).
(2) $1 \leq m \leq k-1$, (m is number of arcs in the central matching).
(3) $1 \leq h = |R| \leq k-m$, (h is number of partners).
(4) $d = |X| \geq 2$.
(5) $n = 2k + h + d \geq 7$, (n is vertex number).
(6) $\lambda \geq 1$, (λ is arc-connectivity number).
(7) a is arc number, where:

$$\begin{aligned} a &\geq |A(D[X \cup C \cup Y]) \cup (\bigcup_{1\leq l \leq h} (\bigcup_{1 \leq i \leq |\overline{M_{C_{r_l}}}|} A(D[\{\overline{u_{l_i}}, \overline{v_{l_i}}, r_l\} \cup C]))) - A(D[C \cup R])| \\ &\geq 2m(m+d) + 10(k-m) \\ &= 2m^2 + (2d-10)m + 10k, \end{aligned}$$

and

$$\begin{aligned} a &\leq |A(D[X \cup C \cup Y]) \cup (\bigcup_{1\leq l \leq h} (\bigcup_{1 \leq i \leq |\overline{M_{C_{r_l}}}|} A(D[\{\overline{u_{l_i}}, \overline{v_{l_i}}, r_l\} \cup C])))| \\ &\leq 2m(m+d) + (\sum_{1 \leq i \leq m} 2(m-i)) + 2(2(k-m)(m+1)) + 2(k-m) + 2(h)(m) \\ &= -m^2 + (2d + 4k + 2h - 7)m + 6k. \end{aligned}$$

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