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The Relative Rank of $\mathcal{OPR}(X)$ Modulo $\mathcal{O}(X)$

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Abstract

In this paper, X is an infinite linearly ordered set and $\mathcal{T}(X)$ is the set of all full transformation semigroups. First, we describe the relative rank of the semigroup $\mathcal{OP}(X)$ of all orientation-preserving transformations modulo the semigroup $\mathcal{O}(X)$ of all order-preserving transformations. Moreover, we get the relative rank of the semigroup $\mathcal{OPR}(X)$ of all orientation-preserving or orientation-reversing transformations modulo the semigroup $\mathcal{O}(X)$. Furthermore, we illustrate our result with an example.

1 Introduction and Preliminaries

Let X be an infinite linearly ordered set and let $x \in X$. Denote by $\mathcal{T}(X)$ the monoid of all the full transformations on X with operation as the composition of functions. In this paper, we write functions from the right, $x\alpha$ rather than $\alpha(x)$ and compose from the left to the right; i.e., $x(\alpha\beta) = (x\alpha)\beta$ rather than $(\alpha\beta)(x) = \alpha(\beta(x))$. Let $\alpha \in \mathcal{T}(X)$. We denote by $im(\alpha)$ the image of α and define $im(\alpha) := X\alpha := \{x\alpha : x \in X\}$ and denote the cardinality of im(α) by $rank(\alpha)$; i.e., $rank(\alpha) := |im(\alpha)|$. For sets $A_1, A_2 \subseteq X$, we write $A_1 < A_2$ if $x_1 < x_2$ for all $x_1 \in A_1$ and for

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all $x_2 \in A_2$. Given a subset A of X, denote by $\alpha|_A$ the transformation $\alpha|_A : A \to X$ with $x(\alpha|_A) := x\alpha$ for all $x \in A$; i.e., $\alpha|_A$ is the transformation α restricted to A.

The generating sets of a semigroup S play an important role with the semigroup. A set G is a generating set of S, denoted by $\langle G \rangle = S$, such that S is the least semigroup containing G. The rank of S is the minimal size of a generating set of S defined by $rank(S) := min\{|G| : G \subseteq S, \langle G \rangle = S\}$. In the case when X is infinite, the size and the rank of the full transformation semigroup $\mathcal{T}(X)$ are infinite. This gives rise to the definition of the relative rank as follows:
The relative rank of S modulo U is the minimal size of a subset $G \subseteq S$ such that $G \cup U$ generates S:
$rank(S : U) := min\{|G| : G \subseteq S, \langle G \cup U \rangle = S\}$. A set $G \subseteq S$ with $\langle G \cup U \rangle = S$ is called a generating set of S modulo U. The concept of a relative rank generalizes the concept rank of a semigroup and was introduced by Howie, Ruškuc and Higgins [10].

Let X be a non-empty set. We consider the set $\mathcal{O}'(X)$ of all order-reversing transformations, the semigroup $\mathcal{O}(X)$ of all order-preserving transformations, the semigroup $\mathcal{OP}(X)$ of orientation-preserving transformations, the set $\mathcal{OR}(X)$ of all orientation-reversing transformations, and the semigroup $\mathcal{OPR}(X)$ of all orientation-preserving or orientation-reversing transformations. A transformation $\alpha \in \mathcal{T}(X)$ is called orientation-preserving (orientation-reversing) if there is a decomposition $X = [\alpha]_1 \cup [\alpha]_2$ with $[\alpha]_1 < [\alpha]_2$, $y_1\alpha \geq y_2\alpha$ ($y_1\alpha \leq y_2\alpha$) for all $y_1 \in [\alpha]_1$ and $y_2 \in [\alpha]_2$, and $x\alpha \leq y\alpha$ ($x\alpha \geq y\alpha$) for all $x \leq y \in [\alpha]_1$ or $x \leq y \in [\alpha]_2$. By the definition, we obtain $\mathcal{O}(X) \subseteq \mathcal{OP}(X) \subseteq \mathcal{OPR}(X) \subseteq \mathcal{T}(X)$ and $\mathcal{O}(X)$, $\mathcal{OP}(X)$, and $\mathcal{OPR}(X)$ are subsemigroups of $\mathcal{T}(X)$. In 2000s, the order-preserving transformation semigroup, the orientation-preserving transformation semigroup and the orientation-preserving or orientation-reversing transformation semigroup caught the interest of many researchers see [1], [2], [3], [4], [5], [7], [8]. The semigroups $\mathcal{O}(X)$ and $\mathcal{OP}(X)$ have been widely studied and investigated for a finite set X. In [2] and [7], the authors have determined the rank of these semigroups on a finite set X. The rank of $\mathcal{O}(X)$ is equal to n and the rank of $\mathcal{OP}(X)$ is equal to two in [7] and in [2], respectively. Additionally, the relative rank of $\mathcal{OP}(X)$ modulo $\mathcal{O}(X)$ is equal to one and it was determined by Catarino and Higgins [2]. In particular, we notice that the rank of semigroups $\mathcal{O}(X)$, $\mathcal{OP}(X)$ and $\mathcal{OPR}(X)$ are infinite when X is an infinite set. In [8], the authors have computed the relative rank of $\mathcal{T}(X)$ modulo $\mathcal{O}(X)$ is equal to one when X is a countably infinite linearly ordered set or X is an arbitrary well-ordered set. They also showed that the relative rank of $\mathcal{T}(X)$ modulo $\mathcal{O}(X)$ is infinite when $X = \mathbb{R}$ under the usual order.

