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Analytical Solution to the Damped Cubic-Quintic Duffing Equation

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Abstract

A new analytical solution to the only integrable damped cubic-quintic Duffing equation in terms of Jacobi elliptic functions is obtained. Also, the obtained solutions is compared with the numerical solution using Runge-Kutta method. These results can be used to explain many phenomena related to plasma physics and quantum mechanics.

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1 Introduction

Many important physical phenomena and engineering problems related to classical mechanics maybe described by Hamiltonian flows. The family of the Duffing oscillator [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] is considered one of the canonical examples of Hamiltonian systems. Duffing's equation was used to model the conservative double-well oscillators which can occur, for example, in magneto-elastic mechanical systems. The system in question consists of a beam placed vertically between two magnets with the top end fixed and the bottom end free to swing. With velocity applied to the beam, the beam starts to oscillate between the two magnets [11]. This system could be modeled by the forced damped cubic Duffing equation

$$\ddot{x} + 2\varepsilon\dot{x} + px + qx^3 = F\cos(\Omega t). \tag{1.1}$$

The addition of a quintic term to (1.1) provides a simple generalization of the Duffing oscillator that has not been as extensively studied as the canonical cubic case. For appropriate parameter choices, this describes a classical particle in a triple-well potential rather than the double-well potentials usually considered. Thus, in this work, we consider the unforced damped Duffing cubic-quintic equation

$$\begin{cases} \mathbb{R} \equiv \ddot{x} + 2\varepsilon\dot{x} + px + qx^3 + rx^5 = 0, \\ x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \end{cases} \tag{1.2}$$

Also, we will prove that this oscillator is integrable for $p = \frac{3}{4}\varepsilon^2$ and $q = 0$. Accordingly, we find a new analytical solution and then compare it with the numerical solution using Runge-Kutta numerical method.

2 The analytical solution to the integrable case of damped cubic-quintic Duffing equation

To find the solution to the damped cubic-quintic Duffing Eq. (1.2), we introduce

$$x(t) = f(t)y(g(t)), \tag{2.3}$$

where $f(t)g(t) \neq 0$ and $y = y(t)$ represents any exact solution to the following undamped cubic-quintic equation

$$\ddot{y} + \alpha y + \beta y^3 + \gamma y^5 = 0. \tag{2.4}$$

Inserting (2.3) into the ode $\mathbb{R} = 0$, we get

$$\mathbb{R} = K_1\dot{y} + K_2 y + f(t)K_3 y^3 + f(t)K_4 y^5, \tag{2.5}$$

with

$$\begin{cases} K_1 = 2f'(t)g'(t) + f(t)g''(t) + 2\varepsilon f(t)g'(t), \\ K_2 = f''(t) + 2\varepsilon f'(t) - \alpha f(t)g'(t)^2 + pf(t), \\ K_3 = qf(t)^2 - \beta g'(t)^2, \\ K_4 = rf(t)^4 - \gamma g'(t)^2. \end{cases}$$

Equating the coefficients $K_1 - K_4$ to zero, we obtain the following system: $K_{1,2,3,4} = 0$.

For $\gamma = r$, the system $K_{1,2,3,4} = 0$ gives

$$\begin{cases} g'(t) = f^2(t), \\ f'(t) = -\varepsilon/2f(t), \\ 4p - 3\varepsilon^2 - 4\alpha f^4(t) = 0, \\ q - \beta f^2(t) = 0. \end{cases} \tag{2.6}$$

Eliminating $f(t)$ from Eq. (2.6), we have $\alpha = \beta = q = 0$ and $p = 3/4\varepsilon^2$. The functions $f(t)$ and $g(t)$ can be determine by solving the first two equation in system (2.6) as follows:

$$\begin{cases} f(t) = \exp\left(-\varepsilon t/2\right), \\ g(t) = \frac{1}{\varepsilon}(1 - \exp\left(-\varepsilon t\right)). \end{cases} \tag{2.7}$$

