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Analytical Approximant to a Damped Pendulum Forced with a Constant Torque

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Abstract

In this work, we obtain an approximate analytical solution to a damped pendulum equation with constant torque. We also compare this solution both graphically and numerically with Runge-Kutta numerical solution.

1 Introduction

Since the time of Galileo [1], the pendulum has constituted a physical object fascinating physicists and becoming one of the paradigms in the study

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of physics and natural phenomena. In the framework of nonlinear dynamics, there is no doubt that the pendulum is one of the objects that have deserved more attention in modeling all kind of phenomena related to oscillations, bifurcations and chaos. The simple pendulum has been used as a physical model to solve problems such as: non-linear plasma oscillations, Duffing oscillators, rigid plates that satisfy the Johanessen performance criteria, transverse vibrations nonlinear of a plate carrying a concentrated mass, a beam supported by a double periodic axial oscillating mount, cracks subjected to concentrated forces, surface waves in a plasma column, coupled modes of nonlinear bending vibrations of a circular ring, double spin spacecraft, motion of spacecraft over slowly rotating asteroids, nonlinear vibration of clasped beams, the nonlinear equation of wave, non-linear mathematical models of DNA, the non-linear Schrodinger equation, among others. In this

work, we give an approximate analytical solution to the damped pendulum forced with a constant torque

$$\ddot{\theta}+\beta\dot{\theta}+\sin\theta=\gamma,\theta(0)=\theta_0 \text{ and } \theta'(0)=\dot{\theta}_0. \tag{1.1}$$

We focus on a mechanical example proposed by Andronow et al. [4] consisting of a damped pendulum forced with a constant torque. This system has been studied previously, in particular as a model of the pull-out torques of synchronous motors 2 or as a model of a single point Josephson Junction [5]. It can be easily built and the physics involved in this device is very simple. Let us write the equation of motion in the form

$$\ddot{\theta}+\beta\dot{\theta} \;=-\frac{\partial V(\theta)}{\partial\theta} \text{ with } V(\theta)=-\gamma\theta-\cos\theta \tag{1.2}$$

We will describe the motion of a ball in the potential $V(\theta)$. The ball is acted upon by its (normalized) weight and a viscous drag. The ball stays in contact with the potential. The motion of the ball is different from that of the pendulum because the kinetic energy has a different form. The kinetic energy of the falling ball is $K = m\ \dot{s}^2/2$, where s is the position of the ball in curvilinear coordinates along the potential. However, the qualitative behavior and, in particular, the equilibria are the same [1].

2 Approximate Analytical solution

In order to solve the i.v.p. (1.1), we first approximate the sine function by means of a cubic polynomial using Chebyshev technique as follows:

$$\sin\theta \approx \theta - \frac{2}{13}\theta^3, \ -75^\circ \leq \theta \leq 75^\circ \tag{2.3}$$

Thus, we solve the following i.v.p.

$$\ddot{\theta} + \beta\dot{\theta} \ + \theta - \frac{2}{13}\theta^3 = \gamma, \ \theta(0) = \theta_0 \text{ and } \theta'(0) = \dot{\theta}_0. \tag{2.4}$$

Define the residual function

$$R(t) = \ddot{\theta} + \beta\dot{\theta} \ + \theta - \frac{2}{13}\theta^3 - \gamma. \tag{2.5}$$

Let $\theta(t) = x(t) + d$, where d is a root to the cubic $\theta - \frac{2}{13}\theta^3 - \gamma = 0$.
We may take the value $d = \arcsin(\gamma)$ for $|\gamma| \leq 1/2$. Then

$$R(t) = x''(t) + \beta x'(t) + \left(1 - \frac{6d^2}{13}\right)x(t) - \frac{6}{13}dx(t)^2 - \frac{2}{13}x(t)^3 \tag{2.6}$$

