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# A note on almost $\alpha$-ideals in semigroups

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**Abstract**

The notion of $\alpha$-ideals in semigroups is a generalization of $(m,n)$-ideals, where $m$ and $n$ are nonnegative integers. This notion was introduced by Miccoli and Pondělíček in 1989. In this paper, we define the notion of almost $\alpha$-ideals in semigroups. Moreover, we provide some properties of this concept and present a characterization of semigroups having no proper almost $\alpha$-ideal.

## 1 Introduction

The notion of $(m,n)$-ideals in semigroups, defined by Lajos [4], is a generalized concept of bi-ideals. Later on, this idea was extended to the concept of $\alpha$-ideal [5] in semigroups by Miccoli and Pondělíček. In 1990, Catino [2] studied the concept of $\alpha$-ideals in semigroups and characterized semigroups

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in which any $\alpha$-ideal is complete.

The concept of ideals in semigroups was generalized to the notion of almost ideals [3] in 1980 by Grošek and Satko who described all semigroups having no proper almost ideals. This result was studied in other algebraic systems; ternary semigroups and semihypergroups in 2019 and 2020, respectively (see [7, 8]). In 1981, Bogdanović [1] introduced the notion of almost bi-ideals in semigroups. In 2018, Wattanatripop et al. [9] gave the notion of quasi-almost-ideal in semigroups and studied some of its properties. Simuen et al. [6] defined almost quasi-$\Gamma$-ideals in $\Gamma$-semigroups and provided some interesting basic properties.

In this paper, we define the notion of almost $\alpha$-ideals in semigroups. It turns out that almost $\alpha$-ideals is a generalization of almost ideals. Moreover, we present interesting properties of this concept.

## 2 Preliminaries

Throughout this paper, we denote the set of all nonnegative integers $\{0, 1, 2, \ldots\}$ and the set of all natural numbers $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$ and $\mathbb{N}^+$, respectively. Let $\{0, 1\}^*$ be the free monoid over the set $\{0, 1\}$. We denote the length of $\alpha \in \{0, 1\}^*$ by $|\alpha|$.

A *semigroup* $\langle S; \circ \rangle$ is an algebra of type (2) in which the binary operation $\circ$ satisfies the associative law. For any elements $x, y \in S$, we simply write $x \circ y$ as $xy$. Moreover, we denote a semigroup $\langle S; \circ \rangle$ by $\mathbf{S}$.

Let $\mathbf{S}$ be a semigroup. For any subsets $A$ and $B$ of $S$, we define

$$AB := \{ab : a \in A \text{ and } b \in B\}.$$

We note that $AB := \emptyset$ if $A$ or $B$ is empty. If $A = \{a\}$ or $B = \{b\}$, we write $AB$ as $aB$ or $Ab$, respectively.

We now recall some important terminologies about $\alpha$-ideals in semigroups (More information about $\alpha$-ideals in semigroups can be found in [5]).

Let $\mathbf{S}$ be a semigroup, and $\alpha \in \{0, 1\}^*$. We define $f_\alpha^S \colon \mathcal{P}(S) \to \mathcal{P}(S)$ by $f_\alpha^S(A) := \emptyset$ if $\alpha$ is the empty word, and if $\alpha = a_1 \cdots a_n$ is not the empty word, then $f_\alpha^S(A) := A_1 \cdots A_n$, where

$$A_i := \begin{cases} A & \text{if } a_i \neq 0, \\ S & \text{if } a_i = 0, \end{cases}$$

for any $A \subseteq S$ and $1 \leq i \leq n$.

By the above definitions, we obtain the following useful lemma:

**Lemma 2.1 ([5]).** *Let* $\mathbf{S}$ *be a semigroup, and* $A$ *and* $B$ *be nonempty subsets of* $S$*. Then for any* $\alpha \in \{0,1\}^*$*, we have* $f_\alpha^S(A) \subseteq f_\alpha^S(B)$ *whenever* $A \subseteq B$*.*

Let $\mathbf{S}$ be a semigroup. A nonempty subset $A$ of $S$ is said to be a *subsemigroup* of $\mathbf{S}$ if $AA \subseteq A$. A subsemigroup $A$ of $\mathbf{S}$ is called an *$\alpha$-ideal* of $\mathbf{S}$ if $f_\alpha^S(A) \subseteq A$. It is not difficult to see that:

1. any left ideal of $\mathbf{S}$ is a 01-ideal of $\mathbf{S}$;
2. any right ideal of $\mathbf{S}$ is a 10-ideal of $\mathbf{S}$;
3. any bi-ideal of $\mathbf{S}$ is a 101-ideal of $\mathbf{S}$;
4. any interior ideal of $\mathbf{S}$ is a 010-ideal of $\mathbf{S}$;
5. any $(m,n)$-ideal of $\mathbf{S}$, where $m,n \in \mathbb{N}$, is a $1^m01^n$-ideal of $\mathbf{S}$.

