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Phragmen-Lindelőf Theorem at infinity

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Abstract

We consider a Phragmen-Lindelőf Theorem yielding the behavior at infinity of solutions of Dirichlet problem for general quasilinear second order elliptic partial differential equation in a domain which has a structure of infinity many connected components.

1 Introduction

In 1908, Edvard Phragmen and Emst Lindelőf introduced their famous Phragmen-Lindelőf Theorem. Lancaster [6] used the barriers found by Serrin [7] to obtain Phragmen-Lindelőf theorems in Slabs where their conclusions hold for solutions in certain domains. We obtain a new Phragmen-Lindelőf Theorem filling the "gap" in the works of Jin and Lancaster [3, 4] and in the existing Literature on the asymptotic behavior at infinity of solutions of the Elliptic Dirichet problem.

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Let $n \geq 2$ be an integer. For each $M > 0$, let S_M denote the set

$$\{(x_1, ..., x_n) \in \mathbb{R}^n / |x_n| < M\} = \{(x_1, ..., x_n) \in \mathbb{R}^n / x_n \in I_M\},$$

where $I_M = (-M, M)$.

Let Ω be an open set in $\mathbb{R}^n$ contained in S_M. Suppose (a_{ij}) is an $n \times n$ (symmetric) positive semidefinite matrix of functions in $C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ with trace one (see for example [3] and [7]). Then

$$\sum_{i=1}^{n} a_n(X, z, p) = 1 \text{ for } (X, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \tag{1.1}$$

Let b be a function in $C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ such that $b(X, z, p)$ is nonincreasing in z and let Q be the nonhyperbolic quasilinear second-order operator defined by

$$Qu(X) = \sum a_{ij}(X, u(X), Du(X))Diju(X) + b(X, u(X), Du(X)). \tag{1.2}$$

Consider the Dirichlet problem of finding a function $f \in C^2(\Omega) \cap C^\circ(\overline{\Omega})$ which satisfies

$$Qf = 0 \text{ in } \Omega \tag{1.3}$$

and

$$f = \phi \text{ on } \partial\Omega. \tag{1.4}$$

We wish to establish a special "Phragmen-Lindelőf Theorem at infinity", for solutions of (1.3) and (1.4) when the domain Ω has a specific form. A typical type of such theorem is the conclusion that if Ω is an unbounded domain in $\mathbb{R}^2$, $f \in C^2(\Omega) \cap C^\circ(\overline{\Omega})$ is a bounded harmonic function, Ω contains the region between tow rays, and $f(x)$ converges to a as $|x|$ tends to infinity with x between the two rays.

Similar theorems in $\mathbb{R}^n, n \geq 2$, for solutions of many elliptic and degenerate elliptic equations have appeared in the Literature.

We will consider our domain Ω to be unbounded and let $T(\Omega)$ denote the set of directions $\omega \in S^{n-2}$ at infinity of the set
$\pi(\Omega) = \{\overrightarrow{x} \in \mathbb{R}^{n-1} / (\overrightarrow{x}, y) \in \Omega \text{ for some } y \in I_M\}$; that is,

$$T(\Omega) = \cap_{N=1}^{\infty} \overline{\cup_{r \geq N} \{\omega \in S^{n-2} / r\omega \in \pi(\Omega)\}}. \tag{1.5}$$

Notice that $\omega \in T(\Omega)$ if and only if there exists a sequence $\{(x_j, y_j)\}$ in Ω with $|x_j| \to \infty$ and $\frac{x_j}{|x_j|} \to \omega$ as $j \to \infty$.

In various papers, Jin and Lancaster have obtained "Phragmen-Lindelőf Theorem at infinity" which show that solution $f(x,y)$ of (1.3) and (1.4) converges at infinity in the direction $\omega \in T(\Omega)$ with the solution $K_\omega(y)$ of an ordinary differential equation. For example, in [5, Theorem 2.2], the following theorem was proven:

Let $M > 0$, $\Omega \subset S_n$, and $\omega \in T(\Omega)$. Suppose that

1. $f \in C^2(\Omega) \cap C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies (1.3) and (1.4);

2. For some open subset O of S^{n-2} with $\omega \in 0$, there exists a function $E \in C^0(O \times I_M \times \mathbb{R}^2)$ such that $E(\frac{X}{|X|}, y, z, q)$ is nonincreasing in z and

$$\frac{b(x,y,z,p,q)}{a_{nn}(x,y,z,p,q)} \to E(\sigma, y, z, q). \tag{1.6}$$

as $|x| \to \infty$ with $\frac{x_j}{|x_j|} \to \sigma$ and $p \to 0$ uniformly for $|y| \to M$,$\sigma \in O$, and $z, q \in \mathbb{R}$.

