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New Result For Large Deviation Principle For Stochastic Evolution Equations in the Besov-Orlicz Space

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Abstract

In this paper, we develop a large deviations principle for random evolution equations to the Besov-Orlicz space $\mathcal{B}^{u,0}_{M_2,w}$ corresponding to the Young function $M_2(t) = \exp(x^2) - 1$.

1 Introduction

In recent years, many results on the Brownian motion and diffusion processes in path spaces with stronger topologies than the usual uniform one have been obtained [9]. In [2], large deviation principles have been developed in Hölder spaces for the Brownian motion. In [5], general diffusion processes were presented. Later on, an extension to Besov spaces was considered in [8], [10], and [15]. The main purpose of this work is to reproduce the results proved in [3] in the case of a process of random evolution. Our method is based on the work of Baldi and Sanz [3], (see also Priouret work in [13]). Consider the

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solution $X^\varepsilon = \{X_t^\varepsilon, 0 \le t \le 1\}$ of the equation:

$$X_t^\varepsilon = x + \int_0^t b(X_s^\varepsilon, Y_s)ds + \varepsilon^{\frac{1}{2}} \int_0^t \sigma(X_s^\varepsilon, Z_s)dW_s, \tag{1.1}$$

where $x \in \mathbb{R}^d$, W is a standard Brownian motion taking values in $\mathbb{R}^k$, Y is a progressively measurable random process which satisfies some integrability conditions and Z is a random process whose topological support is a compact subset of $\mathcal{B}^{u,0}_{M_2,w}$. Moreover, W is independent of (Y, Z) and σ and b satisfy some regularity assumptions which we will describe later.

This paper is organized as follows: In section 2, we present some notions and results on the topology of Besov-Orlicz space. In section 3, we give some definitions and general results. In section 4, we find our main result with its proof given in section 5.

2 Notions and results on the topology of Besov-Orlicz space

Let $I = [0, 1]$ and denote the space of Lebesgue integrable $\mathbb{R}^d$-valued functions with exponent p by $L^p(I)$, $(1 \le p < \infty)$.
Let A_{M_2} be the Orlicz space on I corresponding to the Young function $M_2(x) = \exp(x^2) - 1$ endowed with the norm

$$\| f \|_* = \inf \left\{ \tau > 0, \frac{1}{\tau}[1 + \int_0^1 M_2(\tau |f(t)|)dt] \right\}.$$

For more details on Orlicz spaces, we refer the reader to [7]. In this paper, we use the following equivalence norm in A_{M_2} :

$$\| f \|_{M_2} = \sup_{p \ge 1} \frac{\| f \|_p}{\sqrt{p}}.$$

The proof of the equivalence of the norms $\| f \|_{M_2}$ and $\| f \|_*$ can be found in [7].
Note that, for $p_0 \ge m$,

$$\| f \|_{M_2} = \sup_{p \ge p_0} \frac{\| f \|_p}{\sqrt{p}}.$$

For $f \in L^p(I)$, $1 \le p < \infty$, we consider the following modulus of smoothness in $L^p(I)$ norm:

$$w_p(f, t) := \sup_{0 \le h \le t \le 1} \| \Delta_h f \|_p,$$

where
$$\Delta_h f(x) = 1_{[0,1-h]}(x)(f(x+h) - f(x)), \forall h \in [0,1].$$
The modulus of smoothness for the Orlicz norm is defined by
$$w_{M_2}(f,t) = \sup_{0\leq h\leq t\leq 1} \| \Delta_h f \|_{M_2} .$$
Let $\mathcal{B}^u_{M_2,w}$ denote the Besov-Orlicz space of continuous functions $f : [0,1] \to \mathbb{R}^u$, $u = d, k$ or l such that $\| f \|_{M_2,w} < \infty$. For all $\alpha > 0$, put
$$\| f \|_{M_2,w} = \| f \|_{M_2} + \sup_{0\leq t\leq m} \frac{w_{M_2}(f,t)}{w(t)},$$
where $w(t) = \sqrt{t(1+\log\frac{1}{t})}$. We will use the equivalence in [7]. Let $\chi_1, \chi_{j,k}, j = 0,1..., k = 1...2^j, supp\chi_{j,k} = [(k-1)/2^j, k/2^j]$ be the set of Haar functions over the interval $[0,1]$ and let $\varphi_0(t) = 1, \varphi_1(t) = t, \varphi_{j,k}(t) = \int_0^t \chi_{j,k}(s)ds$ be the set of Schauder functions. For all continuous functions $f : [0,1] \to \mathbb{R}^d$, the Schauder series expansion is:
$$f(t) = f_0\varphi_0(t) + f_1\varphi_1(t) + \sum_{n=2^j+1}^{2^{j+1}} \sum_{j,k} f_{j,k}\varphi_{j,k}(t),$$
where $f_0 = f(0), f_1 = f(1) - f(0)$ and
$$f_{j,k} \quad = \quad 22^{\frac{j}{2}}\big[\big(f(\frac{2k-1}{2^{j+1}}) - \frac{1}{2}\big(f\big(\frac{2k}{2^{j+1}}\big) + f\big(\frac{2k-2}{2^{j+1}}\big)\big)\big].$$
Let $\mathcal{B}^{u,0}_{M_2,w}$ be the subspace of $\mathcal{B}^u_{M_2,w}$ corresponding to the sequences $f_{j,k}$ such that
$$\mathcal{B}^{u,0}_{M_2,w} = \{f \in \mathcal{B}^u_{M_2,w},\ \| f \|_p = o(\sqrt{p})$$
$$when\ p \to \infty,\ w_p(f,t) = o(\sqrt{p}w(t))\ when\ \frac{1}{p} \wedge t \to 0\}.$$

Theorem 2.1. *1) Let $p_0 \geq 0$, $f \in \mathcal{B}^{u,0}_{M_2,w}$ if and only if*
$$\max\big\{|f_0|, |f_1|, \sup_{j\geq 0}\sup_{p\geq p_0} \frac{2^{-\frac{j}{p}}}{p^{\frac{1}{2}}(1+j)^{\frac{1}{2}}} \| f_{j,\cdot} \|_p \big\} < \infty. \tag{2.2}$$

2) $f \in \mathcal{B}^{u,0}_{M_2,w}$ if and only if
$$\lim_{j\vee p\to\infty} \frac{2^{-\frac{j}{p}}}{p^{\frac{1}{2}}(1+j)^{\frac{1}{2}}} \| f_{j,k} \|_p = 0. \tag{2.3}$$

3 Definitions and general results

In this section, we recall some definitions and results.

Definition 3.1. *A rate function is a function $I : \Xi \longrightarrow [0;\infty]$ on a Hausdorff topological space Ξ which is lower semi-continuous; i. e., where all the level set $\Gamma_\lambda = \{x \in \Xi, I(x) \leq \lambda\}$ are closed in Ξ. A rate function $I : \Xi \longrightarrow [0;\infty]$ is called a good rate function if all the level set $\{x \in \Xi, I(x) \leq \lambda\}$ for $\lambda \geq 0$ are compact in Ξ*

Definition 3.2. *A family $\{P^\varepsilon\}_{\varepsilon>0}$ of probability measures on a Hausdorff topological space Ξ satisfies the large deviation principle (or shorter LDP) with rate function $I : \Xi \longrightarrow [0;\infty]$ if the following two estimates hold:*

i) (Lower bound.) For every open subset $\mathcal{O}$ of Ξ

$$\liminf_{\varepsilon\longrightarrow 0} \varepsilon \log P_\varepsilon(\mathcal{O}) \geq -I(\mathcal{O})$$

ii) (Upper bound.) For every closed subset $\mathcal{F}$ of Ξ

$$\limsup_{\varepsilon\longrightarrow 0} \varepsilon \log P_\varepsilon(\mathcal{F}) \leq -I(\mathcal{F})$$

Mellouk [12] has given a proof of the following Schilder theorem:

Theorem 3.3. *Let P^ε be the law of ϵW on $\mathcal{B}^{u,0}_{M_2,w}$ equipped with the norm $\| \, . \, \|_{M_2,w}$. Then P^ε satisfies the LDP with the good rate function λ defined by:*

$$\lambda(g) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{g}(s)|^2 ds & if \ g \in \mathcal{H} \\ \\ \infty & otherwise \end{cases} . \tag{3.4}$$

.