From the initial conditions $x(0) = x_0$ and $x'(0) = \dot{x}_0$, we get

$$y(0) = x_0 \text{ and } y'(0) = \frac{x_0}{2} + \frac{\dot{x}_0}{\varepsilon}. \tag{2.8}$$

The function $y = y(t)$ must be a solution to the below cubic quintic Duffing equation

$$\ddot{y} + ry^5 = 0. \tag{2.9}$$

Direct calculations show that the function

$$w_{w_0}(t) = w_0 \frac{2\sqrt{\frac{2}{\sqrt{3}} - 1} \cdot \mathrm{cn}\left(\frac{w_0^2\sqrt{r}}{\sqrt[4]{3}}t, m\right)}{\sqrt{1 + \left(2 - \frac{4}{\sqrt{3}}\right)\mathrm{cn}^2\left(\frac{w_0^2\sqrt{r}}{\sqrt[4]{3}}t, m\right) - (7 - 4\sqrt{3})\,\mathrm{cn}^4\left(\frac{w_0^2\sqrt{r}}{\sqrt[4]{3}}t, m\right)}} \tag{2.10}$$

is a solution to the i.v.p.

$$\ddot{w} + rw^5 = 0,\ w(0) = w_0, \text{ and } w'(0) = 0, \tag{2.11}$$

where $m = \left(2-\sqrt{3}\right)/4$.

Define

$$y(t) = w_{w_0}(t+C). \tag{2.12}$$

Then $y(t)$ is a solution to $\ddot{y} + ry^5 = 0$. The numbers w_0 and C can be determined from the system

$$\left\{ y(0) = w_{w_0}(C) = x_0,\ y'(0) = w'_{w_0}(C) = \frac{x_0}{2} + \frac{\dot{x}_0}{\varepsilon} \right\}. \tag{2.13}$$

Thus, the function $x \equiv x(t) = f(t)y(g(t))$ is the analytical solution to the only integrable damped cubic-quintic Eq. (**??**).

3 Analysis and Discussion

We obtain the analytical solution to the only integrable damped cubic-quintic equation. For numerical analysis, let us consider the following values of the related parameters:
$(\varepsilon,\, r,\, x_0,\, \dot{x}_0) = (0.025,\, 15,\, 0,\, 1)$. According to these values, the exact solution to the i.v.p.

$$\ddot{x} + 0.05\dot{x} + 0.00046875x + 15x^5 = 0,\ x(0) = 0 \text{ and } x'(0) = 1, \tag{3.14}$$

is given by

$$x(t) = -\frac{1.5471e^{-0.0125(t+0.91801)}\psi(t)}{\sqrt{2\sqrt{3}+3-2\psi^2(t)+\left(3-2\sqrt{3}\right)\psi^4(t)}}, \tag{3.15}$$

where

$$\psi(t) = \operatorname{cn}\left[70.4373\left(1-e^{-0.025(t+0.91801)}\right), \frac{1}{2} - \frac{\sqrt{3}}{4}\right]. \tag{3.16}$$

Figure 1 shows the profile of solution (3.15) versus the Runge-Kutta numerical solution.