3. There exists a function k mapping $\overline{I_M} \times T$ into $\mathbb{R}$ such that

$$\phi(x.y) \to k(y, \omega) \tag{1.7}$$

uniformly as $|x| \to \infty$ and $\frac{x_j}{|x_j|} \to \omega$ for $(x,y) \in \partial\Omega$ and, for each α>0, there exists $\delta = \delta_{\alpha,\omega}$>0 and functions k_1, k_2 in $C^1(\overline{I_M}) \cap C^2(I_M)$ such that for each $y \in I_M$

$$|k_1(y) - k(y,\omega)| \leqslant \alpha\ , \tag{1.8}$$

$$|k_2(y) - k(y,\omega)| \leqslant \alpha\ , \tag{1.9}$$

$$k_1''(y) + E(\omega, y, k_1(y), k_1'(y) \geq \delta, \tag{1.10}$$

and

$$k_2''(y) + E(\omega, y, k_2(y), k_2'(y) \leq -\delta \tag{1.11}$$

4. There exists $L \geq 0$ and a positive continuous function σ_1 on $[1, \infty)$ such that

$$a_{nn}(\mathbf{x}, y, z, \mathbf{p}, q) \geq \sigma_1(|p|^2 + q^2), \tag{1.12}$$

where $\mathbf{x}$,$\mathbf{p} \in \mathbb{R}^{n-1}$, $y, z, q \in \mathbb{R}$ with $|x| \geq L$ and $|y|$ <M.

5. Q satisfies (1.1) and then

$$\lim_{j\to\infty} |f(x_j, y_j) - k(y_j,\omega)| = 0 \tag{1.13}$$

uniformly for sequences $\{(x_j, y_j)\}$ in $\overline{\Omega}$ with $|x_j| \to \infty$ and $\frac{x_j}{|x_j|} \to \omega$ as $j \to \infty$.

In [5, Example 5.4], Jin and Lancaster considered an example in which the hypotheses of this theorem are not satisfied in certain directions in $T(\Omega)$ and behavior at infinity of the solution f in such a direction is unclear.

In this example, $\Omega = U \times I_M$ where U is a subset of $\mathbb{R}^2$ which contains the first quadrant of the plane and whose boundary oscillates sinusoidally in the second and fourth quadrants. The conclusion in [5, Theorem 2.2] holds for directions in the first quadrant.

In our work, the first step toward understanding the behavior at infinity of solutions of (1.3) and (1.4) in directions in which Ω contains infinitely many "voids" are taken. We find here that a "Phragmen-Lindelőf Theorem at infinity" continues to hold but that the solution $f(x, y)$ of (1.3) and (1.4) converges at infinity in the direction $\omega \in T(\Omega)$ to the solution $k_\omega(t, y)$ of a partial differential equation in two variables rather than to the solution k_ω of an ordinary differential equation.

2 Main Result

We may represent an element $X = (x_1, ..., x_n)$ of $\mathbb{R}^n$ in either of the following two ways:

(i) $X = (\overrightarrow{x}, y)$, where $\overrightarrow{x} = (x_1, ..., x_{n-1})$ and $y = x_n$.

(ii) $X = (x_1, x', y)$, where $x' = (x_2, ..., x_{n-1})$ and $y = x_n$.

For $\alpha_0 > 0$ and $\omega \in S^{n-2}$, let

$$S_\omega^{\alpha_0} = \left\{ (\overrightarrow{x}, y) \in \mathbb{R}^{n-1} \times \mathbb{R} / \left| \frac{x_j}{|x_j|} - \omega \right| < \alpha_0 \right\}$$

For the purpose of this investigation, the following choices for Ω and ω will be made. Let $\omega = (1, 0, ..., 0) \in S^{n-2}$ and Λ be a bounded open subset of $\mathbb{R} \times I_M$.

For some α_0 and $\mathbb{R}_0 > 0$, suppose that the set $\Omega_i^{(\alpha_0)} = \{(\overrightarrow{x}, y) \in \Omega \cap S_\omega^{\alpha_0} / |\overrightarrow{x}| > \mathbb{R}_0\}$ is the disjoint union of sets $\Omega_i^{(\alpha_0)}$, $i = 1, 2, ...$ such that there is an increasing sequence (d_i) which satisfies $\lim_{i\to\infty} d_i = \infty$ and $\Omega_i^{(\alpha_0)} = \{(x_i, x', y) \in S_\omega^{\alpha_0} / (x_i - d_i, y) \in \Lambda\}$

