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Submanifolds of a Riemannian manifold endowed with a new type of semi-symmetric non-metric connection in the tangent bundle

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Abstract

In this paper, we study a new type of semi-symmetric non-metric connection on a Riemannian manifold in the tangent bundle. We consider Submanifolds of a Riemannian with respect to the semi-symmetric non-metric connection in the tangent bundle. Finally, we establish certain results on totally geodesic and totally umbilical with respect to the semi-symmetric non-metric connection in the tangent bundle.

1 Introduction

Let M^n be an n-dimensional Riemannian manifold with Riemannian metric g and let ∇ be a Levi-Civita connection on it. A linear connection $\tilde{\nabla}$ on M^n is said to be a symmetric connection if its torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ is of the form

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \tag{1.1}$$

is zero for all X and Y on M^n; otherwise it is non-symmetric . A linear connection $\tilde{\nabla}$ is said to be a semi-symmetric connection if

$$\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y, \tag{1.2}$$

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where

$$\pi(X) = g(P, X), \tag{1.3}$$

for all X and Y on M^n, π is 1-form and P is a vector field.

A linear connection $\tilde{\nabla}$ is said to be a metric on M^n if $\tilde{\nabla} g = 0$ otherwise it is non-metric [20]. Agashe et al. defined a semi-symmetric but non-metric connection on M^n and studied some properties of the curvature tensor with respect to a semi-symmetric non-metric connection [1]. In 2019, Chaubey et al. [3] defined a new type of a semi-symmetric non-metric connection on a Riemannian manifold and studied the submanifold of a Riemannian manifold with respect to semi-symmetric non-metric connection. Numerous investigators studied semi-symmetric non-metric connection and their properties including those in [1, 2, 4, 9, 12, 17, 19, 21, 24].

On the other hand, Yano and Ishihara [25] introduced the notion of the vertical, complete and horizontal lifts of geometric objects and connections from a manifold to its tangent bundles. Tani [22] studied complete and vertical lifts of surface of tangent bundle with respect to a metric tensor. Recently, the author [13, 14] studied the tangent bundle endowed with respect to semi-symmetric non-metric and quarter symmetric non-metric connections on Kähler manifold and an almost Hermitian manifold. The connections and geometric objects such as Almost r-contact structure, metallic structures in the tangent bundle have been studied in [6, 12, 15, 16].

In this paper, we study submanifolds of a Riemannian manifold with respect to the semi-symmetric non-metric connection in the tangent bundle.

2 Preliminaries

Let M_n be an n-dimensional differentiable manifold and let the set of all tangent vectors of M at point p be defined by $T_p(M_n)$. Then the set $TM_n = \bigcup_{p \in M_n} T_p(M_n)$ is called a tangent bundle over the manifold M_n. The projection bundle $\pi_{M_n} : TM_n \to M_n$ which denotes the natural bundle structure of TM_n over M_n. Let $\{U; x^i\}$ be coordinate neighborhood in M_n where $\{x^i\}$ is a system of local coordinates in neighborhood U. Let $\{x^i, y^i\}$ be a system of local coordinates in $\pi^{-1}_{M_n}(U) \subset TM_n$ i.e. $\{x^i, y^i\}$ the induced coordinate in $\pi^{-1}_{M_n}(U)$.

2.1 Vertical and complete lifts

Let f be a function, X a vector field, ω 1-form, F tensor field of type (1,1) and ∇ affine connection in M_n. The vertical and complete lifts of a function f, a vector field X, 1-form *omega*, tensor field of type (1,1) F and affine connection ∇ are given by $f^V, X^V, \omega^V, F^V, \nabla^V$ and $f^C, X^C, \omega^C, F^C, \nabla^C$, respectively [14, 25].

