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Fibonacci and Lucas Numbers of Factorials and Factorials of Fibonacci and Lucas

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Abstract

Let F_n and L_n be the nth Fibonacci number and the nth Lucas number, respectively. In this article, we obtain some relations among $F_{n!}$, $F_n!$, $L_{n!}$, and $L_n!$.

1 Introduction

Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence given by $F_1 = F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$, and let $(L_n)_{n\geq 1}$ be the Lucas sequence given by the same recursive pattern as the Fibonacci sequence but with the initial values $L_1 = 1$ and $L_2 = 3$.

Farrokhi [5] considered the Diophantine equations of the form $F_n = kF_m$, $F_{a_1}F_{a_2}\cdots F_{a_n} = F_b$, and $F_{a_1}F_{a_2}\cdots F_{a_m} = F_{b_1}F_{b_2}\cdots F_{b_n}$. Pongsriiam [13, 14] generalized Farrokhi's result by replacing the Fibonacci numbers by $L_n, F_n \pm 1$, and $L_n \pm 1$. Many other researchers also associated these kinds of problems with the Brocard-Ramanujan equation by replacing the factorials with the product of Fibonacci numbers (for instance, see [12], [15], [16]).

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Arithmetic functions of factorials and of Fibonacci numbers were also studied by some mathematicians [1], [6], [10]. In this article, we are interested in some relations among $F_{n!}$, $F_n!$, $L_{n!}$, and $L_n!$ such as which term is the largest or the smallest, and which divides which? This also leads to the solutions to the Diophantine equations of the form $F_{n!} = kF_n!$, $F_n! = kF_{n!}$, $L_{n!} = kL_n!$, $L_n! = kL_{n!}$, $F_{n!} = kL_n!$, $L_{n!} = kF_n!$, $F_n! = kL_{n!}$, and $L_n! = kF_{n!}$. For example, we see that the solutions to the Diophantine equation $F_{n!} = kF_n!$ are $(n, k) = (1, 1)$, $(2, 1), (3, 4), (4, 7728), (5, 4465299379159138867393\,2)$.

Some of our results can be proved by using Carmichael's primitive divisor theorem [4]. However, like Farrokhi [5], we intend to use only elementary tools in this article.

2 Preliminaries and Lemmas

In this section, we recall some basic definitions and useful results for the reader's convenience. Throughout this article, let n be a positive integer, let p be a prime and let $x \in \mathbb{R}, \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$. The largest integer less than or equal to x is denoted by $\lfloor x \rfloor$. The *p-adic valuation* of n is the exponent of p in the prime factorization of n and is denoted by $\nu_p(n)$. Binet's formulas for F_n and L_n state that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$ for all n. In addition, the *order (or rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is the smallest positive integer k such that $n \mid F_k$. It is well-known that $m \mid F_n$ if and only if $z(m) \mid n$. Legendre's formula for $\nu_p(n!)$ is well-known while $\nu_p(F_n)$ and $\nu_p(L_n)$ for any p can be obtained from the following result, but we only need it when $p = 2, 5$ as follows:

Lemma 2.1. (Lengyel [9]) *For $n \geq 1$, we have $\nu_5(F_n) = \nu_5(n)$, $\nu_5(L_n) = 0$,*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 1, & \text{if } n \equiv 3 \pmod{6}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{6}, \end{cases}$$

$$\nu_2(L_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 2, & \text{if } n \equiv 3 \pmod{6}; \\ 1, & \text{if } n \equiv 0 \pmod{6}. \end{cases}$$

Lemma 2.2. (Legendre's formula) *For every $n \geq 1$ and any prime p, we have*

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Lemma 2.3 appears as an exercise in the book by Nathanson [11, p. 215]. A more general version of Lemma 2.3 is also given by King [7]. A similar version of Lemma 2.4 appears as an exercise in the book by Bartle and Sherbert [2, p. 205]. Lemma 2.5 is a well-known result and can be easily proved by induction. A slightly different version of Lemma 2.5 is also given in the book by Koshy [8, p. 138].