Assume the ansatz form

$$x(t) = \exp(-\beta t/2)y(t), \tag{2.7}$$

where $y = y(t)$ is the analytical solution to some Duffing-Helmholtz equation

$$y''(t) + py(t) + qy^2(t) + ry^3(t), \ y(0) = \theta_0 - d \text{ and } y'(0) = \frac{1}{2}\beta\left(\theta_0 - d\right) + \dot{\theta}_0. \tag{2.8}$$

We have:

$$R(t) = -\tfrac{1}{52}e^{-\frac{\beta t}{2}}y(t)\left(13\beta^2 + 24d^2 + 52(p-1)\right) - \tfrac{1}{13}y(t)^2\left(6de^{-\beta t} + 13qe^{-\frac{\beta t}{2}}\right) - \tfrac{1}{13}y(t)^3\left(13re^{-\frac{\beta t}{2}} + 2e^{-\frac{3\beta t}{2}}\right). \tag{2.9}$$

For $|t\beta| \ll 1$, we have $e^{-\beta t} \approx e^{-\frac{\beta t}{2}} \approx e^{-\frac{3\beta t}{2}} \approx 1$ so that

$$R(t) \approx -\frac{1}{52}y(t)\left(13\beta^2 + 24d^2 + 52p - 52\right) - \frac{1}{13}y(t)^2\left(6d + 13q\right) - \frac{1}{13}y(t)^3\left(13r + 2\right). \tag{2.10}$$

The last approximation suggests the choice

$$p = 1 - \frac{6}{13}d^2 - \frac{\beta^2}{4},\ q = -\frac{6d}{13} \text{ and } r = -\frac{2}{13}. \tag{2.11}$$

The solution to the i.v.p.

$$\begin{array}{l} y''(t) + \left(1 - \frac{6}{13}d^2 - \frac{\beta^2}{4}\right) y(t) - \frac{6d}{13}y^2(t) - \frac{2}{13}y^3(t), \\ y(0) = y_0 := \theta_0 - d \text{ and } y'(0) = \dot{y}_0 := \ \frac{1}{2}\beta\left(\theta_0 - d\right) + \dot{\theta}_0 \end{array} \tag{2.12}$$

may be written in terms of the Weierstrass $\wp$ function or the Jacobian cn function. Using the $\wp$ function, we have

$$y(t) = A + \frac{B}{1 + C\wp(t + D; g_2, g_3)}, \tag{2.13}$$

where

$$B = -\frac{6A\left(A^2r+Aq+p\right)}{3A^2r+2Aq+p}, C = \frac{12}{3A^2r+2Aq+p}.$$

$$g_2 = \frac{1}{12}\left(-3A^4r^2 - 4A^3qr - 6A^2pr + p^2\right), g_3 = \frac{1}{216}\left(p^3 - A^2\left(q^2 - 3pr\right)\left(3A^2r + 4Aq + 6p\right)\right). \tag{2.14}$$

The values of A and D are determined from the initial conditions. The number A is found from the quartic

$$3rA^4 + 4qA^3 + 6pA^2 - (6py_0^2 + 4qy_0^3 + 3ry_0^4 + 6\dot{y}_0^2) = 0. \tag{2.15}$$

and

$$D = \pm\wp^{-1}\left(\frac{y_0 - A - B}{C\left(A - y_0\right)}; g_2, g_3\right). \tag{2.16}$$

The solution to (2.12) may also be expressed in terms of the Jacobian elliptic function cn in the ansatz form

$$y(t) = \bar{A} + \frac{\bar{B}}{1 + c_1\mathrm{cn}(\sqrt{\omega}t + c_2, m)}, \tag{2.17}$$

where

$$\bar{B} = -\frac{3A^4r + 4A^3q + 6A^2p - 6py_0^2 - 4qy_0^3 - 3ry_0^4 - 6\dot{y}_0^2}{3A\left(A^2r + Aq + p\right)},\ c_2 = \mathrm{cn}^{-1}\left(\frac{y_0 - A - B}{c_1(A - y_0)}, m\right). \tag{2.18}$$

$$m = -\frac{Ac_1^2\left(A^2r+Aq+p\right)}{2B\omega},\ \ \omega = \frac{A^3c_1^2(-r)+3A^3r+3A^2Br-A^2c_1^2q+3A^2q+2ABq-Ac_1^2p+3Ap+Bp}{B}. \tag{2.19}$$