The following formulas of complete and vertical lifts are defined by

$$(fX)^V = f^V X^V, (fX)^C = f^C X^V + f^V X^C, \tag{2.4}$$

$$X^V f^V = 0, X^V f^C = X^C f^V = (Xf)^V, X^C f^C = (Xf)^C, \tag{2.5}$$

$$\omega^V(f^V) = 0, \omega^V(X^C) = \omega^C(X^V) = \omega(X)^V, \omega^C(X^C) = \omega(X)^C, \tag{2.6}$$

$$F^V X^C = (FX)^V, F^C X^C = (FX)^C, \tag{2.7}$$

$$[X,Y]^V = [X^C, Y^V] = [X^V, Y^C], [X,Y]^C = [X^C, Y^C]. \tag{2.8}$$

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C, \qquad \nabla^C_{X^C} Y^V = (\nabla_X Y)^V \tag{2.9}$$

2.2 Semi-symmetric non-metric connection

Let M_n be an n-dimensional Riemannian manifold with a Riemannian metric g. A linear connection $\tilde{\nabla}$ on M_n is given by [3]

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\pi(Y)X - \pi(X)Y\}, \tag{2.10}$$

where ∇ is a Levi-Civita connection, X, Y vector fields and π 1-form on M_n. The metric g has the relation

$$(\tilde{\nabla}_X g)(Y,Z) = \frac{1}{2}\{2\pi(X)g(Y,Z) - \pi(Y)g(X,Z) - \pi(Z)g(X,Y)\}. \tag{2.11}$$

The connection $\tilde{\nabla}$ satisfying equations (1.2), (1.3), (2.10) and (2.11) is called a semi-symmetric non-metric connection.

2.3 Semi-symmetric non-metric connection of a Riemannian manifold in the tangent bundle

Let (M_n, g) be an n-dimensional Riemannian manifold with the Riemannian metric g and let TM_n be its tangent bundle. Then g^C is a Riemannian metric in TM_n. Taking complete lifts of equations (1.2), (1.3), (2.10) and (2.11), the obtained equations are

$$\begin{aligned} \tilde{T}^C(X^C, Y^C) &= \pi^C(Y^C)X^V + \pi^V(Y^C)X^C \\ &- \pi^C(X^C)Y^V - \pi^V(X^C)Y^C \end{aligned} \tag{2.12}$$

and

$$\pi^C(X^C) = g^C(X^C, P^C) \tag{2.13}$$

A linear connection $\tilde{\nabla}^C$ defined by

$$\begin{aligned} \tilde{\nabla}^C_{X^C} Y^C = \nabla^C_{X^C} Y^C &+ \frac{1}{2}\{\pi^C(Y^C)X^V + \pi^V(Y^C)X^C \\ &- \pi^C(X^C)Y^V - \pi^V(X^C)Y^C\} \end{aligned} \tag{2.14}$$

is said to be a semi-symmetric non-metric connection if the torsion tensor $\tilde{T}^C$ of TM_n with respect to $\tilde{\nabla}^C$ satisfies equations (2.12) and (2.13) and the Riemannian metric g^C satisfies the relation

$$\begin{aligned} (\tilde{\nabla}^C_{X^C} g^C)(Y^C, Z^C) &= \frac{1}{2}\{2\pi^C(X^C)g^C(Y^V, Z^C) + 2\pi^V(X^C)g^C(Y^C, Z^C) \\ &- \pi^C(Y^C)g^C(X^V, Z^C) - \pi^V(Y^C)g^C(X^C, Z^C) \\ &- \pi^C(Z^C)g^C(X^V, Y^C) - \pi^V(Z^C)g^C(X^C, Y^C)\} \end{aligned} \tag{2.15}$$

where ∇^C is a Levi-Civita connection on TM_n.

3 Submanifolds of a Riemannian manifold endowed with the semi-symmetric non-metric connection in the tangent bundle

Let M_n be an n-dimensional differentiable manifold and M_{n-2} be the submanifold of codimension 2 immersed in M_n by immersion $i : M_{n-2} \to M_n$ such that each $p \in M_{n-2} \to pi \in M_n$. Let J be the differentiability di of the immersion i so that the vector field X in the tangent space of M_{n-2} corresponds to a vector field JX in that of M_n. Let X and Y be vector fields in the tangent space of M_{n-2}, G be a Riemannian metric in M_n, and g be the induced metric on submanifold M_{n-2} from G. Then

$$G(JX, JY) \circ p = g(X, Y).$$

Let N_1 and N_2 be two mutually orthogonal unit normal vector fields to the submanifold M_{n-2} satisfying the following relations:

$$\begin{aligned} (i) \quad G(JX, N_1) &= G(JX, N_2) = G(N_1, N_2) = 0, \\ (ii) \quad G(N_1, N_1) &= G(N_2, N_2) = 1. \end{aligned} \tag{3.16}$$

Let ∇ be the Levi-Civita connection on M_n and ∇^* be the induced connection on M_{n-2} from ∇. Then

$$\nabla_{JX} JY = J(\nabla^*_X Y) + h(X,Y)N_1 + k(X,Y)N_2 \tag{3.17}$$

for all vector fields X and Y on M_{n-2}. Here h and k denote the second fundamental tensors of the submanifold M_{n-2}.