Lemma 2.3. *$n! \geq e^{1-n}n^n$ for every n.*

Lemma 2.4. *$-x > \log(1-x) > \frac{-x}{1-x}$ for every $x \in (0,1)$.*

Lemma 2.5. *$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$ and $\alpha^{n-1} \leq L_n \leq \alpha^{n+1}$ for every n.*

Next, we prove a few more inequalities that are needed for the proof of our main theorems.

Lemma 2.6. *$e^{1-n}n^n \log(1.6) - 1 > \alpha^{2n}$ for every $n \geq 7$.*

Proof. We prove this lemma by induction. It is straightforward to verify that this result holds when $n = 7$. So assume that $n \geq 7$ and that the result holds for n. Then $e^{-n}(n+1)^{(n+1)} \log(1.6) - 1$ is larger than

$$e^{-n}(n+1)n^n \log(1.6) - 1 > e^{-1}(n+1)\left(e^{1-n}n^n \log(1.6) - 1\right) > \alpha^{2n+2},$$

where the last two inequalities are obtained from the induction hypothesis and a straightforward calculation that $n+1 \geq 8 \geq e\alpha^2$. □

Corollary 2.7. *$e^{1-n}n^n \log(1.6) - 1 > F_{n-1}^2$ for $n \geq 4$, and $e^{1-n}n^n \log(1.6) - 1 > L_{n-1}^2$ for $n \geq 6$.*

Proof. It is straightforward to verify that the first inequality holds when $n = 4, 5, 6$. For $n \geq 7$, we obtain by Lemma 2.6 and Lemma 2.5 that

$$e^{1-n}n^n \log(1.6) - 1 > \alpha^{2n} > F_{n-1}^2.$$

The second inequality can be proved similarly. □

Lemma 2.8. *$F_n > \log F_{n+2}$ for $n \geq 3$ and $L_n > \log L_{n+2}$ for $n \geq 2$.*

Proof. It is easy to check that the first inequality holds for $n = 3, 4$. So let $n \geq 4$ and assume that $F_k > \log F_{k+2}$ for all $k = 3, 4, \ldots, n$. Then $F_{n+1} = F_n + F_{n-1}$ is larger than

$$\log F_{n+2} + \log F_{n+1} = \log(F_{n+2}F_{n+1}) > \log(F_{n+2} + F_{n+1}) = \log F_{n+3}$$

which proves the first inequality. The other inequality can be proved similarly. □

3 Main Results

We first study how large the numbers $F_{n!}, F_n!, L_{n!}$, and $L_n!$ are when we compare them to each other. Then we give some divisibility results and the solutions to some Diophantine equations.

Theorem 3.1. *The following statements hold:*

(i) *For $n \geq 3$, $F_{n!} > F_{n+1}^{F_{n-1}}$.*

(ii) *For $n \geq 4$, $L_{n!} > L_{n+1}^{L_{n-1}}$.*

Proof. It is easy to check that (i) holds when $n = 3$. So let $n \geq 4$. We apply Binet's formula and Lemma 2.4 to obtain $\log F_{n!}$ is equal to

$$\log\left(\frac{\alpha^{n!}}{\sqrt{5}}\right) + \log\left(1 - \left(\frac{\beta}{\alpha}\right)^{n!}\right) > n! \log\alpha - \log\sqrt{5} - \frac{\left(\frac{\beta}{\alpha}\right)^{n!}}{1 - \left(\frac{\beta}{\alpha}\right)^{n!}}. \tag{3.1}$$

Since $\frac{\beta}{\alpha} = -\frac{3-\sqrt{5}}{2} \in (-1, 1)$ and $n \geq 4$, we obtain

$$0 \leq \left(\frac{\beta}{\alpha}\right)^{n!} \leq \left(\frac{3-\sqrt{5}}{2}\right)^{24} < \frac{1}{10}.$$

Since the function $x \mapsto \frac{x}{1-x}$ is increasing on (0,1), we obtain

$$\frac{\left(\frac{\beta}{\alpha}\right)^{n!}}{1 - \left(\frac{\beta}{\alpha}\right)^{n!}} < \frac{0.1}{1-0.1} = \frac{1}{9} < 0.12.$$