Assumption 1. There exist $E, a_{11}^*, a_{1n}^* \in C^0(I_M \times \mathbb{R}^3)$ such that $E(y, z, p_1, q)$ is nonincreasing in z,$a_{11}^*(y, z, p_1, q)$ and (y, z, p_1, q) are independent of z, and

$$\frac{b(\overrightarrow{x}, y, z, \overrightarrow{p}, q)}{a_{nn}(\overrightarrow{x}, y, z, \overrightarrow{p}, q)} \to E(y, z, p_1, q), \tag{2.1}$$

$$\frac{a_{11}(\overrightarrow{x}, y, z, \overrightarrow{p}, q)}{a_{nn}(\overrightarrow{x}, y, z, \overrightarrow{p}, q)} \to a_{11}^*(y, z, p_1, q), \tag{2.2}$$

and
$$\frac{a_{1n}(\overrightarrow{x},y,z,\overrightarrow{p},q)}{a_{nn}(\overrightarrow{x},y,z,\overrightarrow{p},q)} \to a^*_{1n}(y,z,p_1,q). \tag{2.3}$$
as $|\overrightarrow{x}| \to \infty$ with $\frac{x_j}{|x_j|} \to \omega$ and $|p'| = |(p_2,...,p_{n-2})| \to 0$ uniformly for $|y| < M$ and $z, p_1, q \in \mathbb{R}$. Let $Q^{(\infty)}$ be the operator on $C^2(\mathbb{R}\times I_M)$ defined by $Q^{(\infty)}v(t,y) = a^*_{11}(t,y,v,Dv)\frac{\partial^2 v}{\partial t^2} + 2a^*_{1n}(t,y,v,Dv)\frac{\partial^2 v}{\partial t \partial y} + \frac{\partial^2 v}{\partial y^2} + E(t,y,v,Dv)$.

Assumption 2. There exist functions $H \in C^0(\overline{I_M})$ and $k \in C^0(\overline{\Lambda}) \cap C^2(\Lambda)$ such that

$$\phi(\overrightarrow{x},y) \to H(y) \text{ uniformly as } |\overrightarrow{x}| \to \infty \text{ and } \frac{x_j}{|x_j|} \to \omega \text{ for } (\overrightarrow{x},y) \in \partial\Omega, \tag{2.4}$$
$$Q^{(\infty)}k = 0 \text{ in } \Lambda \tag{2.5}$$
and
$$k(t,y) = H(y) \text{ for } (t,y) \in \partial\Lambda. \tag{2.6}$$

Assumption 3. For each $\alpha > 0$, there exist $B_0 = B_0(\alpha) > 0$, $\delta = \delta_\alpha > 0$ and functions k_1, k_2 in $C^1(\overline{\Lambda}) \cap C^2(\Lambda)$ such that for each $y \in I_M$

$$|k_1(t,y) - k(t,y)| \le \alpha \ , \ (t,y) \in \Lambda, \tag{2.7}$$

$$|k_2(t,y) - k(t,y)| \le \alpha \ , \ (t,y) \in \Lambda. \tag{2.8}$$
$$Q^{(\infty)}k_1(t,y) \ge \delta \ , \ (t,y) \in \Lambda, \tag{2.9}$$
$$Q^{(\infty)}k_2(t,y) \le -\delta \ , \ (t,y) \in \Lambda, \tag{2.10}$$
and
$$\left|D^2 k_j(t,y)\right| \le B_0 \ , \ (t,y) \in \Lambda \ , \ j = 1,2. \tag{2.11}$$

Theorem 2.1. *Let $\Omega \subset S_M$ and $\omega = (1,0,...,0) \in \mathbb{R}^{n-1}$ be as above suppose*
1. *Q satisfies (1) ;*
2. *$f \in C^2(\Omega) \cap C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies (1.3) and (1.4);*
3. *Assumptions 1-3 are satisfied;*
4. *there exist $L \ge 0$ and a positive continuous function σ_1 on $[1,\infty)$ such that*

$$a_{nn}(\overrightarrow{x},y,z,\overrightarrow{p},q) \ge \sigma_1(|\overrightarrow{p}|^2 + q^2) \tag{2.12}$$

whenever $\overrightarrow{x}, \overrightarrow{p} \in \mathbb{R}^{n-1}$, $y,z,q \in \mathbb{R}$ with $|\overrightarrow{x}| \ge L$ and $|y| < M$.

Let $(x,y) \in \overline{\Lambda}$. *suppose* $(\overrightarrow{x}_{(l)})$ *is a sequence in* $\mathbb{R}^{n-1}$ *and* $(y_{(l)})$ *is a sequence in* $\overline{I}_M$ *such that*

(i) $\left(\overrightarrow{x}_{(l)}, y_{(l)}\right) = (\overrightarrow{x}_{(l),1}, \overrightarrow{x}_{(l),2}, ..., \overrightarrow{x}_{(l),n-1}, y_{(l)}) \in \overline{\Omega_i^{(\alpha_0)}}$ *for* $l \in \mathbb{N}$.