In this paper, we consider X as an infinite linearly densely ordered set that has no both minimal and maximal element, and for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$ holds X_1 has maximal element or X_2 has minimal element. Since X has no both minimal and maximal element and for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, we have all the possibilities of X_1 and X_2 shown in the following cases:

1. X_1 is a half-open interval with maximal element and X_2 is a half-open interval with minimal element,

2. X_1 is an open interval and X_2 is a half-open interval with minimal element; i.e., $X_2 = [a, \infty)$, for some $a \in X$,

3. X_1 is a half open-interval with maximal element, i.e. $X_1 = (-\infty, a]$, for some $a \in X$ and X_2 is an open interval,

4. X_1 and X_2 are open intervals.

Since X is a dense set, case 1 will not happen. Since X_1 has maximal element or X_2 has minimal element, case 4 also will not happen. We can conclude that the possibilities of decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$ which satisfy the condition of X will be only cases 2 and 3. This means that X_1 is an open interval and $X_2 = [a, \infty)$ for some $a \in X$ or $X_1 = (-\infty, a]$ and X_2 is an open interval for some $a \in X$. We can also write $X_1 = (-\infty, a)$ and $X_2 = [a, \infty)$ or $X_1 = (-\infty, a]$ and $X_2 = (a, \infty)$ for some $a \in X$ because X is a dense set and cases (i) and (iv) are impossible as we have already shown. So the purpose of this paper is to determine the relative rank of $\mathcal{OPR}(X)$ modulo $\mathcal{O}(X)$ when X satisfies a condition in case 2 or case 3.

2 Main results

2.1 The relative rank $\mathcal{OP}(X)$ modulo $\mathcal{O}(X)$

In this section, we describe the relative rank of the semigroup $\mathcal{OP}(X)$ orientation-preserving transformations modulo the semigroup $\mathcal{O}(X)$ of all order-preserving transformations as shown in the following propositions.

Proposition 2.1. *[12] Let X be an infinite linearly densely ordered set that has no both minimal and maximal element and, for any decomposition $X =$*

$X_1 \cup X_2$ with $X_1 < X_2$, X_1 has a maximal element or X_2 has a minimal element. If there exists an order-isomorphic transformation between two open intervals, then rank$(\mathcal{OP}(X) : \mathcal{O}(X)) \leq 2$.

Proposition 2.2. *[12] Let X be an infinite linearly densely ordered set that has no both minimal and maximal element and, for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, X_1 has a maximal element or X_2 has a minimal element. If there exists an order-isomorphic transformation between two open intervals, then rank$(\mathcal{OP}(X) : \mathcal{O}(X)) \geq 2$.*

Theorem 2.3. *[12] rank$(\mathcal{OP}(X) : \mathcal{O}(X)) = 2$.*

Example 2.4. *Let $X \in \{\mathbb{Q}, \mathbb{R}\}$. Since $\mathbb{Q}$ and $\mathbb{R}$ are infinite linearly densely ordered set that have neither a minimal nor a maximal element, and for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$ holds X_1 has maximal element or X_2 has minimal element, we have rank$(\mathcal{OP}(X) : \mathcal{O}(X)) = 2$.*

2.2 The relative rank of $\mathcal{OPR}(X)$ modulo $\mathcal{O}(X)$

In this section, we extend the result from Section 2.1 in order to calculate the relative rank of the semigroup $\mathcal{OPR}(X)$ of all orientation-preserving or orientation-reversing transformations modulo the semigroup $\mathcal{O}(X)$ of all order-preserving transformations as follows:

Lemma 2.5. *Let X be an infinite linearly densely ordered set that has no minimal or maximal element, and for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$ holds X_1 has maximal element or X_2 has minimal element. If $\alpha \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$, then $im(\alpha)$ has a maximal or a minimal element.*

Proof. Suppose that X is an infinite linearly densely ordered set that has neither a minimal nor a maximal element and, for any decomposition $X = X_1 \cup X_2$, with $X_1 < X_2$ holds X_1 has maximal element or X_2 has minimal element. Let $\alpha \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$. Since $\alpha \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$, there is a decomposition $X = [\alpha]_1 \cup [\alpha]_2$ with $[\alpha]_1 < [\alpha]_2$ that satisfies the definition of an orientation-reversing transformation. Since X is densely ordered set and, for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, X_1 has a maximal element or X_2 has a minimal element, we have $[\alpha]_1$ is an open interval and $[\alpha]_2 = [a, \infty)$ or $[\alpha]_1 = (-\infty, a]$ and $[\alpha]_2$ is an open interval for some $a \in X$.