Let $\tilde{\nabla}^*$ be the induced connection of the submanifold M_{n-2} corresponding to the semi-symmetric non-metric connection $\tilde{\nabla}$ of M_n defined as (2.10). Then for the unit normal vectors N_1 and N_2,

$$\tilde{\nabla}_{JX} JY = J(\tilde{\nabla}^*_X Y) + m(X,Y)N_1 + n(X,Y)N_2 \tag{3.18}$$

for arbitrary vector fields X and Y of M_{n-2} and m, n denote the tensor fields of type (0, 2) of the submanifold M_{n-2}.

Let TM_{n-2} and TM_n be the tangent bundles of Riemannian manifolds M_{n-2} and M_n respectively. Let $\tilde{G}$ be a Riemannian metric given in M_n. The complete lift G^C of G is a Riemannian metric in TM_n. Then the induced metric from G^C on TM_{n-2} is denoted by g^C, then we can write

$$G^C(JX^C, JY^C) = g^C(X^C, Y^C)$$

Operating complete lift on both sides of the equation (3.16), we get

$$\begin{aligned} \tilde{G}^C(JX^C, N_1^{\bar{C}}) &= \tilde{G}^C(JX^C, N_1^{\bar{V}}) = 0 \\ \tilde{G}^C(JX^C, N_2^{\bar{C}}) &= \tilde{G}^C(JX^C, N_2^{\bar{V}}) = 0 \\ \tilde{G}^C(N_1^{\bar{C}}, N_1^{\bar{C}}) &= \tilde{G}^C(N_1^{\bar{V}}, N_1^{\bar{V}}) = 0 \\ \tilde{G}^C(N_2^{\bar{C}}, N_2^{\bar{C}}) &= \tilde{G}^C(N_2^{\bar{V}}, N_2^{\bar{V}}) = 0 \\ \tilde{G}^C(N_1^{\bar{C}}, N_2^{\bar{C}}) &= \tilde{G}^C(N_1^{\bar{V}}, N_2^{\bar{C}}) = 0 \\ \tilde{G}^C(N_1^{\bar{V}}, N_1^{\bar{C}}) &= \tilde{G}^C(N_2^{\bar{V}}, N_2^{\bar{C}}) = 1 \end{aligned} \tag{3.19}$$

where $N_1^{\bar{C}}, N_1^{\bar{V}}, N_2^{\bar{C}}$ and $N_2^{\bar{V}}$ are complete and vertical lifts of unit normal vectors from N_1 and N_2, respectively along with TM_{n-2}.

Operating complete lifts on both sides of equations (3.17) and (3.18), we get

$$\begin{aligned}\nabla^C_{J\tilde{B}X} CJ\tilde{B}Y^C &= J(\nabla^{*C}_{\tilde{B}X^C}\tilde{B}Y^C) + h^C(\tilde{B}X^C,\tilde{B}Y^C)N_1^{\bar{V}} + k^V(\tilde{B}X^C,\tilde{B}Y^C)N_2^{\bar{C}} \\ &+ h^V(\tilde{B}X^C,\tilde{B}Y^C)N_1^{\bar{C}} + k^C(\tilde{B}X^C,\tilde{B}Y^C)N_2^{\bar{V}} \qquad (3.20)\end{aligned}$$

and

$$\begin{aligned}\tilde{\nabla}^C_{J\tilde{B}X} CJ\tilde{B}Y^C &= J(\tilde{\nabla}^{*C}_{\tilde{B}X^C}\tilde{B}Y^C) + m^C(\tilde{B}X^C,\tilde{B}Y^C)N_1^{\bar{V}} + n^V(\tilde{B}X^C,\tilde{B}Y^C)N_2^{\bar{C}} \\ &+ m^V(\tilde{B}X^C,\tilde{B}Y^C)N_1^{\bar{C}} + n^V(\tilde{B}X^C,\tilde{B}Y^C)N_2^{\bar{V}}, \qquad (3.21)\end{aligned}$$

where $N_1^{\bar{C}}, N_1^{\bar{V}}, N_2^{\bar{C}}$ and $N_2^{\bar{V}}$ are complete and vertical lifts of unit normal vectors N_1 and N_2, respectively, along with submanifold M_{n-2}.