Theorem 3.4. *Let P^ε be the probability measure family on a Polish space E satisfying the LDP with a good rate function λ and let $F : E \to E'$ be a continuous function. Denote by $Q^\varepsilon = P^\varepsilon \circ F^{-1}$ the image measure family of P^ε by F. Then $\{Q^\varepsilon\}$ satisfies the LDP with a good rate function $\tilde{\lambda}$ defined by:*

$$\tilde{\lambda}(y) = \inf_{x:F(x)=y} \lambda(x)$$

with $\inf \varnothing = \infty$.

4 Main results

Let X^ε be the solution of:

$$X_t^\varepsilon = x + \int_0^t b(X_s^\varepsilon, Y_s)ds + \varepsilon^{\frac{1}{2}} \int_0^t \sigma(X_s^\varepsilon, Z_s)dW_s. \quad (4.5)$$

Let $\Omega = \mathcal{C}([0,1], \mathbb{R}^k)$ be the space of trajectories of a standard $\mathbb{R}^k$-valued Brownian motion W, P the Wiener measure and $\mathcal{F}$ the completion of the Borel σ-field of Ω with respect to P. Let $Y = \{Y_t, t \in [0,1]\}$ be a $\mathbb{R}^m$-valued process which is $\{\mathcal{F}_t\}$ progressively measurable. We suppose that Y is a random variable with values in $L^1([0,1], \mathbb{R}^m)$. Let $Z = \{Z_t, t \in [0,1]\}$ be an $\mathcal{F}_t-$ progressively measurable process taking values in $\mathbb{R}^l$. Assume that $suppZ$ is a compact subset in $\mathcal{B}^{l,0}_{M_2,w}$ and that (Y, Z) and W are independent. Now Suppose that the coefficients σ and b satisfy the following hypotheses (L):

i) $\sigma : \mathbb{R}^d \times \mathbb{R}^l \to \mathcal{M}_{d\times k}$, is a space of smooth $d \times k$ matrix and $b : \mathbb{R}^d \times \mathbb{R}^m \to \mathcal{M}_{d\times k}$.

ii) The function $b(x, y)$ is jointly measurable in (x, y) and there exists a constant $C > 0$ such that

$$\mid b(x,y) \mid \leq C(1+ \mid x \mid), \quad \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^m$$

$\mid b(x_1, y_1) - b(x_2, y_2) \mid \leq C(\mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid), \quad \forall (x_1, x_2) \in \mathbb{R}^d, \forall (y_1, y_2) \in \mathbb{R}^m.$

iii) The function $\sigma(x, z)$ is jointly measurable in (x, z) and there exists a constant $C > 0$ such that

$$\mid \sigma(x,z) \mid \leq C, \quad \forall (x,z) \in \mathbb{R}^d \times \mathbb{R}^m$$

$\mid \sigma(x_1, z_1) - \sigma(x_2, z_2) \mid \leq C(\mid x_1 - x_2 \mid + \mid z_1 - z_2 \mid), \quad \forall (x_1, x_2) \in \mathbb{R}^d, \forall (z_1, z_2) \in \mathbb{R}^m.$

The existence of a unique solution of (4.5) which is $(\mathcal{F}_t)_{t\in[0,1]}$-adapted and has continuous sample paths is ensured by our assumptions on σ and b.

Theorem 4.1. *Assume* (L) *is satisfied. Let* X^ε *be a solution of* (4.5)*. Then*

$$\mathbb{P}(X^\varepsilon \in \mathcal{B}^{u,0}_{M_2,w}) = 1.$$

Let $\mathcal{H}$ be the Cameron-Martin space associated with the Brownian motion.

$$\mathcal{H} = \left\{ h \in L^2([0,1]); \exists\, \dot{h} \in L^2([0,1]),\ h_t = \int_0^t \dot{h}_s ds,\ t \in [0,1] \right\}.$$

For $h \in \mathcal{H}([0,1], \mathbb{R}^k)$, $r \in L^p([0,1], \mathbb{R}^m)$ and $u \in suppZ$, we define the Skeleton $S_z(h,r,u) = g$ by

$$g_t = z + \int_0^t b(g_s, r_s) ds + \int_0^t \sigma(g_s, u_s) \dot{h} ds. \tag{4.6}$$