(ii) $\left(x_{(l),1} - d_l, y_{(l)}\right) \to (t,y)$ *as* $l \to \infty$, *Then*

$$\lim_{l\to\infty} f(\overrightarrow{x}_{(l)}, y_{(l)}) = k(t,y) \tag{2.13}$$

3 Barrier construction

The barriers needed here are the same as those in [5] (with $\sigma = 1$). Let

$$X(\alpha) = \begin{cases} \frac{1}{2} - \ln(\alpha) & \text{if} \quad 0 < \alpha < 1 \\ \frac{1}{2\alpha^2} & \text{if} \quad 1 < \alpha < \infty \end{cases}$$

and

$$\eta(\beta) = \begin{cases} \frac{1}{\sqrt{2\beta}} & \text{if} \quad 0 < \beta < \frac{1}{2} \\ e^{\frac{1}{2}-\beta} & \text{if} \quad \frac{1}{2} < \beta < \infty \end{cases}$$

Define

$$A(H) = 2M(\int_1^{e^{x(H)}} \eta(\ln t\ dt))^{-1}\ ,\ H \geq 1 \tag{3.1}$$

Then

$$h_\alpha(r) = a\sqrt{e}(\frac{1}{2H^2} - \frac{1}{2}) + \frac{9}{\sqrt{2}}(\lambda(\sqrt{e}) - \lambda(\frac{r}{a}))\ ,\ a < r < a\sqrt{e}$$

$$a\sqrt{e}(\frac{1}{2H^2} - \ln(\frac{r}{a}))\ ,\ \text{if } a\sqrt{e} \leq r < ae^{x(H)}$$

where λ satisfies $\lambda'(t) = \frac{1}{\sqrt{\ln t}}$, $\omega = \omega_a = \omega_{a,x_0,\Gamma,H}$ and $l = l_a = l_{a,x_0,\Gamma,H}$ are given by

$$\omega_{a,x_0,\Gamma,H}(x,y) = \gamma + ae^{\chi(H)} - \sqrt{(h_a^{-1}(y+M))^2 - |x-x_0|^2} \tag{3.2}$$

and

$$l_{a,x_0,\Gamma,H}(x,y) = \gamma - ae^{x(H)} + \sqrt{(h_a^{-1}(y+M))^2 - |x-x_0|^2}. \tag{3.3}$$

The domain of $\omega_{a,x_0,\Gamma,H}$ and $l_{a,x_0,\Gamma,H}$ is

$$\Omega_{\alpha,x_0,H} = \{(x,y)/\ |y| < M, |x-x_0| < h_0^{-1}(y+M)\}. \tag{3.4}$$

This domain is a relatively compact set in $\mathbb{R}^n$ which has a central axis of symmetry that this is $\{(x_0, y)/\,|y| < tM\}$, as the parameter a becomes larger, the domain $\Omega_{a,x_0,\gamma,H}$ becomes larger but the variation of ω_a along the axis of symmetry decreases and tends to zero as a tends to infinity. Then upper barriers $\omega = \omega_{a,x_0,\gamma,H}$ and lower barrier $l = l_{a,x_0,\gamma,H}$ have the properties that, for any constant b,

$$Q_1(\omega + b) \leq 0 \text{ in } \Omega_{a,x_0,H}, \tag{3.5}$$

$$\omega \geq \vartheta \text{ on } \overline{\Omega}_{a,x_0,H}, \tag{3.6}$$

$$\omega(x_0, y) \leq \vartheta + \frac{2M}{H} \text{ for } |y| \leq M, \tag{3.7}$$

$$Q_1(l + b) \geq 0 \text{ in } \Omega_{a,x_0,H}, \tag{3.8}$$

$$l \leq \vartheta \text{ on } \overline{\Omega}_{a,x_0,H}, \tag{3.9}$$

and

$$l(x_0, y) \geq \vartheta - \frac{2M}{H} \text{ for } |y| \leq M. \tag{3.10}$$

4 Proof of the Main Result

Let $\epsilon > 0$, $\alpha = \epsilon$. Let $\delta = \delta_0$,k_1 and k_2 be as in Assumption 3 . From Assumption 2 and the continuity of $H(y)$, one sees that there exist $\delta_1 > 0$ and R_1 such that if $(\overrightarrow{x}, y) \in \partial\Omega$, $|\overrightarrow{x}| \geq R_1$, $|y| \leq M$ and $\left|\frac{\overrightarrow{x}}{|\overrightarrow{x}|} - \omega\right| < \delta_1$, one has

$$|\phi(\overrightarrow{x}, y) - H(y)| < \epsilon. \tag{4.1}$$

Assumption 1 implies there exist $\delta_2 > 0$ and R_2 such that

$$\frac{b(\overrightarrow{x}, y, z, \overrightarrow{p}, q)}{a_{nn}(\overrightarrow{x}, y, z, \overrightarrow{p}, q)} - E(y, z, p_1, q) \leq \frac{\delta}{16}\ , \tag{4.2}$$

$$\frac{a_{11}(\overrightarrow{x}, y, z, \overrightarrow{p}, q)}{a_{nn}(\overrightarrow{x}, y, z, \overrightarrow{p}, q)} - a^*_{11}(y, z, p_1, q) \leq \frac{\delta}{16B_0}, \tag{4.3}$$

$$\frac{a_{1n}(\overrightarrow{x}, y, z, \overrightarrow{p}, q)}{a_{nn}(\overrightarrow{x}, y, z, \overrightarrow{p}, q)} - a^*_{1n}(y, z, p_1, q) \leq \frac{\delta}{16B_0}. \tag{4.4}$$