In addition, $\alpha = \frac{1+\sqrt{5}}{2} > 1.6$ and $\log\sqrt{5} < 0.805$. Therefore,

$$n! \log(\alpha) - \log(\sqrt{5}) - \frac{\left(\frac{\beta}{\alpha}\right)^{n!}}{1 - \left(\frac{\beta}{\alpha}\right)^{n!}} > n! \log(1.6) - 1. \tag{3.2}$$

By Lemma 2.3, Corollary 2.7, and Lemma 2.8, respectively, we obtain

$$n! \log(1.6) - 1 \geq e^{1-n} n^n \log(1.6) - 1 > F_{n-1}^2 > F_{n-1} \log F_{n+1} = \log F_{n+1}^{F_{n-1}}. \tag{3.3}$$

From (3.1), (3.2), and (3.3), we obtain $\log F_{n!} > \log F_{n+1}^{F_{n-1}}$, which proves (i). Similarly, we first verify that (ii) holds when $n = 4, 5$. For $n \geq 6$, we apply

Binet's formula, Lemma 2.3, Corollary 2.7, and Lemma 2.8, respectively, to obtain $\log L_{n!}$ is larger than

$$n!\log\alpha > n!\log 1.6 \geq e^{1-n}n^n\log 1.6 > L_{n-1}^2 > L_{n-1}\log L_{n+1} = \log L_{n+1}^{L_{n-1}},$$

which proves (ii). □

Theorem 3.2. *The sequences* $\left(\frac{F_n!}{F_{n!}}\right)_{n\geq 2}$, $\left(\frac{L_n!}{L_{n!}}\right)_{n\geq 2}$, $\left(\frac{L_n!}{F_{n!}}\right)_{n\geq 2}$, *and* $\left(\frac{F_n!}{L_{n!}}\right)_{n\geq 2}$ *are strictly decreasing.*

Proof. It is easy to check that for $n = 2, 3$, we have

$$\frac{F_{n+1}!}{F_{(n+1)!}} < \frac{F_n!}{F_{n!}}, \frac{L_{n+1}!}{L_{(n+1)!}} < \frac{L_n!}{L_{n!}}, \frac{L_{n+1}!}{F_{(n+1)!}} < \frac{L_n!}{F_{n!}}, \text{ and } \frac{F_{n+1}!}{L_{(n+1)!}} < \frac{F_n!}{L_{n!}}.$$

Let $n \geq 4$. By Lemma 2.5 and Theorem 3.1, we obtain

$$\begin{aligned} F_{(n+1)!} &\geq \alpha^{(n+1)!-2} > \alpha^{2(n!+1)} \geq (L_{n!})^2 > L_{n+1}^{L_{n-1}} L_{n!} \\ &> (L_n+1)(L_n+2)\cdots(L_n+L_{n-1})L_{n!} = \frac{L_{n+1}!}{L_n!}L_{n!}. \end{aligned} \tag{3.4}$$

We also observe that

$$\frac{L_{n+1}!}{L_n!} = \prod_{1\leq k\leq L_{n-1}}(L_n+k) > \prod_{1\leq k\leq F_{n-1}}(F_n+k) = \frac{F_{n+1}!}{F_n!}. \tag{3.5}$$

From (3.4) and (3.5), we obtain the following inequalities:

$$F_{(n+1)!} > \frac{L_{n+1}!}{L_n!}L_{n!} > \frac{F_{n+1}!}{F_n!}F_{n!} \tag{3.6}$$

$$L_{(n+1)!} > F_{(n+1)!} > \frac{L_{n+1}!}{L_n!}L_{n!} \tag{3.7}$$

$$F_{(n+1)!} > \frac{L_{n+1}!}{L_n!}L_{n!} > \frac{L_{n+1}!}{L_n!}F_{n!} \tag{3.8}$$

$$L_{(n+1)!} > F_{(n+1)!} > \frac{L_{n+1}!}{L_n!}L_{n!} > \frac{F_{n+1}!}{F_n!}L_{n!}. \tag{3.9}$$

The inequalities (3.6), (3.7), (3.8), and (3.9) imply the desired results. □

Corollary 3.3. *We obtain* $F_{n!} > F_n!$ *for* $n \geq 3$, $L_{n!} > L_n!$ *for* $n \geq 4$, $F_{n!} > L_n!$ *for* $n \geq 4$, *and* $L_{n!} > F_n!$ *for* $n \geq 2$.