Define $\bar{\lambda} : \mathcal{B}^{d,0}_{M_2,w} \to [0,\infty]$ by

$$\bar{\lambda}(\bar{h}) = \inf\{\lambda(h); h \in \mathcal{H}([0,1], \mathbb{R}^d) : \exists (r,u) \in suppY \times suppZ, \bar{h} = S(h,r,u)\}. \tag{4.7}$$

Since $\bar{\lambda}$ is not necessarily lsc (see for example [6]), we introduce its lsc regularization $\bar{\lambda}^*$ defined by

$$\bar{\lambda}^*(\bar{h}) = \lim_{a \to 0} \lim_{\rho \in B_X(\bar{h}, a)} \bar{\lambda}(\rho), \tag{4.8}$$

where $B_X(\bar{h}, a)$ is the ball of radius a centered at $\bar{h}$ with respect to the norm $\| \cdot \|_{M_2,w}$.

Theorem 4.2. *Assume* (L)*. Then, the family* X^ε *of solution of* (4.5) *satisfies a large deviation principle on the space* $\mathcal{B}^{d,0}_{M_2,w}$ *with a good rate function* $\bar{\lambda}^*$ *defined in* (4.8)*.*

Proof. For the proof of this result we refer the interested reader to [14] □

5 Proof of the Main Results

In this section, we show that Proposition (5.1) below is indeed equivalent to Theorem (4.2).

Proposition 5.1. *Assuming* (L)*, we have, for any* $r > 0$*,* $\rho > 0$*, there exist* $\varepsilon_0 > 0$*,* $\beta > 0$*,* $\tilde{r} > 0$*, such that if* $\varepsilon \le \varepsilon_0$*,* $| x - z | < \tilde{r}$*, then*

$$P\Big(\| X^\varepsilon - g \|_{M_2,w} > \rho, \| \varepsilon^{\frac{1}{2}} W - h \|_{M_2,w} + \| Y - r \|_{L^1([0,1],\mathbb{R}^m)} + \| Z - u \|_{M_2,w} < \beta \Big) \le \exp\big(-\frac{r}{\varepsilon} \big) \tag{5.9}$$

In what follows we show that Proposition (5.1) can be deduced easily from Theorem (4.2).

Proof. We modify the proof introduced in [3]. Consider the diffusion $\widehat{X}^{\varepsilon}_t = \begin{pmatrix} \tilde{X}^{\varepsilon}_t \\ Y_t \\ Z_t \end{pmatrix}$ where $\tilde{X}^{\varepsilon}_t = \begin{pmatrix} X^{\varepsilon}_t \\ \varepsilon^{\frac{1}{2}} W_t \end{pmatrix}$ and X^{ε}_t is solution of (4.5). We have

$$d\tilde{X}^{\varepsilon}_t = \tilde{b}(\tilde{X}^{\varepsilon}_t, Y_t)dt + \varepsilon^{\frac{1}{2}} \tilde{\sigma}(\tilde{X}^{\varepsilon}_t, Z_t)dW_t$$

where $\tilde{\sigma}(x,z) = \begin{pmatrix} \sigma(x,z) \\ I_d \end{pmatrix}$, $\tilde{b}(x,y) = \begin{pmatrix} b(x,y) \\ 0 \end{pmatrix}$, $\tilde{\sigma}$ is a matrix $(d+k) \times k$. For $h \in \mathcal{H}([0,1], \mathbb{R}^k)$, $r \in L^1([0,1], \mathbb{R}^m)$ and $u \in \mathcal{B}^{l,0}_{M_2,w}$, we define

$$\tilde{g} = \tilde{S}(h, r, u)$$

if and only if

$$\tilde{g}_t = \begin{pmatrix} z \\ 0 \end{pmatrix} + \int_0^t \tilde{b}(\tilde{g}_s, r_s)ds + \int_0^t \tilde{\sigma}(\tilde{g}_s, u_s)\dot{h}_s ds.$$

As the coefficients b and σ satisfy (L), Theorem (4.2) applies to $\tilde{X}^{\varepsilon}_t$ because the hypotheses (L) also satisfy the coefficients $\tilde{b}$ and $\tilde{\sigma}$. So, $\tilde{X}^{\varepsilon}_t$ satisfies LDP with the lsc regularisation rate function of

$$\tilde{\lambda} : \mathcal{B}^{d+k,0}_{M_2,w} \to [0, \infty]$$

$$\tilde{\lambda}(\tilde{h}) = \inf\{\lambda(h); h \in \mathcal{H}([0,1], \mathbb{R}^k) : \exists (r,u) \in suppY \times suppZ, \tilde{h} = \tilde{S}(h,r,u)\}.$$