If $|\overrightarrow{x}| \geq R_1$, $|p'| = |(p_2, ..., p_{n-1})| \leq \delta_2$ and $\left|\frac{\overrightarrow{x}}{|\overrightarrow{x}|} - \omega\right| < 2\delta_2$, consider the compact set

$$K = \{(\overrightarrow{p}, q) \in \mathbb{R}^n/\,|\overrightarrow{p}|^2 + q^2 \leq 2(1 + ||Dk||^2_\infty)\}$$

From (2.12), one sees that there exists $\mu(K)>0$ such that $a_{nn}(\overrightarrow{x},y,z,\overrightarrow{p},q)\geq \mu(K)$ if $(\overrightarrow{p},q)\in K$, $\overrightarrow{x}\in\mathbb{R}^{n-1}$, $y,z\in\mathbb{R}$. set $T_2(f)=||f-k_2||_\infty$. since $E(y,z,p_1,q)$ is uniformly continuous for (y,z,p_1,q) in a fixed compact set, there exist $\delta_3>0$ such that

$$|E(y,z,p_1+s,q+t)-E(y,z,p_1,q)|\leq\frac{\delta}{16B_0}, \tag{4.5}$$

$$|a_{11}^*(y,z,p_1+s,q+t)-a_{11}^*(y,z,p_1,q)|\leq\frac{\delta}{16B_0}, \tag{4.6}$$

and

$$|a_{1n}^*(y,z,p_1+s,q+t)-a_{1n}^*(y,z,p_1,q)|\leq\frac{\delta}{16B_0}. \tag{4.7}$$

for $|s|$,$|t|\lessdot\delta_3$, $|y|\lessdot M$, $|z|\lessdot T_2(f)$, and $|q|^2\leq 2(1+||Dk||_\infty^2)$.

Let us set

$$\delta_0=\min\{1,\frac{1}{2}\alpha_0,\delta_1,\delta_2,\delta_3\}$$

and choose $H\geq 2$ such that $\frac{2M}{H}<\epsilon$, $\varkappa(H)\leq\ln(2)$,

$$A(H)\geq 16T_2(f)\ ,\ A(H)\geq\frac{5}{\mu(K)\delta} \tag{4.8}$$

and

$$\frac{2\sqrt{2T_2(f)A(H)e^{\varkappa(H)}}}{A(H)}+\frac{2}{H}<\delta_0, \tag{4.9}$$

where $A(H)$ is given in (3.1). There exists $R_3>0$ such that if $|\overrightarrow{x}_0|\geq R_3$, $|\overrightarrow{x}-\overrightarrow{x}_0|\leq A(H)e^{\varkappa(H)}$, and $\left|\frac{\overrightarrow{x}_0}{|x_0|}-\omega\right|<\delta_0$ then, $\left|\frac{\overrightarrow{x}}{|\overrightarrow{x}_0|}-\omega\right|<2\delta_0(\leq\alpha_0)$.

Set $R_0=\max\{R_1,R_2,R_3\}+A(H)e^{\varkappa(H)}$, define

$$W=\{\overrightarrow{x}/\,|\overrightarrow{x}|>R_0,\left|\frac{\overrightarrow{x}}{|\overrightarrow{x}_0|}-\omega\right|<\delta_0\}.$$

We claim that if $(\overrightarrow{x}_0,y)\in\overline{\Omega}$ and $\overrightarrow{x_0}=(x_1^{(0)},x_2^{(0)},...,x_{n-1}^{(0)})\in W$, then $(\overrightarrow{x_0},y)\in\overline{\Omega}_l^{(\alpha_0)}$ for some $l=l(\overrightarrow{x_0})\in\mathbb{N}$ and

$$f(\overrightarrow{x}_0,y)<tk(x_1^{(0)}-d_l,y)+2\epsilon. \tag{4.10}$$

Throughout the remainder of this proof, let $\overrightarrow{x}_0$represent a point in W such that $(\overrightarrow{x}_0, y) \in \overline{\Omega}$ for some $y \in I_M$ and let $l(\overrightarrow{x}_0)$ represent the natural number referenced in the previous sentence.

Let $\omega(\overrightarrow{x}, y) = \omega_{a,\overrightarrow{x_0},\gamma,H}(\overrightarrow{x}, y)$ be the upper barrier with $\gamma = 2\epsilon$ and $a = A(H)$. set $l = l(\overrightarrow{x}_0)$. Notice then that $\omega \geq \gamma = 2\epsilon$ on $\Omega_{a,\overrightarrow{x}_0,H}$.