The numbers $\bar{A}$ and c_1 are solutions to the following equations:

$$q\left(2q^2-9pr\right)\bar{A}^6-3\left(9p^2r-2pq^2+18pr^2y_0^2+12qr^2y_0^3+9r^3y_0^4+18r^2\dot{y}_0^2\right)\bar{A}^5- \\ 15qr\left(6py_0^2+4qy_0^3+3ry_0^4+6\dot{y}_0^2\right)\bar{A}^4- \\ 10q^2\left(6py_0^2+4qy_0^3+3ry_0^4+6\dot{y}_0^2\right)\bar{A}^3-15pq\left(6py_0^2+4qy_0^3+3ry_0^4+6\dot{y}_0^2\right)\bar{A}^2- \\ 3\left(6py_0^2+4qy_0^3+3ry_0^4+6\dot{y}_0^2\right)\left(3p^2+6pry_0^2+4qry_0^3+3r^2y_0^4+6r\dot{y}_0^2\right)\bar{A}- \\ q\left(6py_0^2+4qy_0^3+3ry_0^4+6\dot{y}_0^2\right)^2=0. \tag{2.20}$$

$$27\bar{A}^4\left(p+q\bar{A}+r\bar{A}^2\right)^4c_1^4+ \\ 18\bar{A}^2\left(p+q\bar{A}+r\bar{A}^2\right)^2 \\ (-3p^2\bar{A}^2-2pq\bar{A}^3-q^2\bar{A}^4+3pr\bar{A}^4-6p^2y_0^2-12pq\bar{A}y_0^2- \\ 8pr\bar{A}^2y_0^2-4pqy_0^3-8q^2\bar{A}y_0^3-12qr\bar{A}^2y_0^3-3pry_0^4-6qr\bar{A}y_0^4- \\ 19r^2\bar{A}^2y_0^4-6p\dot{y}_0^2-12q\bar{A}\dot{y}_0^2-18r\bar{A}^2\dot{y}_0^2)c_1^2+ \\ \left(3p\bar{A}^2+q\bar{A}^3-6py_0^2-4qy_0^3-3ry_0^4-6\dot{y}_0^2\right) \\ (9p^3\bar{A}^2+9p^2q\bar{A}^3-3pq^2\bar{A}^4+27p^2r\bar{A}^4-3q^3\bar{A}^5+15pqr\bar{A}^5- \\ 2q^2r\bar{A}^6+9pr^2\bar{A}^6+18p^2q\bar{A}y_0^2+18pq^2\bar{A}^2y_0^2- \\ 36p^2r\bar{A}^2y_0^2+6pqr\bar{A}^3y_0^2+12pq^2\bar{A}y_0^3+12q^3\bar{A}^2y_0^3- \\ 24pqr\bar{A}^2y_0^3+4q^2r\bar{A}^3y_0^3+36p^2ry_0^4+9pqr\bar{A}y_0^4+9q^2r\bar{A}^2y_0^4- \\ 18pr^2\bar{A}^2y_0^4+3qr^2\bar{A}^3y_0^4+48pqry_0^5+16q^2ry_0^6+ \\ 36pr^2y_0^6+24qr^2y_0^7+9r^3y_0^8+18pq\bar{A}\dot{y}_0^2+18q^2\bar{A}^2\dot{y}_0^2- \\ 36pr\bar{A}^2\dot{y}_0^2+6qr\bar{A}^3\dot{y}_0^2+72pry_0^2\dot{y}_0^2+48qry_0^3\dot{y}_0^2+36r^2y_0^4\dot{y}_0^2+36r\dot{y}_0^4)=0. \tag{2.21}$$