$\tilde{S}$ is equivalent to S defined in (4.2).
Since $\tilde{\lambda}$ is not necessarily lsc, we introduce its lsc regularisation $\tilde{\lambda}^*$ defined by

$$\tilde{\lambda}^*(\tilde{h}) = \lim_{a \to 0} \lim_{\rho \in B_X(\tilde{h}, a)} \tilde{\lambda}(\rho).$$

$\tilde{S}$ decomposes into two systems, one on the first d coordinates and the other on the k one. Therefore

$$\tilde{S}(h, r, u) = \begin{pmatrix} S(h,r,u) \\ h \end{pmatrix}$$

where $S(h,r,u)$ is defined in (4.6). As in [[16], pp.701-706], $\widehat{X}^{\varepsilon}$ also satisfies a LDP with rate function $\widehat{\lambda}^{*}$, the lsc regularisation of $\widehat{\lambda}$ defined by

$$\widehat{\lambda}(\widehat{g}) = \inf\{\lambda(h); h \in \mathcal{H}([0,1],\mathbb{R}^k) : \exists (r,u) \in suppY \times suppZ \ such\ that \\ \widehat{S}(h,r,u) = (\tilde{S}(h,r,u),r,u) = \widehat{g}\}.$$

Let $A \subset \mathcal{B}^{d,0}_{M_2,w}$. Put

$$\widehat{\lambda}^{*}(\widehat{A}) = \inf\left\{\widehat{\lambda}^{*}(\widehat{g})\ when\ \widehat{g} \in \widehat{A}\right\}.$$

Now, fix $r > 0$, $\rho > 0$. Then

$$P\big(\parallel X^{\varepsilon} - S_z(h,r,u) \parallel_{M_2,w} > \rho, \parallel \varepsilon^{\frac{1}{2}} W - h \parallel_{M_2,w} \\ + \parallel Y - r \parallel_{L^1([0,1],\mathbb{R}^m)} + \parallel Z - u \parallel_{M_2,w} < \beta\big) = P(\widehat{X}^{\varepsilon} \in A_\beta)$$

with A_β being the set of trajectoiries

$$A_\beta = \left\{ \widehat{\nu} = \begin{pmatrix} \begin{pmatrix} \widehat{\nu}_1 \\ \widehat{\nu}_2 \end{pmatrix} \\ \widehat{\nu}_3 \\ \widehat{\nu}_4 \end{pmatrix} \in \big(\mathcal{B}^{d,0}_{M_2,w} \times \mathcal{B}^{k,0}_{M_2,w}\big) \times L^1([0,1],\mathbb{R}^m) \times \mathcal{B}^{l,0}_{M_2,w}, \parallel \widehat{\nu}_{1,x} - \right.$$

$$\left. S_z(h,r,u) \parallel_{M_2,w} \geq \rho, \parallel \widehat{\nu}_2 - h \parallel_{M_2,w} + \parallel \widehat{\nu}_3 - r \parallel_{L^1([0,1],\mathbb{R}^m)} + \parallel \widehat{\nu}_4 - u \parallel_{M_2,w} < \beta \right\}$$

where $\widehat{\nu}_{1,x}$ is a trajectory of $\mathcal{B}^{d,0}_{M_2,w}$ from x at the moment 0 and $S_z(h,r,u)$ is a solution of (4.6) from z at the moment 0. As the mapping $(h,r,u) \to S_z(h,r,u)$ is continuous (see Lemma (5.2) below) in $B_r \times suppY \times suppZ \to \mathcal{B}^{d,0}_{M_2,w}$, there exists $\beta > 0$ such that if $(\widehat{\nu}_1, \widehat{\nu}_2, \widehat{\nu}_3) \in B_r \times suppY \times suppZ$, and

$$\parallel \widehat{\nu}_1 - h \parallel_{B_r \cap \mathcal{B}^{k,0}_{M_2,w}} + \parallel \widehat{\nu}_2 - r \parallel_{L^1([0,1],\mathbb{R}^m)} + \parallel \widehat{\nu}_3 - u \parallel_{\mathcal{B}^{l,0}_{M_2,w}} < \beta.$$

