Non-uniform bound on normal approximation for call function of locally dependent random variables

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Abstract

In this paper, we concentrate on a mean for call function of random variables which is useful and important, especially in finance. We propose a non-uniform bound from a normal approximation under local dependence by using the Stein’s method. Finally, we illustrate an application on a collateralized debt obligation.

1 Introduction and main results

A call function, defined by $(x-k)^+ = \max\{x-k,0\}$, where $x$ and $k$ are real numbers, is useful and important, especially in finance. In 2009, Karoui and Jiao [8] and Karoui, Jiao and Kurtz [9] approximated a loss on each tranche of a collateralized debt obligation (CDO) in the form of $E(W-k)^+$, where $W$ is a sum of independent random variables and $k$ is a real number.

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They used the Stein’s method to obtain error bounds from approximating $E(W - k)^+$ by a mean for call function of Poisson and normal random variables. The refined bounds are proposed by Yonghint and Neammanee [10] for Poisson approximation and Jongpreechaharn and Neammanee [7] for normal approximation under the same assumption. For a normal approximation, Jongpreechaharn and Neammanee [6] extended the assumption to a local dependence motivated by Chen and Shao [4].

Let $X_1, X_2, \ldots, X_n$ be random variables with zero means and finite variances. For $A \subseteq \{1, 2, \ldots, n\}$, let $X_A$ denote $\{X_i : i \in A\}$ and $A^c = \{i \in \{1, 2, \ldots, n\} : i \notin A\}$. We say that $X_1, X_2, \ldots, X_n$ satisfy the local dependence condition if there exists a partition $\{A_i\}_{i=1}^{l_n}$ of $\{1, 2, \ldots, n\}$ such that for each $i \in \{1, 2, \ldots, l_n\}$, $X_{A_i}$ and $X_{A_i^c}$ are independent.

Let $Z$ be a standard normal random variable and let $k$ be a positive real number. Denote $W = \sum_{i=1}^{n} X_i$ and assume that $W$ has a unit variance.

Suppose that $X_1, X_2, \ldots, X_n$ satisfy the local dependence condition. Jongpreechaharn and Neammanee [6] approximated $E(W - k)^+$ by $E(Z - k)^+$ and proposed a uniform bound in the following theorem.

**Theorem 1.1** ([6]). For each $i = 1, 2, \ldots, l_n$, let $Y_i = \sum_{j \in A_i} X_j$. Under the local dependence condition, we have

$$
\sup_{k > 0} |E(W - k)^+ - E(Z - k)^+| \\
\leq 24.97 \sum_{i=1}^{l_n} E|Y_i|^3 + 0.80 \left( \sum_{i=1}^{l_n} EY_i^4 \right)^{1/2} + \left( l_n EW^4 \sum_{i=1}^{l_n} EY_i^6 \right)^{1/2},
$$

where $EW^4 \leq \sum_{i=1}^{l_n} EY_i^4 + 3$.

In this work, we use Stein’s method introduced in Section 2 to obtain a non-uniform bound as follows.

**Theorem 1.2.** For each $i = 1, 2, \ldots, l_n$, let $Y_i = \sum_{j \in A_i} X_j$. Under the local dependence condition and $k \geq 2$, we have

$$
|E(W - k)^+ - E(Z - k)^+| \\
\leq C_1(k) \sum_{i=1}^{l_n} E|Y_i|^3 + C_2(k) \left( \sum_{i=1}^{l_n} EY_i^4 \right)^{1/2} + C_3(k) \left( l_n \sum_{i=1}^{l_n} EY_i^6 \right)^{1/2},
$$

where

$$
C_1(k) = \frac{5.5e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{5.5}{k} + \frac{1}{2k^2}, \quad C_2(k) = \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{1}{k},
$$
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\[ C_3(k) = \frac{3}{k} \sqrt{EW^6} + \frac{15.69}{k}  \sqrt{EW^4} + \frac{18.24}{k^2}, \quad EW^4 \leq \sum_{i=1}^{l_n} EY_i^4 + 3 \]

and

\[ EW^6 \leq \sum_{i=1}^{l_n} EY_i^6 + 15 \sum_{i=1}^{l_n} EY_i^4 + 10 \left( \sum_{i=1}^{l_n} EY_i^3 \right)^2 + 15. \]