So, we consider the first case; that is, $[\alpha]_1$ is an open interval and $[\alpha]_2 = [a, \infty)$ for some $a \in X$. We claim that $a\alpha$ is the maximal element of image α. Let $y \in im(\alpha)$. Then there is $x \in X$ such that $x\alpha = y$. If $x \in [a, \infty)$, then $a\alpha \geq x\alpha$. If $x \in [\alpha]_1$, then $x\alpha \leq c\alpha$ for all $c \in [a, \infty)$; i.e., $x\alpha \leq a\alpha$. Combining, we obtain $a\alpha$ is the maximal element of image α. For the second case, we have $[\alpha]_1 = (-\infty, a]$ for some $a \in X$ and $[\alpha]_2$ is an open interval. We claim that $a\alpha$ is the minimal element of image α. Let $y \in im(\alpha)$. Since $y \in im(\alpha)$, there is $x \in X$ such that $x\alpha = y$. If $x \in (-\infty, a]$, then $x\alpha \geq a\alpha$. If $x \in [\alpha]_2$, then $x\alpha \geq c\alpha$ for all $c \in (-\infty, a]$; i.e.,, $x\alpha \geq a\alpha$. Combining, we obtain $a\alpha$ is the minimal element of image α. Therefore, $im(\alpha)$ has a maximal or minimal element. $\square$

Lemma 2.6. *Let X be an infinite linearly densely ordered set that has no minimal or maximal element and, for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, X_1 has maximal element or X_2 has minimal element. If $\alpha \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$, then there is $p \in X$ such that $p \leq x\alpha$ for all $x \in [\alpha]_2$ and $p \geq x\alpha$ for all $x \in [\alpha]_1$.*

Proof. Let $\alpha \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$. Since $\alpha \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$, there is a decomposition $X = [\alpha]_1 \cup [\alpha]_2$ with $[\alpha]_1 < [\alpha]_2$ that satisfies the definition of an orientation-preserving transformation. Since X is a densely ordered set and, for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, X_1 has a maximal element or X_2 has a minimal element, we have $[\alpha]_1$ is an open interval and $[\alpha]_2 = [a, \infty)$ or $[\alpha]_1 = (-\infty, a]$ and $[\alpha]_2$ is an open interval for some $a \in X$. Since $\alpha \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$ with $\alpha|_{[\alpha]_1}$ and $\alpha|_{[\alpha]_2}$ are order-reversing, there is $p_1 = inf([\alpha]_2\alpha)$ such that $p_1 \leq x\alpha$ for all $x \in [\alpha]_2$ and there is $p_2 = sup([\alpha]_1\alpha)$ such that $p_2 \geq x\alpha$ for all $x \in [\alpha]_1$. Moreover, it is possible that $p_1 \geq p_2$. Therefore, there is $p \in \{p_1, p_2\}$ such that $p \leq x\alpha$ for all $x \in [\alpha]_2$ and $p \geq x\alpha$ for all $x \in [\alpha]_1$. $\square$

Theorem 2.7. *Let X be an infinite linearly densely ordered set that has neither a minimum element nor a maximum element and, for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, X_1 has a maximum element or X_2 has minimum element. If there exist order-isomorphic transformation and anti-isomorphic transformation between two open intervals, then $rank(\mathcal{OPR}(X) : \mathcal{O}(X)) \leq 2$.*

Proof. Suppose that X is an infinite linearly densely ordered set that has neither a minimum element nor a maximum element and, for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, X_1 has a maximum element or X_2 has a minimum element.

Let $\alpha \in \mathcal{OPR}(X) \setminus \mathcal{O}(X)$. Then $\alpha \in \mathcal{OP}(X) \setminus \mathcal{O}(X)$ or $\alpha \in \mathcal{OR}(X)$.
Case 1. $\alpha \in \mathcal{OR}(X)$. We will consider two cases:

Case 1.1. $\alpha \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$. Since $\alpha \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$, there is a decomposition $X = [\alpha]_1 \cup [\alpha]_2$ with $[\alpha]_1 < [\alpha]_2$ that satisfies the definition of an orientation-reversing transformation. Since X is a densely ordered set and, for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, X_1 has a maximal element or X_2 has a minimal element, we have $[\alpha]_1$ is an open interval and $[\alpha]_2 = [a, \infty)$ or $[\alpha]_1 = (-\infty, a]$ and $[\alpha]_2$ is an open interval for some $a \in X$. Hence, we consider two subcases:

Case 1.1.1. $[\alpha]_1$ is an open interval and $[\alpha]_2 = [a, \infty)$ for some $a \in X$. Since X is a densely ordered set and for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$ holds X_1 has maximal element or X_2 has minimal element, we can write $X_1 = (-\infty, m)$ and $X_2 = [m, \infty)$ for some $m \in X$. Since there exists an order-isomorphic transformation between two open intervals, there are two transformations
$\nu_1 : [\alpha]_1 \to (-\infty, m)$ and $\nu_2 : (a, \infty) \to (m, \infty)$ which are order-isomorphic. We define $a\nu_2 := m$. Then we define a transformation θ from X to X by

$$x\theta_1 := \begin{cases} x\nu_1 & \text{if } x \in [\alpha]_1 \\ x\nu_2 & \text{if } x \in [a, \infty). \end{cases}$$

Since $\nu_1 : [\alpha]_1 \to (-\infty, m)$ and $\nu_2 : [\alpha]_2 \to [m, \infty)$ are order-isomorphic transformations, we get $\theta_1 \in \mathcal{O}(X)$ which is a bijective transformation on X.