Theorem 3.1. *Let M_{n-2} be a submanifold of the Riemannian manifold M_n. The induced connection $\tilde{\nabla}^*C$ on TM_{n-2} in TM_n endowed with a semi-symmetric non-metric connection $\tilde{\nabla}^C$ is also a semi-symmetric non-metric connection.*

Proof: From (2.12)

$$\begin{aligned}\tilde{\nabla}^C_{J\tilde{B}X^C} J\tilde{B}Y^C &= \nabla^C_{J\tilde{B}X^C} J\tilde{B}Y^C + \frac{1}{2}\{\hat{\pi}^C(J\tilde{B}Y^C)J\tilde{B}X^V + \hat{\pi}^V(J\tilde{B}Y^C)J\tilde{B}X^C \\ &- \hat{\pi}^C(J\tilde{B}X^C)J\tilde{B}Y^V - \hat{\pi}^V(J\tilde{B}X^C)J\tilde{B}Y^C\} \qquad (3.22)\end{aligned}$$

for arbitrary vector fields X^C and Y^C. Using equations (3.20) and (3.21), then equation (3.22) takes the form

$$\begin{aligned}J(\tilde{\nabla}^{*C}_{\tilde{B}X^C}\tilde{B}Y^C) &+ m^C(\tilde{B}X^C,\tilde{B}Y^C)N_1^{\bar{V}} + n^V(\tilde{B}X^C,\tilde{B}Y^C)N_2^{\bar{C}} \\ &+ m^V(\tilde{B}X^C,\tilde{B}Y^C)N_1^{\bar{C}} + n^C(\tilde{B}X^C,\tilde{B}Y^C)N_2^{\bar{V}} \\ &= J(\nabla^{*C}_{\tilde{B}X^C}\tilde{B}Y^C) + h^C(\tilde{B}X^C,\tilde{B}Y^C)N_1^{\bar{V}} \\ &+ k^V(\tilde{B}X^C,\tilde{B}Y^C)N_2^{\bar{C}} + h^V(\tilde{B}X^C,\tilde{B}Y^C)N_1^{\bar{C}} + k^C(\tilde{B}X^C,\tilde{B}Y^C)N_2^{\bar{V}} \\ &+ \frac{1}{2}\{\hat{\pi}^C(J\tilde{B}Y^C)J\tilde{B}X^V + \hat{\pi}^V(J\tilde{B}Y^C)J\tilde{B}X^C \\ &- \hat{\pi}^C(J\tilde{B}X^C)J\tilde{B}Y^V - \hat{\pi}^V(J\tilde{B}X^C)J\tilde{B}Y^C\}\end{aligned}$$

which gives

$$\begin{aligned}\tilde{\nabla}^{*C}_{\tilde{B}X^C}\tilde{B}Y^C &= \nabla^{*C}_{\tilde{B}X^C}\tilde{B}Y^C \\ &+ \frac{1}{2}\{\hat{\pi}^C(J\tilde{B}Y^C)J\tilde{B}X^V + \hat{\pi}^V(J\tilde{B}Y^C)J\tilde{B}X^C \\ &- \hat{\pi}^C(J\tilde{B}X^C)J\tilde{B}Y^V - \hat{\pi}^V(J\tilde{B}X^C)J\tilde{B}Y^C\} \qquad (3.23)\end{aligned}$$

$$\begin{aligned} &(i) \quad h^C(\tilde{B}X^C, \tilde{B}Y^C)N_1^{\bar{V}} = m^C(\tilde{B}X^C, \tilde{B}Y^C)N_1^{\bar{V}} \\ \text{and} \quad &(ii) \quad k^V(\tilde{B}X^C, \tilde{B}Y^C)N_2^{\bar{C}} = n^V(\tilde{B}X^C, \tilde{B}Y^C)N_2^{\bar{C}} \end{aligned} \tag{3.24}$$

Thus, the relation between the induced connections $\tilde{\nabla^*}^C$ and ∇^{*C} on TM_{n-2} corresponding to the semi-symmetric non-metric and Levi Civita connections of the Riemannian manifold M_n are given by equation (3.23).