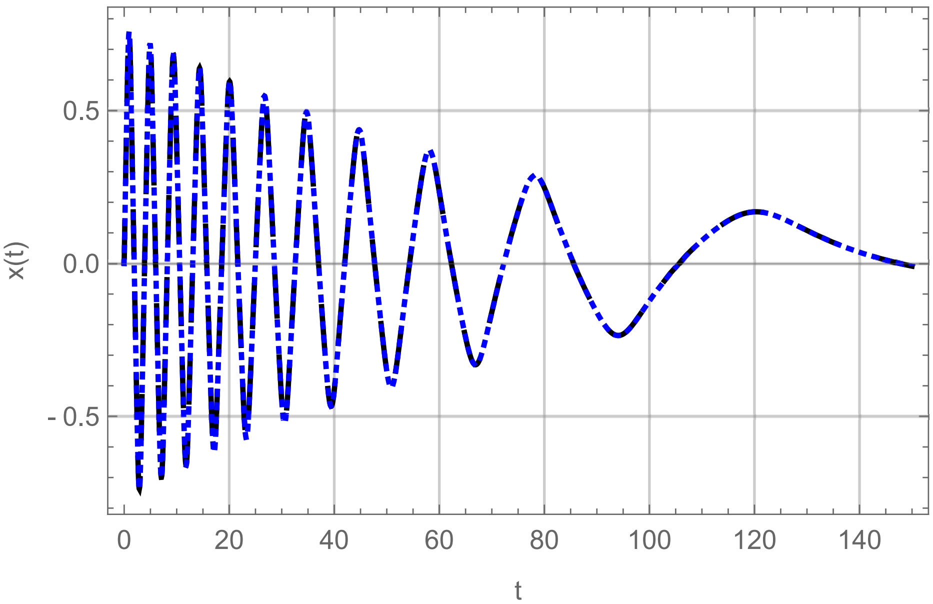

Figure 1: A comparison between the analytical solution (3.15) and the RK numerical solution.

4 Conclusion

We solved the damped cubic-quintic Duffing equation analytically and numerically. We derived the analytical solution in the Jacobi elliptic form with constant value to the modulus (m). Also, the mentioned equation has been solved numerically using the Runge-Kutta method. We gave a comparison between the analytical and numerical solutions. Our results should help many researchers in investigating different phenomena in plasma physicas, especially oscillations in different plasma models and quantum mechanics.

References

[1] Alvaro H. Salas, S. A. El-Tantawy, Castillo H. Jairo, On the approximate and analytical solutions to the fifth-order Duffing oscillator and its physical applications, Waves Random Complex Media, (2021), DOI: 10.1080/17455030.2021.1949072.

[2] Alvaro H. Salas, S. El-Tantawy, M. R. Alharthi, Novel solutions to the (un)damped Helmholtz-Duffing oscillator and its application to plasma

physics: Moving boundary method, Phys. Scr., **96,** (2021), 104003.

[3] Alvaro H. Salas, S. A. El-Tantawy, Analytical Solutions of Some Strong Nonlinear Oscillators, 2021, IntechOpen, DOI: 10.5772/intechopen.97677.

[4] S. A. El-Tantawy, Alvaro H. Salas, M. R. Alharthi, On the Analytical Solutions of the Forced Damping Duffing Equation in the Form of Weierstrass Elliptic Function and its Applications, Math. Probl. Eng., (2021), 6678102.

[5] Alvaro H. Salas, S. A. El-Tantawy, Noufe H Aljahdaly, An exact solution to the quadratic damping strong nonlinearity Duffing oscillator, Math. Probl. Eng., (2021), 8875589.

[6] Noufe H. Aljahdaly, S. A. El-Tantawy, On the multistage differential transformation method for analyzing damping Duffing oscillator and its applications to plasma physics, Mathematics, **9,** (2021), 432.

[7] Alvaro H. Salas, S. A. El-Tantawy, On the approximate solutions to a damped harmonic oscillator with higher-order nonlinearities and its application to plasma physics: semi-analytical solution and moving boundary method, Eur. Phys. J. Plus, **135,** (2020), 833.

[8] A. H. Nayfeh, Problems in Perturbations, Wiley, New York, 1985.

[9] P. A. T. Christopher, An approximate solution to a strongly nonlinear, second order, differential equation, Int. J. Control, **17,** (1973), 597.

[10] A. Beléndez, D. I. Méndez, E. Fernández, S. Marini, I. Pascual, An explicit approximate solution to the Duffing-harmonic oscillator by a cubication method, Phys. Lett. A, **373,** (2009), 2805.

[11] Kim Johannessen, The Duffing Oscillator with Damping for a Softening Potential, Int. J. Appl. Comput. Math., **3,** (2017), 3805.