To define the concept of almost $\alpha$-ideals in semigroups, we need to modify the operator $f_\alpha^S$ as follows:
Let $\alpha = a_1 \cdots a_n \in \{0,1\}^*$ such that $|\alpha| = n$ for some $n \in \mathbb{N}$. We define the set $0(\alpha) := \{j \in \mathbb{N}^+ : a_j = 0\}$. It is clear that if $\alpha$ is the empty word or $\alpha = 1^m$ for all $m \in \mathbb{N}^+$, then $0(\alpha)$ is empty.

Suppose that $\alpha = a_1 \cdots a_n \in \{0,1\}^*$ such that $|\alpha| = n$, for some $n \in \mathbb{N}^+$ and $0(\alpha) = \{i_1, \ldots, i_m\}$. Let $\mathbf{S}$ be a semigroup. For any nonempty subsets $S_1, \ldots, S_m$ of $S$, we define $f_\alpha^{(S_1,\ldots,S_m)} \colon \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$f_\alpha^{(S_1,\ldots,S_m)}(A) := A_1 \cdots A_n,$$

for all $A \subseteq S$, where

$$A_i := \begin{cases} A & \text{if } a_i = 1, \\ S_j & \text{if } a_i = 0 \text{ and } i = i_j, \end{cases}$$

for all $1 \leq i \leq n$.
For any subset $A$ of $S$, it is clear that $f_\alpha^{(S_1,\ldots,S_m)}(A) \subseteq f_\alpha^S(A)$, for all subsets $S_1, \ldots, S_m$ of $S$. We note that the set $0(\alpha)$ may be empty. If for any $1 \leq j \leq m$, $S_j = \{s_j\}$ is a singleton set, then we usually write $f_\alpha^{(S_1,\ldots,S_m)}(A)$ as $f_\alpha^{(s_1,\ldots,s_m)}(A)$.

## 3 Almost $\alpha$-ideals

In what follows, we assume that $\alpha \in \{0,1\}^* \setminus \{1\}^*$ with $|\alpha| = n \geq 2$ and the cardinality of $0(\alpha)$ is $m$ such that $m < n$, where $m, n \in \mathbb{N}^+$. Now, we are ready to define our concept.

**Definition 3.1.** *Let* $\mathbf{S}$ *be a semigroup. A nonempty subset* $A$ *of* $S$ *is said to be an* almost $\alpha$-ideal *of* $\mathbf{S}$ *if* $f_\alpha^{(s_1,\ldots,s_m)}(A) \cap A \neq \emptyset$ *for any* $s_1, \ldots, s_m \in S$.

**Remark 3.2.** *Every almost left (resp., right, bi-, interior,* $(m,n)$*-) ideal is an almost* 01- *(resp.,* 10-, 101-, 010-, $1^m01^n$*-) ideal, respectively.*

**Proposition 3.3.** *Let* $\mathbf{S}$ *be a semigroup. Then any* $\alpha$*-ideal of* $\mathbf{S}$ *is an almost* $\alpha$*-ideal of* $\mathbf{S}$.

*Proof.* Suppose that $A$ is an $\alpha$-ideal of $\mathbf{S}$. Let $s_1, \ldots, s_m \in S$. Since $f_\alpha^{(s_1,\ldots,s_m)}(A) \subseteq f_\alpha^S(A) \subseteq A$, we have $f_\alpha^{(s_1,\ldots,s_m)}(A) \cap A \neq \emptyset$. This shows that $A$ is an almost $\alpha$-ideal of $\mathbf{S}$. □

The converse of Proposition 3.3 does not hold in general as shown in the following example:

**Example 3.4.** *Consider the semigroup* $\mathbf{S}$ *defined by the following Cayley table:*

| $\circ$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $b$ | $c$ | $a$ |
| $c$ | $c$ | $a$ | $b$ |

*Suppose that* $A = \{b, c\}$. *Then* $A$ *is an almost* 0101*-ideal of* $\mathbf{S}$ *since* $s_1As_2A \cap A = S \cap A \neq \emptyset$, *for any* $s_1, s_2 \in S$. *On the other hand, it is not difficult to calculate that* $A$ *is not a* 0101*-ideal of* $\mathbf{S}$ *since* $SASA = S \not\subseteq A$.