Using equation (3.23), the torsion tensor T^{*C} of $\tilde{\nabla^*}^C$ is given by

$$\begin{aligned} T^{*C}(\tilde{B}X^C, \tilde{B}Y^C) &= \tilde{\nabla^*}^C_{\tilde{B}X^C}\tilde{B}Y^C - \tilde{\nabla^*}^C_{\tilde{B}Y^C}\tilde{B}X^C - [\tilde{B}X^C, \tilde{B}Y^C] \\ &= (\hat{\pi}^C(\tilde{B}Y^C))(\tilde{B}X^V) + (\hat{\pi}^V(\tilde{B}Y^C))(\tilde{B}X^C) \\ &- (\hat{\pi}^C(\tilde{B}X^C))(\tilde{B}Y^V) + (\hat{\pi}^V(\tilde{B}X^C))(\tilde{B}Y^C) \end{aligned} \tag{3.25}$$

Equation (3.25) shows that the induced connection $\tilde{\nabla^*}^C$ of TM_{n-2} is semisymmetric.

Moreover, we prove that the connection $\tilde{\nabla^*}^C$ is non-metric; i.e., $\tilde{\nabla^*}^C\hat{g}^C \neq 0$. We have

$$\begin{aligned} \tilde{B}X^C\hat{g}^C(\tilde{B}Y^C, \tilde{B}Z^C) &= (\tilde{\nabla^*}^C_{\tilde{B}X^C}\hat{g}^C)(\tilde{B}Y^C, \tilde{B}Z^C) \\ &+ \hat{g}^C(\tilde{\nabla^*}^C_{\tilde{B}X^C}\tilde{B}Y^C, \tilde{B}Z^C) + \hat{g}^C(\tilde{B}Y^C, \tilde{\nabla^*}^C_{\tilde{B}X^C}\tilde{B}Z^C) \\ &= \hat{g}^C(\tilde{\nabla^*}^C_{\tilde{B}X^C}\tilde{B}Y^C, \tilde{B}Z^C) + \hat{g}^C(\tilde{B}Y^C, \tilde{\nabla^*}^C_{\tilde{B}X^C}\tilde{B}Z^C), \end{aligned}$$

which gives

$$\begin{aligned} (\tilde{\nabla^*}^C_{\tilde{B}X^C}\hat{g}^C)(\tilde{B}Y^C, \tilde{B}Z^C) &= \frac{1}{2}\{2\hat{\pi}^C(\tilde{B}X^C)\hat{g}^C(\tilde{B}Y^V, \tilde{B}Z^C) - \hat{\pi}^C(\tilde{B}Y^C)\hat{g}^C(\tilde{B}X^V, \tilde{B}Z^C) \\ &- \hat{\pi}^C(\tilde{B}Z^C)\hat{g}^C(\tilde{B}X^V, \tilde{B}Y^C)\} \neq 0. \end{aligned}$$

Hence the induced connection $\tilde{\nabla^*}^C$ of TM_{n-2} corresponding to the semi-symmetric non-metric connection $\tilde{\nabla}^C$ is also semi-symmetric non-metric. This completes the proof.

Let $\tilde{\nabla^*}^C$ be the induced connection of TM_{n-2} corresponding to the semi-symmetric non-metric connection $\tilde{\nabla}^C$. Then $\tilde{\nabla^*}^C$ and $\tilde{\nabla}^C$ defined by

$$(\nabla^{*C}J)(\tilde{B}X^C, \tilde{B}Y^C) = (\nabla^{*C}_{\tilde{B}X^C}J)(\tilde{B}Y^C) = \nabla^{*C}_{J\tilde{B}X^C}J\tilde{B}Y^C - J(\nabla^{*C}_{\tilde{B}X^C}\tilde{B}Y^C)$$

$$(\tilde{\nabla^*}^C J)(\tilde{B}X^C, \tilde{B}Y^C) = (\tilde{\nabla^*}^C_{\tilde{B}X^C}J)(\tilde{B}Y^C) = \tilde{\nabla^*}^C_{J\tilde{B}X^C}J\tilde{B}Y^C - J(\tilde{\nabla^*}^C_{\tilde{B}X^C}\tilde{B}Y^C)$$