Let $n \in X$ with $m < n$. Since there exists an anti-isomorphic transformation between two open intervals, there are two transformations $\mu_1 : (-\infty, m) \to (m, -\infty)$ and $\mu_2 : (m, \infty) \to (n, m)$ which are anti-isomorphic. We define $m\mu_2 := n$. Then we define a transformation γ_1 from X to X by

$$x\gamma_1 := \begin{cases} x\mu_1 & \text{if } x \in (-\infty, m) \\ x\mu_2 & \text{if } x \in [m, \infty). \end{cases}$$

It is clear that $X = (-\infty, m) \cup [m, \infty)$. Since $\mu_1 : (-\infty, m) \to (m, -\infty)$ and $\mu_2 : [m, \infty) \to [n, m)$ are anti-isomorphic transformations and $(-\infty, m)\gamma_1 = (-\infty, m)\mu_1 = (m, -\infty) < [n, m) = [m, \infty)\mu_1 = [m, \infty)\gamma_1$, we obtain that $\gamma_1 \in \mathcal{OR}(X)$ which is an injective transformation on X. Since the product of an order-preserving transformation and an orientation-preserving transformation is an orientation-preserving transformation, we obtain $\theta_1\gamma_1$ which is an injective transformation and $im(\theta_1\gamma_1) = (m, -\infty) \cup [n, m)$.

Next, we define a transformation $\theta_2 : (m, -\infty) \cup [n, m) \to im(\alpha)$ by $x\theta_2 := x\gamma_1^{-1}\theta_1^{-1}\alpha$ for all $x \in (m, -\infty) \cup [n, m)$. So, we need to extend the transformation θ_2 to be a transformation $\theta_2' \in \mathcal{O}(X)$. Let us consider the two cases $x > n$ and $x = m$. For $x > n$, we define $x\theta_2' := a\alpha := \max\{x\alpha : x \in X\}$. For $x = m$, there exists $p \in X$ such that $p \leq y_2\alpha$ for all $y_2 \in [\alpha]_2$ and $p \geq y_1\alpha$

for all $y_1 \in [\alpha]_1$. So, we define $m\theta_2' := p$. Hence, we define a transformation θ_2' from X to X by

$$x\theta_2' := \begin{cases} a\alpha & \text{if } x > n \\ p & \text{if } x = m \\ x\theta_2 & \text{if } x \in (-\infty, m) \cup (m, n]. \end{cases}$$

Next, we will show that θ_2' is an order-preserving transformation; i.e., $\theta_2' \in \mathcal{O}(X)$. First, let $x \in (-\infty, n]$ and $y \in (n, \infty)$ with $x \leq y$; i.e., $x \in (-\infty, m) \cup (m, n]$ or $x = m$. If $x = m$, then $x\theta_2' = p \leq a\alpha = y\theta_2'$; i.e., $x\theta_2' \leq y\theta_2'$. If $x \in (-\infty, m) \cup (m, n]$, then $x\theta_2' = x\theta_2 = x\beta_1^{-1}\theta_1^{-1}\alpha \leq a\alpha = y\theta_2'$; i.e., $x\theta_2' \leq y\theta_2'$. Next, let $x, y \in (-\infty, m)$ or $x, y \in (m, n]$ with $x \leq y$. Since $\gamma_1 \in \mathcal{OR}(X)$ which is injective, we have $x\gamma_1^{-1}, y\gamma_1^{-1} \in (-\infty, m)$ or $x\gamma_1^{-1}, y\gamma_1^{-1} \in [m, \infty)$ such that $x\gamma_1^{-1} \geq y\gamma_1^{-1}$. Since $\theta_1 \in \mathcal{O}(X)$ is bijective, we have $x\gamma_1^{-1}\theta_1^{-1}, y\gamma_1^{-1}\theta_1^{-1} \in [\alpha]_1$ or $x\gamma_1^{-1}\theta_1^{-1}, y\gamma_1^{-1}\theta_1^{-1} \in [\alpha]_2$ and so $x\gamma_1^{-1}\theta_1^{-1} \geq y\gamma_1^{-1}\theta_1^{-1}$. Since $\alpha \in \mathcal{OR}(X)$ which is injective, we have $x\gamma_1^{-1}\theta_1^{-1}\alpha \leq y\gamma_1^{-1}\theta_1^{-1}\alpha$; i.e., $x\theta_2 \leq y\theta_2 \Rightarrow x\theta_2' \leq y\theta_2'$. Finally, let $x \in (-\infty, m)$ and $y \in (m, n]$ with $x < y$. Since $\gamma_1 \in \mathcal{OR}(X)$ which is injective, $y\gamma_1^{-1} \in [m, \infty)$ and $x\gamma_1^{-1} \in (-\infty, m)$ such that $x\gamma_1^{-1} < y\gamma_1^{-1}$. Since $\theta_1 \in \mathcal{O}(X)$ which is bijective, we have $x\gamma_1^{-1}\theta_1^{-1} \in [\alpha]_1$ and $y\gamma_1^{-1}\theta_1^{-1} \in [\alpha]_2$ such that $x\gamma_1^{-1}\theta_1^{-1} < y\gamma_1^{-1}\theta_1^{-1}$. Since $\alpha \in \mathcal{OR}(X)$, we get $x\gamma_1^{-1}\theta_1^{-1}\alpha < y\gamma_1^{-1}\theta_1^{-1}\alpha \Rightarrow x\theta_2 < y\theta_2 \Rightarrow x\theta_2' < y\theta_2'$. Combining, we can conclude that $\theta_2' \in \mathcal{O}(X)$. Next, we show that $\theta_1\gamma_1\theta_2'$. Let $x \in X$. Then