In addition, applying Theorem 1.2 with independent and identically distributed random variables gives a convergence rate of the non-uniform bound. Let \(X_1, X_2, \ldots, X_n\) be independent and identically distributed random variables with \(\Pr(X_1 = 1/\sqrt{n}) = \Pr(X_1 = -1/\sqrt{n}) = 1/2\) and \(W = \sum_{i=1}^{n} X_i\). Thus, by choosing \(A_i = \{i\}\) for \(i = 1, 2, \ldots, n\) in Theorem 1.2, we obtain the following corollary.

**Corollary 1.3.** Adopting the notations in Theorem 1.2, we have for \(k \geq 2\) that

\[ \left| E(W - k)^+ - E(Z - k)^+ \right| \leq \frac{a_k}{\sqrt{n}}, \]

where

\[ a_k = \frac{6.5 e^{1-k^2/2}}{\sqrt{2\pi k^2}} + \frac{1}{2k^2} + \frac{24.74}{k} + \frac{3}{k} \left( \frac{1}{n^2} + \frac{25}{n} + 15 \right)^{1/2} + \frac{15.69}{k} \left( \frac{1}{n} + 3 \right)^{1/2}. \]

Observe that, a non-uniform bound steadily declines as \(k\) gets larger.

### 2 Stein’s method on normal approximation for call function

In 1972, Stein [11] discovered a powerful method for finding a bound from approximation (see also [1] and [3]). The foundation of Stein’s method is the differential equation called the Stein equation. Let \(Z\) be a standard normal random variable. The Stein equation on normal approximation for a given function \(h\) is

\[ xf(x) - f'(x) = h(x) - Eh(Z). \quad (2.1) \]

In this work, we let \(h(x) = (x - k)^+\) for a fixed positive real number \(k\). Thus, (2.1) becomes

\[ xf(x) - f'(x) = (x - k)^+ - E(Z - k)^+. \quad (2.2) \]

From (2.2), we substituting \(x\) by any random variable \(W\) and taking an expectation on both sides of the equation, we obtain

\[ EWf_k(W) - Ef'_k(W) = E(W - k)^+ - E(Z - k)^+, \quad (2.3) \]
where $f_k$ is the Stein solution of (2.2). From (2.3), a targeted bound for $|E(W - k)^+ - E(Z - k)^+|$ can be verified from $|EF_k(W) - Ef_k'(W)|$ instead. This is a procedure of the Stein’s method on normal approximation. In order to verify a bound for $|EF_k(W) - Ef_k'(W)|$, the properties of the Stein solution $f_k$ and its derivative $f'_k$ are essential. Notice that

$$f_k(x) = \begin{cases} \sqrt{2\pi e^{x^2/2}}E(Z - k)^+\Phi(x), & \text{if } x \leq k, \\ 1 - \sqrt{2\pi e^{x^2/2}}[k + E(Z - k)^+]\Phi(-x), & \text{if } x > k \end{cases}$$

and

$$f'_k(x) = \begin{cases} E(Z - k)^+ \left(1 + \sqrt{2\pi x\Phi(x)e^{x^2/2}}\right), & \text{if } x \leq k, \\ [k + E(Z - k)^+] \left(1 - \sqrt{2\pi x\Phi(-x)e^{x^2/2}}\right), & \text{if } x > k, \end{cases}$$

(see also [6] and [7]), where $\Phi$ is the cumulative distribution function of $Z$.

Next, we propose some properties of them in Propositions 2.1 and 2.2 which are used to prove the main theorem.

