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On the Diophantine Equation $2^{2nx} - p^y = z^2$, where p is a prime

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Abstract

In this paper, we study the Diophantine equation $2^{2nx} - p^y = z^2$*, where* n *is a positive integer and* p *is a prime number. For* $p = 2$*, we find the set of all solutions in non-negative integers* x, y *and* z *given by* $\{(x, y, z)\} = \{(0, 0, 0)\} \cup \{(t, 2nt, 0)\}$*. For* p *an odd prime number, we find the set of all solutions in non-negative integers* x, y *and* z *given by* $\{(x, y, z)\} = \{(0, 0, 0)\} \cup \{(\frac{q-1}{n}, 1, 2^{q-1} - 1)\}$ *for prime* $p = 2^q - 1$*, where* q *is also a prime number. For* $p \equiv 3(mod4)$ *not of the form* $2^q - 1$*, we only have the trivial solution* $(x, y, z) = (0, 0, 0)$*.*

1 Introduction

Diophantine equations have grabbed the interest of many mathematicians for a long time. Over the last decade, a number of exponential Diophantine equations have been studied since Catalan presented his conjecture [1] which

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was proved in 2004 by Mihailescu [2]. In 2011, Suvarnamani [3] considered $2^x+p^y=z^2$ and found solutions of this equation according to the values of p (for example, $(3,0,3)$ is a solution for $p>2$ and $(4,2,5)$ is another solution for $p=3$ and for $p=2$, the solutions consist of three types). In 2012, Chotchaisthit [4] studied the Diophantine equation $4^x+p^y=z^2$, where x,y and z are non-negative integers and p is any prime number and revealed that the equation does not have solutions. In the same year, Peker and Cenberci [5] showed that the Diophantine equation $8^x+19^y=z^2$ has no solution. The results have been generalized via the Diophantine equation equation $(4n)^x+p^y=z^2$, where p is an odd prime, $n\in N$, and x,y and $z\in N_0$ [6]. In 2015, Bacani and Rabago [7] obtained all solutions of the Diophantine equation $p^x+q^y=z^2$, where p and q are twin primes under some additional assumptions on p and q. In the same year Lan and Xiaoxue [8] studied the equation $8^x+p^y=z^2$ and found that it has infinitely many solutions. In 2017, Asthana [9] showed that the Diophantine equation $8^x+113^y=z^2$, where x,y and z are non-negative integers has only three solutions $(1,0,3),(1,1,11)$ and $(3,1,25)$. In 2018, Fergy and Rabago [10] found all solutions of $4^x-7^y=z^2$ and $4^x-11^y=z^2$ to complement the results and found the two Diophantine equations $4^x-7^y=3z^2$ and $4^x-19^y=3z^2$ have exactly two solutions (x,y,z) in non-negative integers; i.e., $(x,y,z)\in\{(0,0,0),(1,0,1)\}$. In fact, the Diophantine equation $4^x-p^y=3z^2$ has the two solutions $(0,0,0)$ and $(1,0,1)$ under some additional assumption on p. In 2019, Laipaporn et al. [11] showed that $3^x+p5^y=z^2$, where p is a prime number has infinitely many solutions. In 2021, Dokchan and Pakapongpun [12] showed that the Diophantine equation $p^x+(p+20)^y=z^2$, where p and $p+20$ are primes has no solutions in positive integers x,y and z.