Proof. Using Theorem 3.2, if $n \geq 3$, then

$$1 > \frac{1}{4} = \frac{F_3!}{F_{3!}} \geq \frac{F_n!}{F_{n!}}.$$

So the first inequality holds. The other proofs are similar. $\square$

Corollary 3.4. *We have*

$$F_{n!} \mid F_n! \text{ if and only if } n \leq 2, \tag{3.10}$$
$$L_{n!} \mid L_n! \text{ if and only if } n \leq 2, \tag{3.11}$$
$$L_{n!} \mid F_n! \text{ if and only if } n = 1, \tag{3.12}$$
$$F_{n!} \mid L_n! \text{ if and only if } n \leq 3, \tag{3.13}$$

Proof. By Corollary 3.3, we obtain $F_{n!} \nmid F_n!$ for any $n \geq 3$ and it is is easy to check that $F_{n!} \mid F_n!$ for $n = 1, 2$. So (3.10) holds. Similarly, the other proofs follow from Corollary 3.3 with a verification of small values of n. $\square$

Proposition 3.5. *The following relations hold:*

$$F_n! \mid L_n! \text{ for every } n \geq 1, \tag{3.14}$$
$$L_n! \mid F_n! \text{ if and only if } n = 1, \tag{3.15}$$
$$F_{n!} \mid L_{n!} \text{ if and only if } n \leq 2, \tag{3.16}$$
$$L_{n!} \mid F_{n!} \text{ if and only if } n = 1. \tag{3.17}$$

Proof. By induction it is easy to prove that $L_n > F_n$ for every $n \geq 2$. Therefore, for $n \geq 2$, we have $F_n! \mid L_n!$, $L_n! \nmid F_n!$, and $L_{n!} \nmid F_{n!}$. This implies (3.14), (3.15), and (3.17). For $n \geq 3$, we have $\nu_2(F_{n!}) = \nu_2(n!) + 2 \geq 3 > \nu_2(L_{n!})$, which implies (3.16). $\square$

Since $F_{n!} > F_n!$, we need another result to check whether or not $F_n!$ divides $F_{n!}$. This can be done by using the p-adic valuation as follows.

Theorem 3.6. *For any $n \geq 1$, the following statements hold:*

(i) $\nu_2(F_n!) \leq \nu_2(F_{n!})$ *for* $n \leq 5$, *and* $\nu_2(F_n!) > \nu_2(F_{n!})$ *for* $n \geq 6$.

(ii) $\nu_5(F_n!) = \nu_5(F_{n!})$ *for* $n \leq 6$, *and* $\nu_5(F_n!) > \nu_5(F_{n!})$ *for* $n \geq 7$.

(iii) $\nu_2(L_1!) = \nu_2(L_{1!})$ *and* $\nu_2(L_n!) > \nu_2(L_{n!})$ *for* $n \geq 2$.

(iv) $\nu_2(F_n!) = \nu_2(L_{n!})$ *for* $n \leq 4$, *and* $\nu_2(F_n!) > \nu_2(L_{n!})$ *for* $n \geq 5$.

(v) $\nu_2(L_n!) \le \nu_2(F_{n!})$ *for* $n = 1, 3, 4,$ *and* $\nu_2(L_n!) > \nu_2(F_{n!})$ *for* $n = 2$ *or* $n \ge 5$.

(vi) $\nu_5(L_n!) = \nu_5(F_{n!})$ *for every* $n \le 3$, *and* $\nu_5(L_n!) > \nu_5(F_{n!})$ *for* $n \ge 4$.