On the other hand, the functions $\wp$ and cn are related by the relations

$$\operatorname{cn}(t|m)=1-\frac{6}{4m+1+12\wp\left(t;\frac{1}{12}\left(16m^2-16m+1\right),\frac{1}{216}(2m-1)\left(32m^2-32m-1\right)\right)}. \tag{2.22}$$

$$\wp(t;g_2,g_3)=-\frac{\sqrt{g_2}(4m+1)}{2\sqrt{3}\sqrt{16m^2-16m+1}}+\frac{\sqrt{3g_2}}{\sqrt{16m^2-16m+1}\left(1-\operatorname{cn}\left(\omega t|\,m\right)\right)}, \tag{2.23}$$

where

$$m=\tfrac{1}{2}(1\pm\sqrt{4z-3}),\ \ \omega=\sqrt{2}\sqrt[4]{\tfrac{3g_2}{16m^2-16m+1}}.$$

and

$$4096\left(g_2^3-27g_3^2\right)z^3-11520\left(g_2^3-27g_3^2\right)z^2+972\left(11g_2^3-300g_3^2\right)z-27\left(121g_2^3-3375g_3^2\right)=0. \tag{2.24}$$

Letting $\beta\to 0$ gives an approximate analytical solution to the undamped and constantly forced pendulum $\ddot{\theta}+\sin\theta=\gamma,\ \theta(0)=\theta_0$ and $\theta'(0)=\dot{\theta}_0$. Letting $\gamma\to 0$ gives an approximate analytical solution to the damped and unforced pendulum $\ddot{\theta}+\beta\dot{\theta}\ +\sin\theta=0,\ \theta(0)=\theta_0$ and $\theta'(0)=\dot{\theta}_0$.

In the case when $-1 \leq m \leq 0.55$, we may obtain a good trigonometric approximant by means of the formula

$$\operatorname{cn}(t,m) \approx \cos_m(t) := \frac{\sqrt{\kappa - m + 2}\cos(w(t))}{\sqrt{14 + (\kappa - m - 12)\cos^2(w(t))}}, \text{ where} \quad (2.25)$$

$$w(t) = \sqrt{\frac{\kappa - m + 2}{14}}t \text{ and } \kappa = \sqrt{m^2 - 144m + 144}. \quad (2.26)$$

The respective errors are shown in Table 1 $(T = 4K(m))$.

m	$\max_{-T/2\leq t\leq T/2} \lvert\operatorname{cn}(t,m) - \cos_m(t)\rvert$	m	$\max_{-T/2\leq t\leq T/2} \lvert\operatorname{cn}(t,m) - \cos_m(t)\rvert$
0.1	0.00673646	−1.	0.00673646
0.15	0.00032743	−0.9	0.00574835
0.2	0.000611528	−0.8	0.00478114
0.25	0.00100518	−0.7	0.0038748
0.3	0.00152492	−0.6	0.0030145
0.35	0.00219766	−0.5	0.00222486
0.4	0.00304477	−0.4	0.00151738
0.45	0.00410158	−0.3	0.000912756
0.5	0.00541633	−0.2	0.000434754
0.55	0.00704757	−0.1	0.000117442

Table 1.