**Proposition 2.1.** For $x \leq k$ and $k \geq 1$, we have $|f_k(x)| \leq \frac{1}{k^2}$.

**Proof.** Let $x \leq k$ and $k \geq 1$. Note that $f_k(x) \geq 0$. If $|x| \leq k$, then by the fact that

$$E(Z - k)^+ \leq \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}} \text{ for } k \geq 1, \quad (2.4)$$

(see [7], p.3502), we have $f_k(x) \leq \frac{1}{k^2}$. Suppose that $x < -k$. By the fact that

$$\Phi(-a) \leq \frac{e^{-a^2/2}}{\sqrt{2\pi a}} \text{ for } a > 0, \quad (2.5)$$

(see [11], p.23) and (2.4), we have $f_k(x) \leq \frac{E(Z - k)^+}{-x} < \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}}$. Hence $|f_k(x)| \leq \frac{1}{k^2}$ for $x \leq k$.

Let $\|g\| = \sup_{x \in \mathbb{R}} |g(x)|$ for every real-valued function $g$ on $\mathbb{R}$.

**Proposition 2.2.** For $k \geq 1$, we have $\|f'_k\| \leq \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{1}{k}$.

**Proof.** Let $x \leq k$. If $x < 0$, by (2.5), we have $0 \leq 1 + \sqrt{2\pi x\Phi(x)e^{x^2/2}}$. Hence, $0 \leq 1 + \sqrt{2\pi x\Phi(x)e^{x^2/2}} \leq 1$. By this inequality and (2.4), we have $0 \leq f'_k(x) \leq \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}}$ for $x < 0$. Suppose that $x \geq 0$. Then $f'_k(x) \geq 0$. By (2.4),
we have $f'_k(x) \leq E(Z - k)^+ \left(1 + \sqrt{2\pi} e^{k^2/2} \right) \leq \frac{e^{-k^2/2}}{\sqrt{2\pi} k^2} + \frac{1}{k}$. Thus, for $x \geq 0$, we have $0 \leq f'_k(x) \leq \frac{e^{-k^2/2}}{\sqrt{2\pi} k^2} + \frac{1}{k}$. Therefore,

$$0 \leq f'_k(x) \leq \frac{e^{-k^2/2}}{\sqrt{2\pi} k^2} + \frac{1}{k} \quad \text{for } x \leq k.$$ 

Assume that $x > k$. By (2.5), we obtain $\sqrt{2\pi} \Phi(-x)e^{x^2/2} \leq 1$. Then $f'_k(x) \geq 0$. On the other hand, note that $\Phi(-a) \geq e^{-a^2/2 - \frac{1}{2k^2}}$ for $a > 0$, (see [7], p.3502). Thus $\sqrt{2\pi} \Phi(-x)e^{x^2/2} \geq 1 - \frac{1}{x^2} > 1 - \frac{1}{k^2}$. From this fact and (2.4), we obtain $f'_k(x) \leq \frac{k + E(Z-k)^+}{k^2} \leq \frac{1}{k} + \frac{e^{-x^2/2}}{\sqrt{2\pi} k^2}$. Hence

$$0 \leq f'_k(x) \leq \frac{e^{-k^2/2}}{\sqrt{2\pi} k^4} + \frac{1}{k} \quad \text{for } x > k.$$ 

Combining these two cases, we obtain $\|f'_k\| \leq \frac{e^{-k^2/2}}{\sqrt{2\pi} k^2} + \frac{1}{k}$ for $k \geq 1$. \hfill \Box

Next, we use the Stein’s method and properties of the Stein solution in Propositions 2.1 and 2.2 to provide a non-uniform bound for $|E(W - k)^+ - E(Z - k)^+|$.

3 Proof of the results

Proof of Theorem 1.2. By the proof of Theorem 1.1, we have

$$EW f_k(W) - Ef'_k(W) = R_1 + R_2 + R_3,$$  \hspace{1cm} (3.6)

where

$$|R_1| \leq \|f'_k\| \left[ \left( \sum_{i=1}^{l_n} EY_i^4 \right)^{1/2} + 2 \sum_{i=1}^{l_n} E|Y_i|^3 \right],$$

$$|R_2| \leq 2\|f'_k\| \sum_{i=1}^{l_n} E|Y_i|^3,$$

$$|R_3| \leq E \int_{|t| \leq 1} |f'_k(W + t) - f'_k(W)| \tilde{K}(t) dt$$

and

$$\tilde{K}(t) = \sum_{i=1}^{l_n} Y_i \left[ \mathbb{I}(-Y_i \leq t < 0) - \mathbb{I}(0 \leq t \leq -Y_i) \right].$$
From Proposition 2.2, we obtain