Then

$$\parallel S_z(\widehat{\nu}_1), \widehat{\nu}_2, \widehat{\nu}_3 - S_z(h,r,u) \parallel_{M_2,w} < \rho,$$

where

$$B_r = \left\{ h \in \mathcal{H}([0,1],\mathbb{R}^k) : \left\{ \frac{1}{2}\int_0^t \mid \dot{h}_s \mid^2 ds \leq 2r \right\} \cap \mathcal{B}^{k,0}_{M_2,w} \right\}$$

and S is defined in (4.6).
For such a value of β, there is no trajectory $\widehat{\nu} \in A_\beta$ such that

$$\frac{1}{2}\int_0^1 \mid \dot{\widehat{\nu}}_2(s) \mid ds \leq 2r \ et \ \widehat{\nu} = \begin{pmatrix} S(\widehat{\nu}_2, \widehat{\nu}_3, \widehat{\nu}_4) \\ \widehat{\nu}_2 \\ \widehat{\nu}_3 \\ \widehat{\nu}_4 \end{pmatrix}.$$

So $\widehat{\lambda}^*(A_\beta) \geq r$.
By Theorem (4.2), we have, for all $r > 0$, $\rho > 0$, there exists $\varepsilon_0 > 0$, $\beta > 0$, $\tilde{r} > 0$ such that if $\varepsilon \leq \varepsilon_0$, $\mid x - z \mid < \tilde{r}$. We get

$$P\big(\parallel X^\varepsilon - S_z(h,r,u) \parallel_{M_2,w} > \rho, \parallel \varepsilon^{\frac{1}{2}} W - h \parallel_{M_2,w} + \parallel Y - r \parallel_{L^1([0,1],\mathbb{R}^m)} + \parallel Z - u \parallel_{M_2,w} < \beta\big) \leq \exp\big(-\frac{r}{\varepsilon}\big) \quad (5.10)$$

□

Lemma 5.2. *For all $a \geq 0$, the mapping $\overline{\mathbb{R}^d} \times \left(\mathcal{B}^{k,0}_{M_2,w} \cap \left\{h \in \mathcal{H}([0,1],\mathbb{R}^k) : \frac{1}{2}\int_0^1 \mid \dot{h}_s \mid^2 ds \leq a\right\}\right) \times L^1([0,1],\mathbb{R}^m) \times \mathcal{B}^{l,0}_{M_2,w} \to \mathcal{B}^{d,0}_{M_2,w}$ which to $(x,(h,r,u)) \to g$ solution of (4.2) is continuous.*

Proof. Let $x_n \to x \in \overline{R}^d$, $h_n \to h$ in $\mathcal{B}^{k,0}_{M_2,w} \cap \{h \in \mathcal{H}([0,1],\mathbb{R}^k) : \frac{1}{2}\displaystyle\int_0^1 \mid \dot{h}_s \mid^2 ds \leq a\}$, $r_n \to r$ in $L^1([0,1],\mathbb{R}^m)$ and $u_n \to u$ in $\mathcal{B}^{l,0}_{M_2,w}$. Let g^n_t be the solution of (4.6) relative x^n, h^n, r^n and u^n. We have

$$\begin{aligned} &\parallel g^n_t - g_t \parallel_{M_2,w} \\ &= \parallel \int_0^t \{b(g^n_v, r^n_v) + \sigma(g^n_v, u^n_v)\dot{h}^n_v\} dv \\ &- \int_0^t \{b(g_v, r_v) + \sigma(g_v, u_v)\dot{h}_v\} dv \parallel_{M_2,w} \\ &= \parallel \int_s^t \{(\sigma(g^n_v, u^n_v) - \sigma(g_v, u_v))\dot{h}^n_v \\ &+ b(g^n_v, u^n_v) - b(g_v, u_v) + \sigma(g_v, u_v)(\dot{h}^n_v - \dot{h}_v)\} dv \parallel_{M_2,w} \end{aligned}.$$