Now, set

$$\Omega_{1,l} = \{(\overrightarrow{x}, y) \in \Omega_{a,\overrightarrow{x}_0,H} \cap \Omega_l^{(\alpha_0)} / |\overrightarrow{x} - \overrightarrow{x}_0| < \sqrt{2T_2(f)A(H)e^{\varkappa(H)} - T_2(f)^2}\} \tag{4.11}$$

and define $u_2 \in C^1(\overline{\Omega}_{1,l}) \cap C^2(\Omega_{1,l})$ by

$$u_2(\overrightarrow{x}, y) = \omega(\overrightarrow{x}, y) + k_2(x_1 - d_l, y).$$

Notice that if $(\overrightarrow{x}_0, y) \in \Omega_{1,l}$, then $|\overrightarrow{x}| \geq \max\{R_1, R_2, R_3\}$, $h_a'(h_a^{-1}(y+M)) \geq H \geq 2$, $A(H) < h_a^{-1}(y+M) < A(H)e^{\varkappa(H)}$, and $\left|\frac{\overrightarrow{x}}{|\overrightarrow{x}|} - \omega\right| < 2\delta_0$.

Let $\zeta \geq 0$. We claim that

$$Q(u_2 + \zeta) < 0 \text{ in } \Omega_{1,l} . \tag{4.12}$$

From [4, section 7], it follows that

$$\frac{\partial^2 \omega}{\partial x_i \partial x_j}(\overrightarrow{x}, y) = \frac{\delta_{ij} S^2 + (x_i - x_i^{(0)})(x_j - x_j^{(0)})}{S^3} \text{ for } 1 \leq i \text{ , } j \leq n-1 \text{ ,}$$

$$\frac{\partial^2 \omega}{\partial x_i \partial y}(\overrightarrow{x}, y) = \frac{-(x_i - x_i^{(0)})Z}{S^3 h_a'(Z)} \text{ for } 1 \leq i \leq n-1 \text{ ,}$$

$$\frac{\partial^2 \omega}{\partial y^2}(\overrightarrow{x}, y) = \frac{S^2(Zh_a''(Z) - h_a'(Z) + Z^2 h_a'(Z)}{S^3(h_a'(Z))^3}$$

where $\overrightarrow{x_0} = (x_1^{(0)}, ..., x_{n-1}^{(0)})$, $Z = h_a'(y+M)$, and $S = \sqrt{(h_a'(y+M))^2 - |\overrightarrow{x} - \overrightarrow{x}_0|^2} = \sqrt{Z^2 - |\overrightarrow{x} - \overrightarrow{x}_0|^2}$.

Since $A(H) < Z < 2A(H)$, $|\overrightarrow{x} - \overrightarrow{x}_0|^2 < 2T_2(f)A(H)e^{\varkappa(H)} - T_2(f)^2 \leq 4T_2(f)A(H) - T_2(f)^2$, and $A(H) \geq 16T_2(f)$, it is easy to see that $2S^2 \geq (A(H))^2$. Notice that $|D\omega(\overrightarrow{x}, y)| \leq \frac{|\overrightarrow{x} - \overrightarrow{x}_0|}{S} + \frac{Z}{S|h_a'(Z)|} \leq \frac{2\sqrt{2T_2(f)A(H)e^{\varkappa(H)} - T_2(f)^2}}{A(H)} + \frac{2}{H}$ and so (4.9) implies $|D\omega(\overrightarrow{x}, y)| < \delta_0$.

If one sets $\xi_i = \frac{x_i - x_i^{(0)}}{S}$ for $1 \leq i \leq n-1$, $\xi_n = \frac{-Z}{Sh_a'(Z)}$, and $\overrightarrow{\xi} = (\xi_{1,...,}\xi_n)$, then $\left|\overrightarrow{\xi}\right| \leq 1$ and $\frac{1}{S}\sum_{i,j=1}^n a_{ij}(\overrightarrow{x}, y, u_2 + \zeta, Du_2)\xi_i\xi_j \leq \frac{1}{S}$.

Since $\sum_{i,j=1}^{n-1} a_{ij}(\overrightarrow{x}, y, u_2+\zeta, Du_2)\frac{\delta_{ij}}{S} = \frac{1}{S}(1 - a_{nn}(\overrightarrow{x}, y, u_2+\zeta, Du_2))$ and $\frac{Zh''_a(Z)}{S(h'_a(Z))^3} = \frac{-1}{S}$, then $\sum_{i,j=1}^{n} a_{ij}(\overrightarrow{x}, y, u_2+\zeta, Du_2)D_{ij}\omega(\overrightarrow{x}, y) \leq \frac{2}{S} - \frac{1}{S}(2 + \frac{1}{(h'_a(Z))^2})a_{nn}(\overrightarrow{x}, y, u_2+\zeta, Du_2) < \frac{2}{S}$.
Since $|D\omega(\overrightarrow{x}, y)| < \delta_0 \leq 1$ when $(\overrightarrow{x}, y) \in \Omega_1$, $Du_2(\overrightarrow{x}, y) \in k$ and so $a_{nn}(\overrightarrow{x}, y, u_2+\zeta, Du_2) \geq \mu(K)$ if $(\overrightarrow{x}, y) \in \Omega_{1,l}$.
Set $\mu = \mu(K)$. From (4.8), one obtains $\frac{2}{S} \leq \frac{M\delta}{2}$. Notice that

$$E(y, u_2+\zeta, p_1, q) \leq E(y, u_2, p_1, q) \leq E(y, k_2, p_1, q). \tag{4.13}$$

For all $y \in I_M$ and $p_1, q \in \mathbb{R}$, since $\zeta \geq 0$ and $u_2 = \omega + k_2 \geq 2\epsilon + k_2 > k_2$.