$$x\theta_1\gamma_1\theta_2' = x\theta_1\gamma_1\theta_2 = x\theta_1\gamma_1(\gamma_1^{-1}\theta_1^{-1}\alpha) = x\theta_1(\gamma_1\gamma_1^{-1})\theta^{-1}\alpha = x(\theta_1\theta_1^{-1})\alpha = x\alpha;$$

i.e., $\theta_1\gamma_1\theta_2' = \alpha$.

Case 1.1.2. $[\alpha]_1 := (-\infty, a]$ and $[\alpha]_2$ is an open interval for some $a \in X$, the proof is analogous to the Case 1.1 but we use a transformation γ_2 that is defined as follows Since X is a densely ordered set and, for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, X_1 has maximal element or X_2 has minimal element, we can write $X_1 = (-\infty, m]$ and $X_2 = (m, \infty)$ for some $m \in X$. Let $l \in X$ with $l < m$. Since there exist an anti-isomorphic transformation between two open intervals, there are two transformations $\delta_1 : (-\infty, m) \to (m, l)$ and $\delta_2 : (m, \infty) \to (\infty, m)$ are anti-isomorphic. We define $m\delta_1 := l$. Then we define a transformation γ_2 from X to X by

$$x\gamma_2 := \begin{cases} x\delta_1 & \text{if } x \in (-\infty, m] \\ x\delta_2 & \text{if } x \in (m, \infty). \end{cases}$$

As the transformation γ_1, we can similarly show that $\gamma_2 \in \mathcal{OR}(X)$which is an injective transformation. In particular, we can show $\alpha = \theta_3\gamma_2\theta_4'$, where $\theta_3, \theta_4' \in \mathcal{O}(X)$.

Let γ_3 be an order-reversing bijective transformation on X. Next, we show that $\gamma_2 \in \langle \gamma_1, \gamma_3 \rangle$. Put $\ker(\gamma_2) = \ker(\gamma_3)$ and define $(-\infty, m]\gamma_3 := (\infty, m]$ and $(m, \infty)\gamma_3 := (m, -\infty)$, where $m \in X$. So $(-\infty, m]\gamma_3\gamma_1 = (\infty, m]\gamma_1 = (m, n]$ and $(m, \infty)\gamma_3\gamma_1 = (m, -\infty)\gamma_1 = (-\infty, m)$. Let $l \in X$ and define $n\gamma_3 := l$. Hence $(-\infty, m]\gamma_3\gamma_1\gamma_3 = (m, n]\gamma_3 = (m, l]$ and $(m, \infty)\gamma_3\gamma_1\gamma_3 = (-\infty, m)\gamma_3 = (\infty, m)$; i.e., $\gamma_2 = \gamma_3\gamma_1\gamma_3$. From Case 1.1.2, we have $\alpha = \theta_3\gamma_2\theta_4'$. We know that $\gamma_2 = \gamma_3\gamma_1\gamma_3$. Thus $\alpha = \theta_3\gamma_2\theta_4' = \theta_2\gamma_3\gamma_1\gamma_3\theta_4'$.

Case 1.2. $\alpha \in \mathcal{O}'(X)$. Let $dom(\theta) = X := \{x\gamma_3 : x \in X\}$ and define a transformation θ by $x\theta = x\gamma_3^{-1}\alpha$ for all $x \in X$. Let $a, b \in X$ with $a < b$. Since γ_3 is bijective, $a\gamma_3^{-1} > b\gamma_3^{-1}$. Since $\alpha \in \mathcal{O}'(X)$, $a\gamma_3^{-1}\alpha \le b\gamma_3^{-1}\alpha$; i.e., $a\theta \le b\theta$. Therefore, $\theta \in \mathcal{O}(X)$. Let $x \in X$. Then $x\gamma_3\theta = x\gamma_3(\gamma_3^{-1}\alpha) = x(\gamma_3\gamma_3^{-1})\alpha = x\alpha$; i.e., $\gamma_3\theta = \alpha$.