Thus, the trigonometric approximant reads

$$\theta_{\text{trigo}}(t) = d + \exp(-\beta t/2)\left(A + \frac{B}{1 - \frac{C\sqrt{g_2}(4m-5-(4m+1)\Psi(t))}{2\sqrt{48m^2-48m+3}(1-\Psi(t))}}\right),$$
$$\Psi(t) = \cos_m\left(\frac{\sqrt{2}\sqrt[4]{3g_2}}{\sqrt[4]{16m^2-16m+1}}(t+D)\right). \quad (2.27)$$

Yet we have another formula:

$$\theta_{\text{trigo}}(t) = d + \exp(-\beta t/2)\left(\bar{A} + \frac{\bar{B}}{1 + c_1\cos_m\left(\sqrt{\omega}t + c_2\right)}\right), \quad (2.28)$$

3 The Moving Boundary Method

The approximate solution won't be good in some cases. In order to improve the accuracy, we introduce a moving boundary method as follows:
Given a set X, let χ be its characteristic function: $\chi_X(t) = 1$ if $t \in X$

and 0 otherwise. Denote by $\theta(y_0, \dot{y}_0)(t)$ an approximate analytical solution satisfying the initial conditions $\theta(y_0, \dot{y}_0)(0) = \theta_0$ and $\theta(y_0, \dot{y}_0)'(0) = \dot{\theta}_0$. For our purposes, choose a suitable positive number τ and define a sequence of analytical approximants in the following way:

$$\begin{aligned} \theta_0(t) &= \theta(y_0, \dot{y}_0)(t) \text{ for } 0 \leq t \leq \tau \\ \theta_n(t) &= \theta(\theta_{n-1}(n\tau), \theta'_{n-1}(n\tau))(t - n\tau),\ \tau n < t \leq (n+1)\tau;\ n = 1, 2, 3, ... \end{aligned} \tag{3.29}$$

The analytical approximant is

$$\theta(t) = \theta_0(t) + \sum_{n=1}^{+\infty} \chi_{(n\tau,(n+1)\tau]}(t)\theta_n(t), t \geq 0. \tag{3.30}$$

The sequence of successive aproximants is defined as

$$\Theta_j(t) = \theta_0(t) + \sum_{n=1}^{j-1} \chi_{(n\tau,(n+1)\tau]}(t)\theta_n(t) + \chi_{(j\tau,\infty)}(t)\theta_j(t), t \geq 0,\ \tau > 0 \text{ and } j = 1, 2, 3, ... \tag{3.31}$$

Alternatively, we may define the functions $\theta_n(t)$ as follows:
Let $\varphi(t)$ be the Runge-Kutta numerical solution to the i.v.p. (1.1) for $0 \leq t \leq T$. Suppose that $\tau_1 < \tau_2 < \cdots < \tau_N$ the points on $(0, T)$ at which either $\varphi(t_j) = d$ or $\varphi'(t_j) = 0$ $(j = 1, 2, 3, ..., N)$. Define

$$\begin{aligned} &\theta_0(t) = \theta(y_0, \dot{y}_0)(t) \text{ for } 0 \leq t \leq \tau_1. \\ &\theta_n(t) = \theta(\varphi(\tau_n), \varphi'(\tau_n))(t - \tau_n),\ \tau_n < t \leq \tau_{n+1};\ n = 1, 2, 3, ..., N-1. \\ &\theta_N(t) = \theta(\varphi(\tau_N), \varphi'(\tau_N))(t - \tau_N),\ \tau_N < t \leq T. \end{aligned} \tag{3.32}$$

4 Analysis and Discussion

We have obtained an approximate analytical solution to a damped and constantly forced pendulum. The solution also applies when either $\beta = 0$ or $\gamma = 0$. Let us consider some illustrative examples.
In the case when $\theta_0 = d = \arcsin(\gamma)$ and $\dot{\theta}_0 = 0$, the exact solution is $\theta(t) = d$. Taking the initial values near the equilibrium point $P_e(d, 0)$ yields a good approximate analytical solution. If the analytical solution is not good, then we may apply the Moving Boundary Method.
Example 1. Let $\beta = 0.1$, $\gamma = 0.3$, $\theta_0 = 30°$, $\dot{\theta}_0 = 0$ and $0 \leq t \leq 80$. See Figure 1. The picture on the left corresponds to the Runge-Kutta numerical

solution compared with the analytical approximant (dashed curve). The picture on the right shows the phase portrait for the given data. The dashed horizontal line represents the equilibrium point $\theta = d = 0.304337$. The approximate analytical solution is $\theta_{\text{approx}}(t) =$

$$0.304337+e^{-0.05t}\left(-0.214896+\frac{1.26865}{1+12.0748\wp(t+3.17128;0.0794131,0.00345555)}\right) \tag{4.33}$$