$$|R_1| \leq \left( \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{1}{k} \right) \left[ \left( \sum_{i=1}^{l_n} EY_i^4 \right)^{1/2} + 2 \sum_{i=1}^{l_n} E|Y_i|^3 \right]$$  \hspace{1cm} (3.7)$$

and

$$|R_2| \leq 2 \left( \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{1}{k} \right) \sum_{i=1}^{l_n} E|Y_i|^3.$$  \hspace{1cm} (3.8)$$

For $R_3$, we use the truncation technique to rewrite $R_3$ by

$$|R_3| \leq R_{3,1} + R_{3,2} + R_{3,3},$$  \hspace{1cm} (3.9)$$

where

$$R_{3,1} = E \int_{|t| \leq 1} |f'_k(W + t) - f'_k(W)| \mathbb{I}(W > k) \hat{K}(t) dt,$$

$$R_{3,2} = E \int_{|t| \leq 1} |f'_k(W + t) - f'_k(W)| \mathbb{I}(W > k, W \leq k) \hat{K}(t) dt$$

and

$$R_{3,3} = E \int_{|t| \leq 1} |f'_k(W + t) - f'_k(W)| \mathbb{I}(W \leq k, W \leq k) \hat{K}(t) dt.$$ 

By Proposition 1 of [6], we have $|f'_k(x + t) - f'_k(x)| \leq 2x^2|t| + 10.46|x||t| + 12.16|t|$ for $|t| \leq 1$ and $x \in \mathbb{R}$. This implies that

$$R_{3,1} \leq E \left( 2W^2 + 10.46|W| + 12.16 \right) \mathbb{I}(W > k) \int_{|t| \leq 1} |t| \hat{K}(t) dt$$

and

$$R_{3,2} \leq E \left( 2W^2 + 10.46|W| + 12.16 \right) \mathbb{I}(W > k - 1) \int_{|t| \leq 1} |t| \hat{K}(t) dt.$$ 

By the Hölder’s inequality, the Markov’s inequality and the fact in [6] that $E \left( \int_{|t| \leq 1} |t| \hat{K}(t) dt \right)^2 \leq \frac{l_n}{4} \sum_{i=1}^{l_n} EY_i^6$, we obtain

$$R_{3,1} \leq \left[ 2 \left( EW^4 \mathbb{I}(W > k) \right)^{1/2} + 10.46 \left( EW^2 \mathbb{I}(W > k) \right)^{1/2} \right.$$

$$+ 12.16 \left( \Pr(W > k) \right)^{1/2} \left[ E \left( \int_{|t| \leq 1} |t| \hat{K}(t) dt \right)^2 \right]^{1/2} \left[ l_n \sum_{i=1}^{l_n} EY_i^6 \right]^{1/2} \right) \frac{1}{k} \left( \sqrt{EW^6} + 5.23\sqrt{EW^4} + 6.08 \right) \left( l_n \sum_{i=1}^{l_n} EY_i^6 \right)^{1/2}. \hspace{1cm} (3.10)$$
Using the same argument for bounding $R_{3,1}$, we obtain

$$R_{3,2} \leq \frac{1}{k-1} \left( \sqrt{E W^6} + 5.23 \sqrt{E W^4} + 6.08 \right) \left( \sum_{i=1}^{n} E Y_i^6 \right)^{1/2}.$$  

Since $k \geq 2$, $\frac{1}{k-1} \leq \frac{2}{k}$. From this fact, we have

$$R_{3,2} \leq \frac{2}{k} \left( \sqrt{E W^6} + 5.23 \sqrt{E W^4} + 6.08 \right) \left( \sum_{i=1}^{n} E Y_i^6 \right)^{1/2}. \tag{3.11}$$