The relation between an almost $\alpha$-ideal and a subset containing such almost $\alpha$-ideal is presented as follows:

**Theorem 3.5.** *Let* $\mathbf{S}$ *be a semigroup, and* $A$ *be an almost* $\alpha$*-ideal of* $\mathbf{S}$. *Then for any subset* $B$ *of* $S$ *containing* $A$ *is also an almost* $\alpha$*-ideal of* $\mathbf{S}$.

*Proof.* For any $s_1, \ldots, s_m \in S$, by Lemma 2.1, we have

$$\emptyset \neq f_\alpha^{(s_1,\ldots,s_m)}(A) \cap A \subseteq f_\alpha^{(s_1,\ldots,s_m)}(B) \cap B$$

This means that $B$ is an almost $\alpha$-ideal of $\mathbf{S}$. □

As a consequence, we obtain the following result:

**Corollary 3.6.** *Any union of almost $\alpha$-ideals in a semigroup* $\mathbf{S}$ *is also an almost $\alpha$-ideal of* $\mathbf{S}$.

*Proof.* Let $\{A_i : i \in I\}$ be a collection of almost $\alpha$-ideals of $\mathbf{S}$. Since $A_i \subseteq \bigcup_{i\in I} A_i$ and $A_i$ is an almost $\alpha$-ideal of $\mathbf{S}$ for all $i \in I$, by Theorem 3.5, $\bigcup_{i\in I} A_i$ is an almost $\alpha$-ideal of $\mathbf{S}$. □

By Example 3.4, we can calculate that $A = \{a, b\}$ and $B = \{b, c\}$ are almost 0101-ideal of $\mathbf{S}$. But $\{b\} = A \cap B$ is not an almost 0101-ideal of $\mathbf{S}$ since $a\{b\}b\{b\} \cap \{b\} = \{a\} \cap \{b\} = \emptyset$. This shows that the intersection of two almost $\alpha$-ideals may not be an almost $\alpha$-ideal if it is not empty.

The following result provides a characterization of semigroups having no proper almost $\alpha$-ideal.

**Theorem 3.7.** *Let* $\mathbf{S}$ *be a semigroup such that* $|S| > 1$ *and* $\alpha \in \{0,1\}^* \setminus \{1\}^*$*, where* $|\alpha| = n \geq 2$ *and* $|0(\alpha)| = m$ *such that* $m < n$*, where* $m, n \in \mathbb{N}^+$*. Then* $\mathbf{S}$ *has no proper almost $\alpha$-ideal if and only if for any proper subset* $I$ *of* $S$ *there exits a subset* $\{s_1, \ldots, s_m\}$ *of* $S$ *(depending on the set* $I$*) such that* $f_\alpha^{(s_1,\ldots,s_m)}(S \setminus I) = I$.

*Proof.* Assume that $\mathbf{S}$ has no proper almost $\alpha$-ideal. Let $I$ be a proper subset of $S$. Then $S \setminus I$ is not an almost $\alpha$-ideal of $\mathbf{S}$. This implies that there exist $s_1, \ldots, s_m \in S$ with $f_\alpha^{(s_1,\ldots,s_m)}(S \setminus I) \cap (S \setminus I) = \emptyset$. This means that $f_\alpha^{(s_1,\ldots,s_m)}(S \setminus I) = I$.

Conversely, suppose that $\mathbf{S}$ contains a proper almost $\alpha$-ideal $J$. Let $K \subseteq S \setminus J$ be a nonempty set. By our assumption, $f_\alpha^{(s_1,\ldots,s_m)}(S \setminus K) = K$, for some $s_1, \ldots, s_m \in S$. That is, $f_\alpha^{(s_1,\ldots,s_m)}(S \setminus K) \cap (S \setminus K) = \emptyset$. On the other hand, by Theorem 3.5, we have $S \setminus K$ is an almost $\alpha$-ideal of $\mathbf{S}$ since $S \setminus K$ contains an almost $\alpha$-ideal $J$ of $\mathbf{S}$. This implies that $f_\alpha^{(t_1,\ldots,t_m)}(S \setminus K) \cap (S \setminus K) \neq \emptyset$ for any $t_1, \ldots, t_m \in S$. This contradicts the existence of $s_1, \ldots, s_m \in S$. Therefore, $\mathbf{S}$ has no proper almost $\alpha$-ideal. □

# 4 Conclusions

In this paper, we introduced the notion of almost $\alpha$-ideals as a generalization of left almost ideals, right almost ideals, almost bi-ideals, and almost interior ideals in semigroups. Moreover, We discussed some properties of almost $\alpha$-ideals. Furthermore, we characterized semigroups having no proper almost $\alpha$-ideal. For future work, we plan to define the notion of almost fuzzy $\alpha$-ideals in semigroups and investigate some of their properties.

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