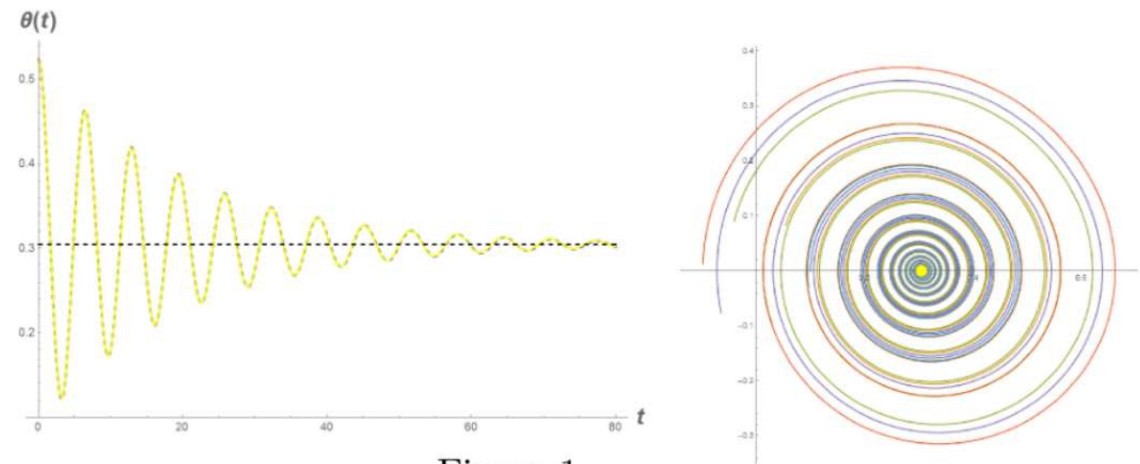

Figure 1.

The error of the approximant compared with Runge-Kutta numerical solution is $E = 0.00162852$. Making use of (2.23)-(2.24) gives

$$\wp(t;0.0794131,0.00345555) = -0.0776374+\frac{0.473306}{1-\text{cn}(0.97294t|-0.00395178)}. \tag{4.34}$$

and so

$$\theta_{\text{approx}}(t) = 0.304337+e^{-0.05t}\left(20.0686-\frac{1853.38}{92.374-\text{cn}(0.97294t+3.17128,-0.00395178)}\right). \tag{4.35}$$

We get a good trigonometric solution from (2.25) which is given by

$$\theta_{\text{trigo}}(t) = 0.304337+e^{-0.05t}\left(20.0686-\frac{1853.38}{92.374-\frac{3.74535\cos(0.9739t+3.17441)}{\sqrt{14.0138+0.0138199\cos(1.9478t+6.34882)}}}\right). \tag{4.36}$$

Example 2. Let $\beta = 0.1$, $\gamma = 0.6$, $\theta_0 = 90°$, $\dot{\theta}_0 = 0$ and $0 \leq t \leq 80$. The analytical approximant reads

$$\theta_{\text{approx}}(t) = 0.640407 + e^{-0.05t}\left(-0.771331 + \frac{4.4179}{1 + 12.1263\psi(t)}\right) \quad (4.37)$$
$$\psi(t) = \wp(t + 3.34388; 0.0962838, -0.00451338).$$

Figure 2 shows the approximant as dashed curve. The other curve corresponds to the Runge-Kutta numerical solution. Clearly, the approximate solution is not good.