Case 2. $\alpha \in \mathcal{OP}(X) \setminus \mathcal{O}(X)$. By Proposition 2.1, there are two transformations $\beta_1, \beta_2 \in \mathcal{OP}(X) \setminus \mathcal{O}(X)$ such that $\langle \mathcal{O}(X), \beta_1, \beta_2 \rangle = \mathcal{OP}(X)$. From Proposition 2.1, β_1 and β_2 are transformations from X to X which are defined as follows:

$$x\beta_1 := \begin{cases} x\phi_1 & \text{if } x \in (-\infty, m') \\ x\phi_2 & \text{if } x \in [m', \infty) \end{cases}$$

such that $\phi_1 : (-\infty, m') \to (m', \infty)$ and $\phi_2 : [m', \infty) \to [l', m')$, where $l' < m' \in X$ are order-isomorphic transformations and

$$x\beta_2 := \begin{cases} x\eta_1 & \text{if } x \in (-\infty, m'] \\ x\eta_2 & \text{if } x \in (m', \infty) \end{cases}$$

such that $\eta_1 : (-\infty, m] \to (m', n']$ and $\eta_2 : (m', \infty) \to (-\infty, m')$, where $m' < n' \in X$, are order-isomorphic transformations.

Next, we show that $\beta_1, \beta_2 \in \langle \mathcal{O}(X), \gamma_1, \gamma_3 \rangle$. Let $l' < m' \in X$ and $a' < b' < c' \in X$. Since there exists an order-isomorphic transformation between open intervals, there are two transformations $\xi_1 : (-\infty, m') \to (a', b')$ and $\xi_2 : (m', \infty) \to (b', c')$ which are order-isomorphic. We define $m'\xi_1 := b'$. Then we define a transformation θ_5 from X to X by

$$x\theta_5 := \begin{cases} x\xi_1 & \text{if } x \in (-\infty, m') \\ x\xi_2 & \text{if } x \in [m', \infty). \end{cases}$$

It is easy to see that $\theta_5 \in \mathcal{O}(X)$ by the definitions of transformations ξ_1 and ξ_2. Let $m' < i' < h' < g' \in X$. Since there exists an anti-isomorphic transformation between two open intervals, there are two transformations $\mu_1 : (a', b') \to (m', -\infty)$ and $\mu_2 : (b', c') \to (g', h')$ which are anti-isomorphic.

We define $b'\mu_2 := g'$. Then we define a transformation ρ_1 from X to X by

$$x\rho_1 := \begin{cases} m' & \text{if } x \in (-\infty, a'] \\ x\mu_1 & \text{if } x \in (a', b') \\ x\mu_2 & \text{if } x \in [b', c') \\ i' & \text{if } x \in (c', \infty). \end{cases}$$

It is clear that $\rho_1 \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$. Indeed, let us put $A_1 := (-\infty, b')$ and $A_2 := [b', \infty)$ for some $b' \in X$. So $A_1\rho_1 = \{m'\} \cup (-\infty, m') > (g', h') \cup \{i\}$ and $x\rho_1 \geq y\rho_1$ whenever $x < y \in A_1$ or $x < y \in A_2$. By the same argument, there are two transformations $\zeta_1 : (-\infty, m') \to (\infty, m')$ and $\zeta_2 : (h', g') \to (m', l')$ which are anti-isomorphic. We define $g'\zeta_2 := l'$. Then we define a transformation δ_1 from X to X by

$$x\delta_1 := \begin{cases} x\zeta_1 & \text{if } x \in (-\infty, m') \\ m' & \text{if } x \in [m', h'] \\ x\zeta_2 & \text{if } x \in (h', g'] \\ l' & \text{if } x \in (g', \infty). \end{cases}$$

It is easy to see that $\delta_1 \in \mathcal{O}'(X)$ by the definitions of transformations of ζ_1 and ζ_2. Now, we show that $\beta_1 \in \langle \mathcal{O}(X), \gamma_1, \gamma_3 \rangle$. So, we have $(-\infty, m')\theta_1\rho_1\delta_1 = (a', b')\rho_1\delta_1 = (-\infty, m')\delta_1 = (m', \infty) = (-\infty, m')\beta_1$ and we have $[m', \infty)\theta_1\rho_1\delta_1 = [b', c')\rho_1\delta_1 = (h, g']\delta_1 = (l, m') = [m', \infty)\beta_1$, i.e., $\beta_1 = \theta_5\rho_1\delta_1$. Since $\rho_1 \in \mathcal{OR}(X) \setminus \mathcal{O}'(X) \subseteq \mathcal{OR}(X) = \langle \mathcal{O}(X), \gamma_1, \gamma_3 \rangle$, there are $\theta_6, \theta_7 \in \mathcal{O}(X)$ such that $\rho_1 = \theta_6\gamma_1\theta_7$. Since $\delta_1 \in O^{'}(X) = \langle \mathcal{O}(X), \gamma_3 \rangle$, there exists $\theta_8 \in \mathcal{O}(X)$ such that $\delta_1 = \gamma_3\theta_8$.