Thus, it remains to consider $R_{3,3}$. By the proof of Proposition 1 of [6], we have $|f_k(x + t) - f_k(x)| \leq \|f_k\| |x||t| + |f_k(x + t)||t|$ for $x + t \leq k$ and $x \leq k$. By this inequality and Propositions 2.1 and 2.2, we have

$$|f_k(x + t) - f_k(x)| \leq \left( \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{1}{k} \right) |x||t| + \frac{|t|}{k^2}.$$  

This implies that

$$R_{3,3} \leq E \left[ \left( \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{1}{k} \right) |W| + \frac{1}{k^2} \right] \int_{|t| \leq 1} |t| \hat{K}(t) dt.$$  

By the fact that $\int_{|t| \leq 1} |t| \hat{K}(t) dt \leq \frac{1}{2} \sum_{i=1}^{n} |Y_i|(Y_i^2 \land 1)$, where $a \land b = \min\{a, b\}$ for any real numbers $a$ and $b$, (see [6]), we obtain

$$R_{3,3} \leq \frac{1}{2} \left( \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{1}{k} \right) \sum_{i=1}^{n} E |W Y_i| (Y_i^2 \land 1) + \frac{1}{2k^2} \sum_{i=1}^{n} E |Y_i|^3.$$  

By modification of the proof of $r_4$ in Theorem 2.2 of [4] on p.2013, we obtain $E |W Y_i|(Y_i^2 \land 1) \leq 3E |Y_i|^3$. This implies that

$$R_{3,3} \leq \left( \frac{3e^{-k^2/2}}{2\sqrt{2\pi k^2}} + \frac{3}{2k} + \frac{1}{2k^2} \right) \sum_{i=1}^{n} E |Y_i|^3. \tag{3.12}$$

Combining (3.6)–(3.12), we obtain

$$|E(W - k)^+ - E(Z - k)^+|$$

$$\leq \left( \frac{5.5e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{5.5}{k} + \frac{1}{2k^2} \right) \sum_{i=1}^{n} E |Y_i|^3 + \left( \frac{e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{1}{k} \right) \left( \sum_{i=1}^{n} E Y_i^4 \right)^{1/2}$$

$$+ \left( \frac{3}{k} \sqrt{E W^6} + \frac{15.69}{k} \sqrt{E W^4} + \frac{18.24}{k} \right) \left( \sum_{i=1}^{n} E Y_i^6 \right)^{1/2}. \tag{3.13}$$
Next, we find bounds for $EW^4$ and $EW^6$. Notice that
\[
EW^4 \leq \sum_{i=1}^{l_n} EY_i^4 + 3,  \tag{3.14}
\]
(see [6]). For $EW^6$, we first notice that $Y_i$ and $Y_j$ are independent for distinct $i$ and $j$ and $EY_i = 0$. Then $\sum_{i=1}^{l_n} EY_i^2 = EW^2 = 1$, (see [6]). This implies that
\[
EW^6 = \sum_{i=1}^{l_n} EY_i^6 + 15 \sum_{i=1}^{l_n} \sum_{j=1, j \neq i}^{l_n} EY_i^4 EY_j^2 + 10 \sum_{i=1}^{l_n} \sum_{j=1, j \neq i}^{l_n} EY_i^3 EY_j^3
\]
\[+ 15 \sum_{i=1}^{l_n} \sum_{j=1, j \neq i}^{l_n} \sum_{l=1, l \neq i, l \neq j}^{l_n} EY_i^2 EY_j^2 EY_l^2
\]
\[
\leq \sum_{i=1}^{l_n} EY_i^6 + 15 \sum_{i=1}^{l_n} EY_i^4 + 10 \left( \sum_{i=1}^{l_n} EY_i^3 \right)^2 + 15.  \tag{3.15}
\]
Combining (3.13)–(3.15), the proof is complete. \hfill \Box

**Proof of Corollary 1.3.** For $r \geq 1$, we have $E|Y_i|^r = E|X_i|^r = \frac{1}{n^{r/2}}$. This implies that $EW^4 \leq \frac{1}{n} + 3$ and $EW^6 \leq \frac{1}{n^2} + \frac{25}{n} + 15$. From these inequalities and Theorem 1.2, we have Corollary 1.3 as required. \hfill \Box