Using (1.38), (1.41)–(1.43) and (1.49), one has

$$\begin{aligned}
Qu_2(\overrightarrow{x}, y) &= Q(\omega + k_2 + \zeta)(\overrightarrow{x}, y) \\
&= \sum_{i,j=1}^{n} a_{ij}D_{ij}\omega + a_{11}D_{11}k_2 + 2a_{1n}D_{1n}k_2 + a_{nn}D_{nn}k_2 + b \\
&< \frac{M\delta}{2} + (\frac{a_{11}}{a_{nn}}D_{11}k_2 + 2\frac{a_{1n}}{a_{nn}}D_{1n}k_2 + D_{nn}k_2 + \frac{b}{a_{nn}})a_{nn} \\
&= \frac{M\delta}{2} + a_{nn}[\frac{b(u_2+\zeta, Du_2)}{a_{nn}(u_2+\zeta, Du_2)} - E(u_2+\zeta, D_1u_2, D_nu_2) + \\
E(u_2+\zeta, D_1u_2, D_nu_2) &\quad -E(u_2, D_1u_2, D_nu_2) + E(u_2, D_1u_2, D_nu_2) - E(u_2, D_1k_2, D_nk_2) \\
&\quad +E(u_2, D_1k_2, D_nk_2) - E(k_2, D_1k_2, D_nk_2)] \\
&\quad +a_{nn}D_{11}k_2[\frac{a_{11}(Du_2)}{a_{nn}(Du_2)} - a^*_{11}(D_1u_2, D_nu_2) + a^*_{11}(D_1u_2, D_nu_2) \\
-a^*_{11}(D_1k_2, D_nk_2)] &\quad +2a_{nn}D_{1n}k_2[\frac{a_{1n}(Du_2)}{a_{nn}(Du_2)} - a^*_{1n}(D_1u_2, D_nu_2) + a^*_{1n}(D_1u_2, D_nu_2) \\
-a^*_{1n}(D_1k_2, D_nk_2)] &\quad +a_{nn}[a^*_{11}D_{11}k_2 + 2a^*_{1n}D_{1n}u_2 + D_{nn}u_2 + E(k_2, D_1k_2, D_nk_2)] \\
&\leq \frac{M\delta}{2} + [\frac{\delta}{8} + \frac{|D_{11}k_2|\,\delta}{8B_0} + \frac{2\,|D_{1n}k_2|\,\delta}{8B_0} - \delta]a_{nn}(Du_2) \\
&\leq \frac{M\delta}{2} - \frac{M\delta}{2} = 0,
\end{aligned}$$

where $a_{ij} = a_{ij}(\overrightarrow{x}, y, u_2+\zeta, Du_2)$, $a^*_{11}(D_1u_2, D_nu_2) = a^*_{11}(y, D_1u_2, D_nu_2)$, $b = b(u_2+\zeta, Du_2) = b(\overrightarrow{x}, y, u_2+\zeta, Du_2)$, etc.

If $(\overrightarrow{x}, y) \in \partial\Omega_{1,l} \cap \partial\Omega_l^{(\alpha_0)}$, from (2.6), (2.8) and (4.1) one has

$$\begin{aligned}
f(\overrightarrow{x}, y) &= \phi(\overrightarrow{x}, y) < H(y) + \epsilon \leq k_2(x_1 - d_l, y) + 2\epsilon \\
&= k_2(x_1 - d_l, y) + \gamma \leq k_2(x_1 - d_l, y) + \omega(\overrightarrow{x}, y).
\end{aligned}$$

Thus

$$f(\overrightarrow{x},y)-u_2(\overrightarrow{x},y)\lessdot 0 \text{ on } \partial\Omega_{1,l}\cap\partial\Omega_l^{(\alpha_0)}\ .$$

If $(\overrightarrow{x},y)\in\Omega_l^{(\alpha_0)}\cap\partial\Omega_{1,l}$, then $|\overrightarrow{x}-\overrightarrow{x}_0|\lessdot\sqrt{2T_2(f)A(H)e^{\varkappa(H)}-T_2(f)^2}$ and so

$$\begin{aligned}\omega(\overrightarrow{x},y) &= 2\epsilon+A(H)e^{\varkappa(H)}-\sqrt{(h_a^{-1}(y+M))^2-|\overrightarrow{x}-\overrightarrow{x}_0|^2}\\ &\geq 2\epsilon A(H)e^{\varkappa(H)}-\sqrt{(A(H)e^{\varkappa(H)})^2-2T_2(f)A(H)e^{\varkappa(H)}-T_2(f)^2}\\ &= 2\epsilon+T_2(f)\end{aligned}$$

Hence

$$\begin{aligned}f(\overrightarrow{x},y)-k_2(y) &\leq ||f-k_2||_\infty\\ &= T_2(f)\\ &\leq \omega(\overrightarrow{x},y)-2\epsilon\\ &< \omega(\overrightarrow{x},y)\end{aligned}$$

and so $f(\overrightarrow{x},y)\lessdot u_2(\overrightarrow{x},y)$ for $(\overrightarrow{x},y)\in\Omega_l^{(\alpha_0)}\cap\partial\Omega_{1,l}$.