In this paper, we present a different Diophantine equation $2^{2nx}-p^y=z^2$, where n is a positive integer and p is a prime number. For $p=2$, we find the set of all solutions in non-negative integers x,y and z given by $\{(x,y,z)\}=\{(0,0,0)\}\cup\{(t,2nt,0)\}$. For p an odd primes number, we verify that $2^{2nx}-11^y=z^2$ has no solutions in non-negative integers except possibly when $x=y=z=0$. In addition, the Diophantine equation $2^{2nx}-7^y=z^2$ has solutions $\{(x,y,z)\}$ given by $\{(x,y,z)\}=\{(0,0,0)\}\cup\{(\frac{2}{n},1,3)\}$, where $\frac{2}{n}$ is a positive integer. At the end of our paper, we state and prove a generalization of these two previous results. Most precisely, we show that $2^{2nx}-p^y=z^2$ has the set of all solutions $\{(x,y,z)\}$ given by $\{(x,y,z)\}=\{(0,0,0)\}\cup\{(\frac{q-1}{n},1,2^{q-1}-1)\}$ for prime $p=2^q-1$ (with q also a prime). For $p\equiv 3(mod4)$ not of the form 2^q-1, the Diophantine equation $2^{2nx}-p^y=z^2$ has only the trivial solution $(x,y,z)=(0,0,0)$.

2 Main results

Proposition 2.1. *(Catalan's conjecture)* $(3, 2, 2, 3)$ *is the unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$*, where* a, b, x *and* y *are integers such that* $min\{a, b, x, y\} > 1$.

Proof. This was proved in 2004 by Mihailescu [2].

Theorem 2.2. *The Diophantine equation* $2^{2nx} - 2^y = z^2$ *has the set of all solutions* $\{(x, y, z)\}$ *given by* $\{(x, y, z)\} = \{(0, 0, 0)\} \cup \{(t, 2nt, 0)\}$*, where* n *is a positive integer.*

Proof. First, we consider the two cases: $z = 0$ and $z > 0$.
Case 1. If $z = 0$, then we have $2^{2nx} - 2^y = 0$. Thus, $y = 2nx$.
Case 2. If $z > 0$, then we separate z into three subcases:
Subcase 1. If $x = 0$, then $1 - 2^y = z^2$ which implies that $z = y = 0$.
Subcase 2. If $y = 0$, then $2^{2nx} - z^2 = 1$ which is obviously impossible because of Catalan's conjecture.
Subcase 3. If $x, y > 0$, then $2^{2nx} - z^2 = 2^y$ or equivalently $2^y = (2^{nx} + z)(2^{nx} - z)$. Thus there exist non-negative integers α, β such that $2^\alpha = 2^{nx} - z$, and $2^\beta = 2^{nx} + z$, where $\alpha < \beta$ and $\alpha + \beta = y$. Therefore, $2^{2n+1} = 2^\alpha(2^{\beta-\alpha} + 1)$ or $2^{2n+1-\alpha} - 2^{\beta-\alpha} = 1$ which has no solution because of Catalan's conjecture.

Theorem 2.3. *The Diophantine equation* $2^{2nx} - 7^y = z^2$ *has solutions* $\{(x, y, z)\}$ *given by* $\{(x, y, z)\} = \{(0, 0, 0)\} \cup \{(\frac{2}{n}, 1, 3)\}$*, where* $\frac{2}{n}$ *is a positive integer.*

Proof. Evidently, the case when $z = 0$ yields $(x, y, z) = (0, 0, 0)$. So we may assume that $z > 0$. For $z > 0$, we consider three cases.
Case 1. $x = 0$. This case is trivial.
Case 2. $y = 0$. If $y = 0$, then $2^{2nx} - z^2 = 1$ which has no solution for $min\{z\} > 1$ by Catalan's conjecture. For $z = 1$, we get $x = \frac{1}{2n}$ which is obviously impossible.
Case 3. $x, y > 0$. For this case, we have $2^{2nx} - z^2 = 7^y$ or equivalently $7^y = (2^{nx} + z)(2^{nx} - z)$. Thus there exist non-negative integers α, β such that $7^\alpha = 2^{nx} - z$, and $7^\beta = 2^{nx} + z$, where $\alpha < \beta$ and $\alpha + \beta = y$. Therefore, $2^{nx+1} = 7^\alpha(7^{\beta-\alpha} + 1)$. Thus, $\alpha = 0$ and $2^{nx+1} - 7^y = 1$ which has no solution for $min\{y\} > 1$ by Catalan's conjecture. For $y = 1$, we get $x = \frac{2}{n}$ and $z = 3$.