### 4 Application on collateralized debt obligation

Consider a collateralized debt obligation (CDO) containing $n$ assets. A total loss of a CDO at time $T$ is defined by $L(T) = \frac{1}{n} \sum_{i=1}^{n} (1 - R_i) \mathbb{I}(\tau_i \leq T)$, where $\mathbb{I}(A)$ is an indicator function of $A$, $R_i$ and $\tau_i$ are deterministic recovery rate and stochastic default time for the $i$th asset, respectively. One is interested in calculation of a loss on each tranche of the CDO given by $(L(T) - AP)^+ - (L(T) - DP)^+$, where $AP$ and $DP$ are attachment and detachment points of the tranche, respectively, (see [2] and [5] for more details). Therefore, we concentrate on $E(L(T) - a)^+$, where $a$ is a positive real number.

Let $X_i = \frac{(1-R_i)\mathbb{I}(\tau_i \leq T) - E\mathbb{I}(\tau_i \leq T)}{n \sqrt{\text{Var} L(T)}}$ and $W = \sum_{i=1}^{n} X_i$. Then $W$ has zero mean.
and unit variance. We classify assets in a CDO by their corresponding workplace. Let $l_n$ be a number of workplaces and let $A_i$ be a group of assets from the same workplace for $i = 1, 2, \ldots, l_n$. Then $\{I(\tau_i \leq T)\}_{i=1}^n$ satisfies the local dependence condition.

To simplify the verification, assume that $\{I(\tau_i \leq T)\}_{i=1}^n$ are identically Bernoulli random variables with $\Pr(I(\tau_i \leq T) = 1) = \Pr(I(\tau_i \leq T) = 0) = 1/2$, $E[I(\tau_i \leq T)I(\tau_j \leq T)] = 1/2$ and $R = R_i$. Suppose that every group $A_i$ contains $c$ assets. Thus $l_n = n/c$.

**Theorem 4.1.** Adopting notations defined above, we have

1. $\sup_{k>0}|E(W-k)^+ - E(Z-k)^+| \leq b\sqrt{\frac{c}{n}}$, where $b = 25.77 + (\frac{c}{n} + 3)^{1/2}$,

2. for $k \geq 2$, we have $|E(W-k)^+ - E(Z-k)^+| \leq b_k\sqrt{\frac{c}{n}}$,

where $b_k = \frac{6.5e^{-k^2/2}}{\sqrt{2\pi k^2}} + \frac{1}{2k^2} + \frac{24.74}{k} + \frac{3}{k} \left( \frac{c^2}{n^2} + \frac{25}{n} + 15 \right)^{1/2} + \frac{15.69}{k} \left( \frac{c}{n} + 3 \right)^{1/2}$.

**Proof.** Note that $E[I(\tau_i \leq T)] = 1/2$ and $\text{Var} L(T) = \frac{(1-R)^2}{n^2} \sum_{i=1}^{l_n} \sum_{j \in A_i} \sum_{l \in A_i} \text{Cov}(I(\tau_j \leq T), I(\tau_l \leq T)) = \frac{c(1-R)^2}{4n}$.

Thus $X_i = \frac{2(\tau_i \leq T)^{-1}}{\sqrt{cn}}$ and hence $E[Y_i]^r = E\left[\sum_{j \in A_i} X_j\right]^r = \left( \frac{c}{n} \right)^{r/2}$ for $r \geq 1$.

This implies that $EW^4 \leq \frac{c}{n} + 3$ and $EW^6 \leq \frac{c^2}{n^2} + \frac{25c}{n} + 15$. From these inequalities and Theorems 1.1 and 1.2, we have Theorem 4.1 as required.

Table 1 shows constants $b$ and $b_k$, where we bound $\frac{1}{n^2} \leq \frac{1}{n} \leq 1$ and vary the values of $c$ as shown. Observe that, when the number of correlated random variables $c$ is increased, we always can find a constant $k$ such that a non-uniform bound is sharper than a uniform bound.

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Table 1: Comparison of the constant $b$ for uniform bound and $b_k$ for non-uniform bound.
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