Let $U_0=\{(\overrightarrow{x},y)\in\Omega_{1,l}/f(\overrightarrow{x},y)<u_2(\overrightarrow{x},y)\}$ since $f<u_2$ on $\partial\Omega_{1,l}$, U_0 is relatively compact subset of $\Omega_{1,l}$ and $f=u_2$ on $\Omega_{1,l}\cap\partial U_0=\partial U_0$.

Now, define

$$Ru(\overrightarrow{x},y)=\sum_{i,j=1}^{n}\overline{a}_{ij}(\overrightarrow{x},y,Du)D_{ij}u(\overrightarrow{x},y)+\overline{b}(\overrightarrow{x},y,Du).$$

Set $\overline{a}_{ij}(\overrightarrow{x},y,\overrightarrow{p},q)=a_{ij}(\overrightarrow{x},y,f(\overrightarrow{x},y),\overrightarrow{p},q)$ and $\overline{b}(\overrightarrow{x},y,\overrightarrow{p},q)=b(\overrightarrow{x},y,f(\overrightarrow{x},y),\overrightarrow{p},q)$.

Let $(\overrightarrow{x},y)$ be an arbitrary point in U_0 and set $\xi=f(\overrightarrow{x},y)-u_2(\overrightarrow{x},y)>0$.

Since $Q(u_2+\zeta)<0$ on $\Omega_{1,l}$, it follows that

$$Ru_2(\overrightarrow{x},y)=Q(u_2+\zeta)(\overrightarrow{x},y)<0.$$

Since $(\overrightarrow{x}_1,y_1)$ is an arbitrary point in U_0, one sees that

$$Ru_2<0 \text{ in } U_0 \tag{4.14}$$

A standard argument (e.g section 3 of [6]) then implies $U_0=\phi$ and so $f(\overrightarrow{x},y)\leq u_2(\overrightarrow{x},y)$ on Ω_1.

Therefore,

$$\begin{aligned}f(\overrightarrow{x}_0,y) &\leq \omega(\overrightarrow{x}_0,y)+k_2(x_1-d_l,y)\\ &\leq \frac{2M}{H}+k_2(x_1-d_l,y)\\ &< 2\epsilon+k(x_1-d_l,y) \quad \text{or}\end{aligned}$$

$$f(\overrightarrow{x}_0, y) - k(x_1 - d_l, y) < 2\epsilon.$$

Together with a similar argument using lower barriers and $k_1(y)$; (i.e., $u_1(\overrightarrow{x}, y) = l_a(\overrightarrow{x}, y) + k_1(x_1 - d_l, y)$ with $\Psi(\rho) = 1$), one finds that

$$\left| f(\overrightarrow{x}_0, y) - k(x_1^{(0)} - d_l, y) \right| < 2\epsilon \ .$$

Since $\overrightarrow{x}_0 \in W$ is arbitrary,

$$\left| f(\overrightarrow{x}, y) - k(x_1 - d_{l(\overrightarrow{x})}, y) \right| < 2\epsilon \tag{4.15}$$

for $(\overrightarrow{x}, y) \in \Omega$ with $\overrightarrow{x} \in W$.

Suppose $(\overrightarrow{x}_{(l)}, y(l)) \in \overline{\Omega_l^{(\alpha_0)}}$ for each $l \in \mathbb{N}$ with $\frac{\overrightarrow{x}_{(l)}}{\left|\overrightarrow{x}_{(l)}\right|} \to \omega$ and $(\overrightarrow{x}_{(l),1} - d_l, y(l)) \to (t, y)$ as $l \to \infty$. Then there exists $N > 0$ such that

$$\overrightarrow{x}_{(l)} \in W \text{ and } \left| k(x_{(l),1} - d_l, y(l)) - k(t, y) \right| < t\epsilon,$$

when $l \geq N$.

If $l > N$, then (4.15) yields

$$\begin{aligned} \left| f(\overrightarrow{x}_{(l)}, y(l)) - k(t, y) \right| &\leq \left| f(\overrightarrow{x}_{(l)}, y(l)) - k(x_{(l),1} - d_l, y(l)) \right| + \left| k(x_{(l),1} - d_l, y(l)) - k(t, y) \right| \\ &\leq 3\epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the conclusion of Theorem 2.1 follows.

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