Therefore, $\beta_1 = \theta_5\rho_1\delta_1 = \theta_5\theta_6\gamma_1\theta_7\gamma_3\theta_8 = \theta_1'\gamma_1\theta_4\gamma_3\theta_5$, where $\theta_1' = \theta_5\theta_6 \in \mathcal{O}(X)$; i.e., $\beta_1 \in \langle \mathcal{O}(X), \gamma_1, \gamma_3 \rangle$. We can show similarly to obtain that $\beta_2 \in \langle \mathcal{O}(X), \gamma_1, \gamma_3 \rangle$. So, we have $\alpha \in \mathcal{OP}(X) \setminus \mathcal{O}(X) \in \langle \mathcal{O}(X), \beta_1, \beta_2 \rangle \subseteq \langle \mathcal{O}(X), \gamma_1, \gamma_3 \rangle$.

Altogether, we obtain $\mathcal{OPR}(X) = \langle \mathcal{O}(X), \gamma_1, \gamma_3 \rangle$; i.e., $rank(\mathcal{OPR}(X) : \mathcal{O}(X)) \leq 2$. ◻

Theorem 2.8. *Let X be an infinite linearly densely ordered set that has neither a minimum element nor a maximum element, and for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$ holds X_1 has maximum element or X_2 has minimum element. If there exist order-isomorphic transformation and anti-isomorphic transformation between two open intervals, then $rank(\mathcal{OPR}(X) : \mathcal{O}(X)) \geq 2$.*

Proof. Let X be an infinite linearly densely ordered set that has neither a minimum element nor a maximum element, and for any decomposition

$X = X_1 \cup X_2$ with $X_1 < X_2$ such that X_1 has maximum element or X_2 has minimum element.

Suppose that $A \subseteq \mathcal{OPR}(X) \setminus \mathcal{O}(X)$ with $\langle \mathcal{O}(X), A\rangle = \mathcal{OPR}(X)$. Let $P(X)$ be the power set of X. Let B be the set of all sets B' that there is no $u \in X$ such that $u > B'$ and let C be the set of all sets C' that there is no $v \in X$ such that $v < C'$. Let $a', a, b \in X$ with $a < b \in X$. Since there exists anti-isomorphic transformation between two-open intervals, there are $\theta' : (-\infty, a') \to (b, a)$ and $\theta'' : (a', \infty) \to (\infty, b)$ which are anti-isomorphic. We define $a'\theta' := a$. Then we define a transormation β from X to X by

$$x\beta := \begin{cases} x\theta' & \text{if } x \in (-\infty, a'] \\ x\theta'' & \text{if } x \in (a', \infty). \end{cases}$$

Clearly, β is an injective transformation by the definition of transformation of θ' and θ''. Next, we will show that $\beta \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$. Let $[\beta]_1 = (-\infty, a']$ and $[\beta]_2 = (a', \infty)$. It is easy to see that $[\beta]_1\beta < [\beta]_2\beta$ and $y_1\beta \geq y_2\beta$ for all $y_1, y_2 \in [\beta]_1$ and $y_1, y_2 \in [\beta]_2$ because θ' and θ'' are anti-isomorphic. Therefore, $\beta \in \mathcal{OR}(X) \setminus \mathcal{O}^{'}(X)$.

Since $\beta \in \mathcal{OR}(X) \setminus \mathcal{O}'(X) \subseteq \mathcal{OPR}(X) = \langle \mathcal{O}(X), A\rangle$, there are $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathcal{O}(X) \cup A$, where $k \in \mathbb{N}$ such that $\beta = \alpha_1\alpha_2 \cdots \alpha_k$. Assume there is no $j \in \{1, 2, \ldots, k\}$ with $im(\alpha_j|_Y) \in B$ for some $Y \in P(X) \setminus B$. Since $(a', b') \in P(X) \setminus B$, where $a' < b'$, we have $(a', b')\alpha_1\alpha_2 \cdots \alpha_k \notin B$ which contradicts $(a', b')\beta \in B$. So there is $j \in \{1, 2, \ldots, k\}$ with $im(\alpha_j|_Y) \in B$ for some $Y \in P(X) \setminus B$. It is clear that $\alpha_j \notin \mathcal{O}(X)$. Then there is $\alpha_B \in A$ with $im(\alpha_B|_Y) \in B$ for some $Y \in P(X) \setminus B$.