Theorem 2.4. $(0, 0, 0)$ *is the unique solution* (x, y, z) *of the Diophantine equation* $2^{2nx} - 11^y = z^2$*, where* x, y *and* z *are non-negative integers.*

Proof. The theorem can be shown easily by utilizing Catalan's conjecture and is similar to the proof of the previous theorem. The case when $z = 0$ and $x = 0$ are both trivial. So we may assume without loss of generality that $min\{x, z\} > 0$. If this is the case, then we have $2^{2nx} - z^2 = 11^y$ or equivalently $11^y = (2^{nx} + z)(2^{nx} - z)$. Thus there exist non-negative integers α, β such that $11^\alpha = 2^{nx} - z$ and $11^\beta = 2^{nx} + z$, where $\alpha < \beta$ and $\alpha + \beta = y$. Therefore, $2^{2n+1} = 11^\alpha(11^{\beta-\alpha} + 1)$. Thus $\alpha = 0$ and $2^{2n+1} - 11^y = 1$ which has no solution for $min\{y\} > 1$ by Catalan's conjecture. For $y = 1$, we get $2^{nx+1} = 12$ which is impossible.

Theorem 2.5. *The Diophantine equation $2^{2nx} - p^y = z^2$ has the set of all solutions $\{(x, y, z)\}$ given by $\{(x, y, z)\} = \{(0, 0, 0)\} \cup \{(\frac{q-1}{n}, 1, 2^{q-1} - 1)\}$ for prime $p = 2^q - 1$ (with q also a prime). For $p \equiv 3(mod 4)$ not of the form $2^q - 1$, the Diophantine equation $2^{2nx} - p^y = z^2$ has only the trivial solution $(x, y, z) = (0, 0, 0)$.*

Proof. We consider three cases: $x = 0, y = 0$ and $x, y > 0$.
Case 1. $x = 0$. If $x = 0$, then $1 - p^y = z^2$ which implies that $z = y = 0$ and p is any prime number.
Case 2. $y = 0$. If $y = 0$, then $2^{2nx} - z^2 = 1$ which has no solution for $min\{z\} > 1$ by Catalan's conjecture. For $z = 1$, we get $x = \frac{1}{2n}$ which is impossible.
Case 3. $x, y > 0$. For this case we have $2^{2nx} - z^2 = p^y$ or equivalently $p^y = (2^{nx} + z)(2^{nx} - z)$. Thus there exist non-negative integers α, β such that $p^\alpha = 2^{nx} - z$, and $p^\beta = 2^{nx} + z$, where $\alpha < \beta$ and $\alpha + \beta = y$. Therefore, $2^{nx+1} = p^\alpha(p^{\beta-\alpha} + 1)$. Thus $\alpha = 0$ and $2^{nx+1} - p^y = 1$ which has no solution for $min\{y\} > 1$ by Catalan's conjecture. For $y = 1$, we get $p = 2^{nx+1} - 1$. Note that $2^{nx+1} - 1$ is a prime if and only if $nx + 1$ is also a prime. Thus, we get a family of solutions to $2^{2nx} - p^y = z^2$ given by $\{(x, y, z)\} = \{(\frac{q-1}{n}, 1, 2^{q-1} - 1) \mid q$ is a prime$\}$ for $p = 2^q - 1$. On the other hand, if $p \equiv 1(mod 4)$ not of the form $2^q - 1$ (with $y = 1$), then we get $-1 \equiv 1(mod 4)$ and this is clearly a contradiction. Thus we only have the trivial solution $(0, 0, 0)$ to $2^{2nx} - p^y = z^2$, for $p \equiv 3(mod 4)$.

Remark 2.6. *Theorem 2.3 (respectively, Theorem 2.4) agrees with Theorem 2.5 since $7 = 2^3 - 1$ (respectively, $11 \equiv 3(mod 4)$).*

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