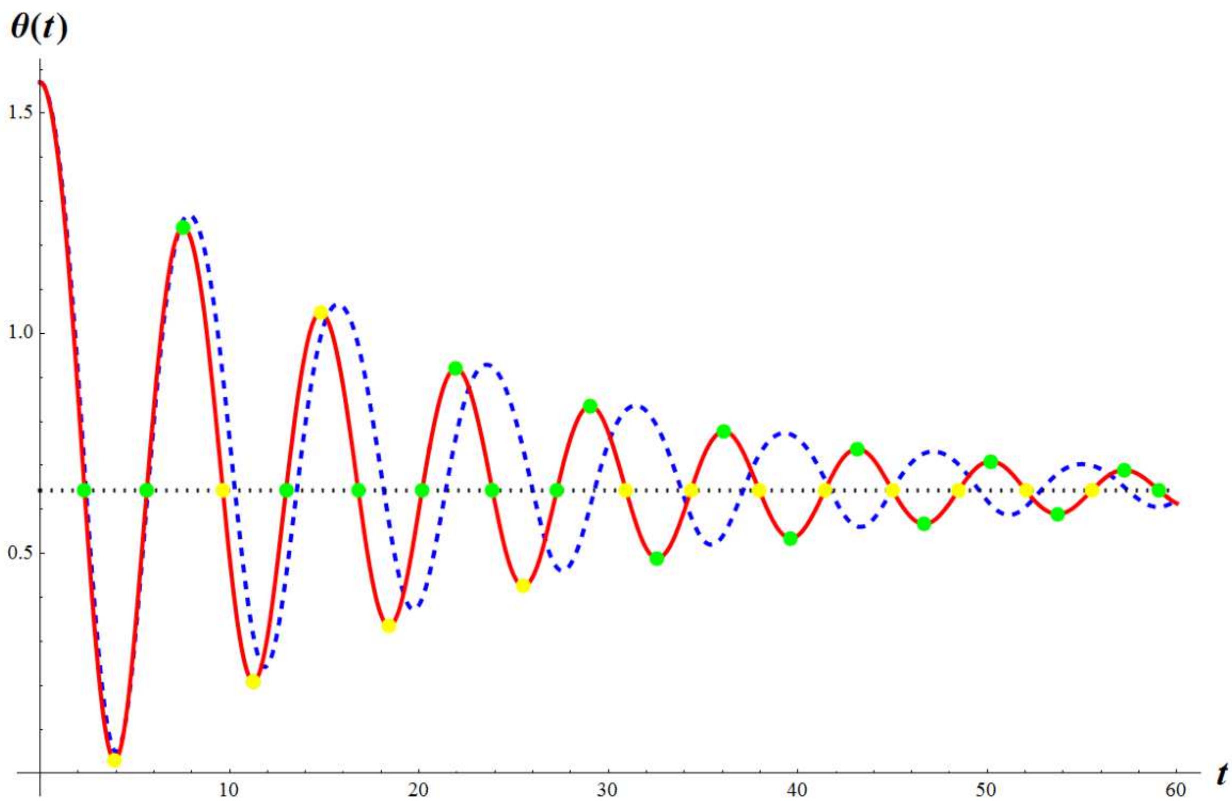

Figure 2

Clearly, the analytical approximant is not good. In order to improve the analytical solution, we apply the Moving Boundary Method by means of (3.32). The points t_j are colored in green and yellow and are presented in Table 1.