By the triangular inequality, we get

$$\begin{aligned}\| g_t^n - g_t \|_{M_2,w} &\leq \| \int_0^t \{b(g_v^n, r_v^n) - b(g_v, r_v)\}dv \|_{M_2,w} \\ &+ \| \int_0^t \{\sigma(g_v^n, u_v^n) - \sigma(g_v, u_v)\}\dot{h}_v^n dv \|_{M_2,w} \\ &+ \| \int_0^t \{\sigma(g_v, u_v)(\dot{h}_v^n - \dot{h}_v)\}dv \|_{M_2,w} \\ &= T_1 + T_2 + T_3\end{aligned}$$

Since T_1 is a particular case of T_2 if we replace in T_2 σ by b and $\dot{h}_v^n$ by 1, it suffices to prove that T_2 and T_3 tend to zero.
Set

$$V = \int_0^t \{\sigma(g_v^n, u_v^n) - \sigma(g_v, u_v)\}\dot{h}_v^n dv$$

Hence

$$T_2 = \| V \|_{M_2} + \sup_{-1\leq t\leq 1} \frac{w_{M_2}(V,t)}{w(t)}$$

Using the Lipschitz property of σ, we get

$$\| V \|_{M_2} \leq C(\| g^n - g \| + \| u_n - u \|) \| h^n \|_{\mathcal{H}} .$$

For the last term, recall that

$$w_p(V,t) = \sup_{|h|\leq t} \| \Delta_h V \|_p .$$

We have

$$\| \Delta_h V \|_p = (\int_{I_h} |V(t+h) - V(t)|^p dt)^{\frac{1}{p}}, \ where\ I_h = \{t \in I,\ t+h \in I\}$$

$$\begin{aligned}|V(t+h) - V(t)| &\leq |\int_t^{t+h} \{\sigma(g_v^n, u_v^n) - \sigma(g_v, u_v)\}\dot{h}_v^n dv| \\ &\leq C(\| g^n - g \| + \| u^n - u \|) \| h^n \|_{\mathcal{H}} |h|.\end{aligned}$$

Therefore,

$$w_p(V,t) = \sup_{|h|\leq t} \| \Delta_h V \|_p \leq C(\| g^n - g \| + \| u^n - u \|) \| h^n \|_{\mathcal{H}} |t|.$$

This implies that

$$\begin{aligned}\sup_{t\in[0,1]}\frac{w_{M_2}(V,t)}{w(t)} &\leq C(\| g^n-g \| + \| u^n-u \|) \| h^n \|_{\mathcal{H}} \frac{|t|}{\sqrt{t(1+\log\frac{1}{t})}}\\ &\leq K(\| g^n-g \| + \| u^n-u \|).\end{aligned}$$

So

$$\| T_2 \|_{M_2,w} \leq C_0(\| g^n-g \| + \| u^n-u \|_{M_2,w}). \tag{5.11}$$

We have

$$T_1 \leq C_1(\| g^n-g \| + \| r^n-r \|_{L^1([0,1],\mathbb{R}^m)}). \tag{5.12}$$

To complete the proof of the proposition, it remains to prove that T_3 tends to zero when $\| h^n-h \|_{M_2,w}\to 0$. We have

$$\begin{aligned}T_3(t) &\leq \| \int_0^t \{[\dot{h}_v^n-\dot{h}_v]d\sigma(g_v,u_v) \|_{M_2,w}\\ &+ \| \{\sigma(g_v,u_v)\}\{[h_v^n-h_v]\} \|_{M_2,w}\\ &\leq \| h^n-h \|\| \int_0^t d|\sigma(g_v,u_v)| \|_{M_2,w}\\ &+ C \| h^n-h \|_{M_2,w},\end{aligned}$$

where $|\sigma(g_v,u_v)|$ is the variation of $\sigma(g_v,u_v)$ on $[0,t]$.
Thus

$$\| T_2 \|_{M_2,w} \leq C \| h^n-h \|_{M_2,w} . \tag{5.13}$$

Combining (5.11), (5.12) and (5.13), we now get

$$\begin{aligned}\| g^n-g \|_{M_2,w} &\leq \Big(C_0(\| g^n-g \| + \| u^n-u \|_{M_2,w})\\ &+ C_1(\| g^n-g \| + \| r^n-r \|_{L^1([0,1],\mathbb{R}^m)} + C_2 \| h^n-h \|_{M_2,w} \Big)\end{aligned}$$

by virtue of the continuity of the map $g\in\mathcal{H}$ with respect to the uniform norm. Letting n tend to infinity, the result follows.

□

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