Proof. For (i), we first check by using Lemma 2.1 that $\nu_2(F_n!) \le \nu_2(F_{n!})$ for $n \le 5$, and $\nu_2(F_6!) > \nu_2(F_{6!})$. Next, we apply Legendre's formula and Lemma 2.1 repeatedly throughout this proof without reference. It is also useful to note that $L_n - 5 > F_n > n + 4$ for every $n \ge 7$, which can be easily proved by induction. Then for $n \ge 7$, $\nu_2(F_n!)$ is equal to

$$\left\lfloor \frac{F_n}{2} \right\rfloor + \sum_{k=2}^{\infty} \left\lfloor \frac{F_n}{2^k} \right\rfloor > \left\lfloor \frac{n}{2} \right\rfloor + 2 + \sum_{k=2}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor + 2 = \nu_2(F_{n!}),$$

which proves (i). Similarly, it is easy to verify the statements (ii), (iii), (iv), (v), and (vi) for $n \le 6$. So we assume throughout that $n \ge 7$. Then the following equalities hold:

(ii) $\nu_5(F_n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{F_n}{5^k} \right\rfloor > \sum_{k=1}^{\infty} \left\lfloor \frac{n}{5^k} \right\rfloor = \nu_5(n!) = \nu_5(F_{n!}).$

(iii) $\nu_2(L_n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{L_n}{2^k} \right\rfloor > \sum_{k=1}^{\infty} \left\lfloor \frac{4}{2^k} \right\rfloor > 1 = \nu_2(L_{n!}).$

(iv) $\nu_2(F_n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{F_n}{2^k} \right\rfloor > \sum_{k=1}^{\infty} \left\lfloor \frac{4}{2^k} \right\rfloor > 1 = \nu_2(L_{n!}).$

(v) $\nu_2(L_n!) = \left\lfloor \frac{L_n}{2} \right\rfloor + \sum_{k=2}^{\infty} \left\lfloor \frac{L_n}{2^k} \right\rfloor > \left\lfloor \frac{n}{2} \right\rfloor + 2 + \sum_{k=2}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor = \nu_2(F_{n!}).$

(vi) $\nu_5(L_n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{L_n}{5^k} \right\rfloor > \sum_{k=1}^{\infty} \left\lfloor \frac{n+5}{5^k} \right\rfloor > \sum_{k=1}^{\infty} \left\lfloor \frac{n}{5^k} \right\rfloor = \nu_5(F_{n!}).$

□

Corollary 3.7. *The following relations hold:*

$$F_n! \mid F_{n!} \text{ if and only if } n \le 5, \tag{3.18}$$
$$L_n! \mid L_{n!} \text{ if and only if } n = 1, \tag{3.19}$$
$$F_n! \mid L_{n!} \text{ if and only if } n \le 3, \tag{3.20}$$
$$L_n! \mid F_{n!} \text{ if and only if } n = 1, \tag{3.21}$$

Proof. By Theorem 3.6(i), we obtain $F_n! \nmid F_{n!}$ for any $n \geq 6$ and it is easy to check using Lemma 2.1 that $F_n! \mid F_{n!}$ for every $n \leq 5$. So (3.18) holds. Similarly, (3.19) and (3.20) follow, respectively, from Theorems 3.6(iii) and 3.6(iv) with some verification of small values of n. Finally, (3.21) follows from Theorems 3.6(v) and 3.6(vi), and the verification of the cases $n = 1, 3$. □

Corollary 3.8. *The only solutions to the Diophantine equation* $F_{n!} = kF_n!$ *are* $(n, k) = (1, 1)$, $(2, 1), (3, 4), (4, 7728), (5, 446529937915913886739 32)$.

Proof. Since $F_{n!} = kF_n!$ if and only if $F_n! \mid F_{n!}$, we easily obtain this corollary from (3.18). The values $F_{24} = 46368$ and $F_{120} = 5358359254990966640871840$ can also be obtained from the Fibonacci Tables [3]. □

Similar to Corollary 3.8, other divisibility relations can also be given in the form of Diophantine equations. We leave the details to the reader. For future work, we will investigate the relations among $F_{n!}$, $F_n!$, $L_{n!}$, $L_n!$, the Fibotorials, and Lucatorials defined by $n_F! = F_1F_2 \ldots F_n$ and $n_L! = L_1L_2 \ldots L_n$.

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