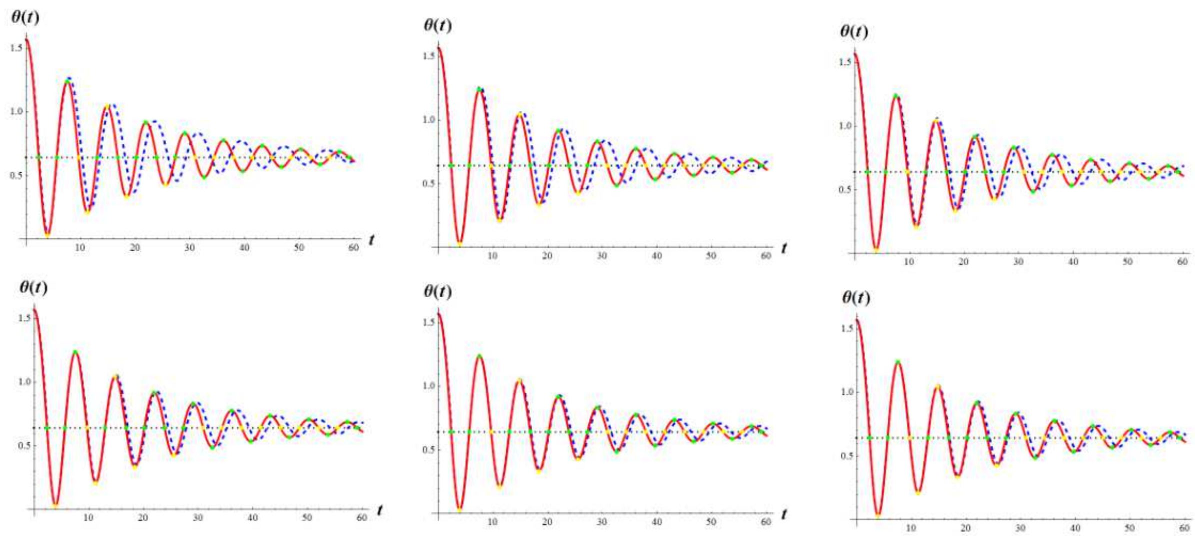

Figure 3

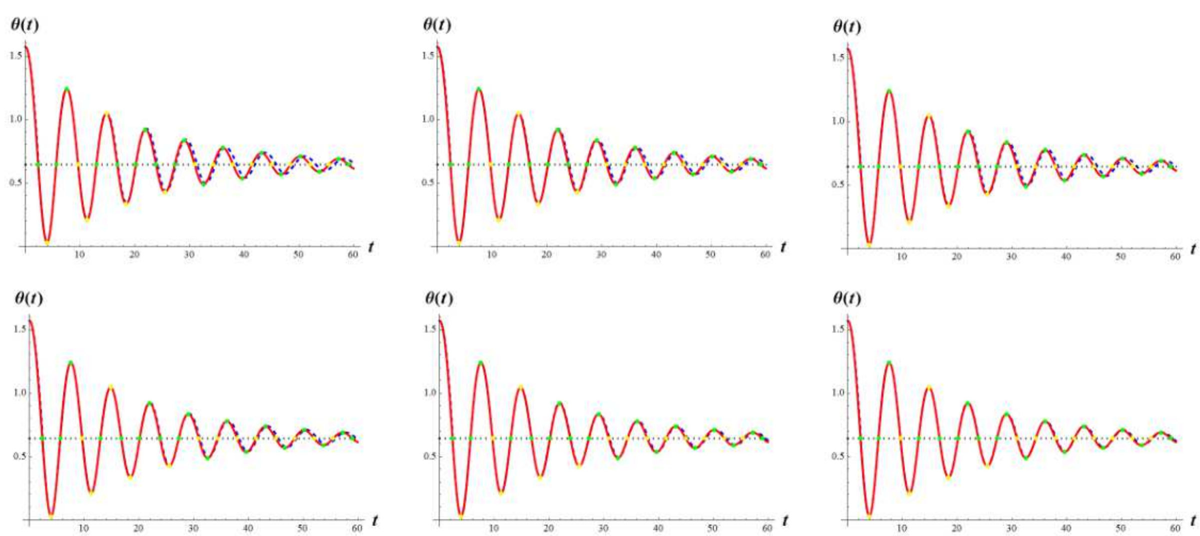

Figure 4

Figures 3 and 4 show the plots for successive approximants evaluated using (3.31).

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