Using equations (3.20) and (3.21), the above equations take the forms

$$\begin{aligned}
(\nabla^{*C}_{\tilde{B}X^C} J)(\tilde{B}Y^C) &= h^C(\tilde{B}X^C, \tilde{B}Y^C)N_1^{\bar{V}} + k^V(\tilde{B}X^C, \tilde{B}Y^C)N_2^{\bar{C}} \\
&+ h^V(\tilde{B}X^C, \tilde{B}Y^C)N_1^{\bar{C}} + k^C(\tilde{B}X^C, \tilde{B}Y^C)N_2^{\bar{V}} \\
(\tilde{\nabla}^{*C}_{\tilde{B}X^C} J)(\tilde{B}Y^C) &= m^C(\tilde{B}X^C, \tilde{B}Y^C)N_1^{\bar{V}} + n^V(\tilde{B}X^C, \tilde{B}Y^C)N_2^{\bar{C}} \\
&+ m^V(\tilde{B}X^C, \tilde{B}Y^C)N_1^{\bar{C}} + n^C(\tilde{B}X^C, \tilde{B}Y^C)N_2^{\bar{V}},
\end{aligned}$$

where h^C, k^C, m^C, n^C and h^V, k^V, m^V, n^V are complete and vertical lifts of second fundamental tensors h, k, m, n of submanifold M_{n-2}.

Let $e_1, e_2, ..., e_{n-2}$ be $(n-2)$ orthonormal local vector fields in M_{n-2}. Then the function

$$H = \frac{1}{n-2}\sum_{i=1}^{n-2} h(e_i, e_i), \tag{3.26}$$

is called the mean curvature H of M_{n-2} with respect to ∇^* and the function

$$H^* = \frac{1}{n-2}\sum_{i=1}^{n-2} m(e_i, e_i), \tag{3.27}$$

is called the mean curvature H^* of M_{n-2} with respect to ∇^*.

Taking complete and vertical lifts of equations (3.26), the obtained equations are

$$H^C = \frac{1}{n-2}\sum_{i=1}^{n-2} h^C(e_i^C, e_i^C)$$

$$H^V = \frac{1}{n-2}\sum_{i=1}^{n-2} h^V(e_i^C, e_i^C)$$

are called mean curvatures of TM_n with respect to ∇^{*C}.

Taking complete and vertical lifts of equations (3.27), the obtained equations are

$$H^{*C} = \frac{1}{n-2}\sum_{i=1}^{n-2} m^C(e_i^C, e_i^C)$$

$$H^{*V} = \frac{1}{n-2}\sum_{i=1}^{n-2} m^V(e_i^C, e_i^C)$$

are called the mean curvatures of TM_{n-2} with respect to $\tilde{\nabla}^*C$.

Definition 3.1 The submanifold TM_{n-2} is called minimal for ∇^{*C} if $H^C = 0$ and $H^V = 0$.

Definition 3.2 The submanifold TM_{n-2} is called minimal for $\tilde{\nabla}^* C$ if $H^* C$ and H^{*V}.

Definition 3.3 The submanifold TM_{n-2} is called totally geodesic with respect to the Levi-Civita connection ∇^{*C} if h^C, h^V, k^C, k^V vanish identically.

Definition 3.4 The submanifold TM_{n-2} is called totally umbilical with respect to the Levi-Civita connection ∇^{*C} if h^C, h^V, k^C, k^V are proportional to the metric g^C; i.e., $h^C = H^C g^C$, $h^V = H^V g^C$, $k^C = H^C g^C$, $k^V = H^V g^C$.

Similarly, the submanifold TM_{n-2} is called totally geodesic and totally umbilical with respect to $\tilde{\nabla}^* C$ according to whether h^C, h^V, k^C, k^V vanish and proportional to g^C respectively. The following theorem is obtained:

Theorem 3.2. *Let M_n be the Riemannian manifold and let M_{n-2} be its submanifold. Let TM_{n-2} and TM_n be the tangent bundles of M_{n-2} and M_n respectively. Then*

(i) The mean curvatures of TM_{n-2} corresponding to the induced connections $\tilde{\nabla^*}^C$ and ∇^{*C} coincide.

(ii) TM_{n-2} will be totally geodesic with respect to $\tilde{\nabla^*}^C$ if and only if it is totally geodesic for ∇^{*C}.

(iii) TM_{n-2} is totally umbilical with respect to $\tilde{\nabla^*}^C$ if and only if it is totally umbilical for ∇^{*C}.

(iv) TM_{n-2} is minimal corresponding to $\tilde{\nabla^*}^C$ if and only if it is also minimal for ∇^{*C}.

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