Assume there is no $p \in \{1, 2, \ldots, k\}$ with $im(\alpha_p|_Y) \in B \cup C$ for some $Y \in P(X) \setminus (B \cup C)$. Since $(a', b') \in P(X) \setminus B$, where $a' < b'$, we have $(a', b')\alpha_1\alpha_2 \cdots \alpha_k \notin B \cup C$, i.e. $(a', b')\alpha_1\alpha_2 \cdots \alpha_k \notin B$ that is a contradiction with $(a', b')\beta \in B$. So there is $p \in \{1, 2, \ldots, k\}$ with $im(\alpha_p|_Y) \in B \cup C$ for some $Y \in P(X) \setminus (B \cup C)$. It is clearly that $\alpha_p \notin \mathcal{O}(X)$. Then there is $\alpha_{B\cup C} \in A$ with $im(\alpha_{B\cup C}|_Y) \in B \cup C$ for some $Y \in P(X) \setminus (B \cup C)$.

Let $c', c, d \in X$ with $c < d$. Since there exists an anti-isomorphic transformation between two open intervals, there are two transformations $\nu' : (-\infty, c') \to (c, -\infty)$ and $\nu'' : (c', \infty) \to (d, c)$ which are anti-isomorphic. We define $c'\nu' := d$. Then we define a transformation β from X to X by

$$x\xi := \begin{cases} x\nu' & \text{if } x \in (-\infty, c') \\ x\nu'' & \text{if } x \in [c', \infty). \end{cases}$$

Clearly, ξ is an injective transformation by the definitions of transformations ν' and ν''. Therefore, we can show similarly as a transformation β and we have $\xi \in \mathcal{OR}(X) \setminus \mathcal{O}'(X)$. Since $\xi \in \mathcal{OR}(X) \setminus \mathcal{O}'(X) \subseteq \mathcal{OPR}(X) = \langle \mathcal{O}(X), A \rangle$, there are $\gamma_1, \gamma_2, \ldots, \gamma_l$, where $l \in \mathbb{N}$ such that $\xi = \gamma_1\gamma_2\cdots\gamma_l$. Assume there is no $q \in \{1, 2, \ldots, l\}$ with $im(\gamma_q|_Y) \in C$ for some $Y \in P \setminus C$. Since $(k, c') \in P(X) \setminus C$, where $k < c'$, we have $(k, c')\gamma_1\gamma_2\cdots\gamma_l \notin C$ which contradicts $(k, c')\xi \in C$. So there is $q \in \{1, 2, \ldots, l\}$ with $im(\gamma_q|_Y) \in C$ for some $Y \in P(X) \setminus C$. It is clear that $\gamma_p \notin \mathcal{O}(X)$. Then there is $\gamma_C \in A$ with $im(\gamma_C|_Y) \in C$ for some $Y \in P(X) \setminus C$.

Next, we assume that $\alpha_B = \alpha_{B\cup C} = \gamma_C$. Then there is $\alpha_{B\cup C} \in A$ with $im(\alpha_{B\cup C}|_Y) \in B \cup C$ for some $Y \in P(X) \setminus (B \cup C)$. Since $\alpha_B \in \mathcal{OPR}(X) \setminus \mathcal{O}(X)$, there is a decomposition $X = [\alpha_B]_1 \cup [\alpha_B]_2$ with $[\alpha_B]_1 < [\alpha_B]_2$ which satisfies the definition of orientation-preserving or orientation-reversing transformation. We consider two cases:

Case 1. $im(\alpha_{B\cup C}|_Y) \in B$, i.e. $Y\alpha_{B\cup C} \in B$. Since there is $M \in P(X) \setminus C$ such that $M\gamma_C \in C$ and $\alpha_{B\cup C} = \gamma_C$, we have $M\alpha_{B\cup C} \in C$. We consider the following cases:

Case 1.1. $Y \subseteq [\alpha_B]_1$. So, we consider two subcases:

Case 1.1.1. $\alpha_{B\cup C}|_Y$ is order-preserving. We consider again two possibilities:

(a) $M \subseteq [\alpha_B]_1$. That is $Y \cup M \subseteq [\alpha_B]_1$ and $\alpha_{B\cup C}|_{Y\cup M}$ is order-preserving because $\alpha_{B\cup C}|_Y$ is order-preserving. Since $Y \in P(X) \setminus (B \cup C)$, there is $k \in X$ such that $k > Y$. Therefore, $k \in [\alpha_B]_1$ or $k \in [\alpha_B]_2$. If $k \in [\alpha_B]_1$, then $k\alpha_{B\cup C} \geq Y\alpha_{B\cup C}$ contradicting $Y\alpha_{B\cup C} \in B$. If $k \in [\alpha_B]_2$, then $(Y \cup M)\alpha_{B\cup C} > k\alpha_{B\cup C}$; i.e., $M\alpha_{B\cup C} > k\alpha_{B\cup C}$ contradicting $M\alpha_{B\cup C} \in C$. Combining, $\alpha_{B\cup C} \notin \mathcal{OPR}(X) \setminus \mathcal{O}(X)$ which is a contradiction since $\alpha_{B\cup C} \in A \subseteq \mathcal{OPR}(X) \setminus \mathcal{O}(X)$.

(b) $M \subseteq [\alpha_B]_2$. Then $Y < M$. Since $Y \in P(X) \setminus (B \cup C)$ with $Y\alpha_{B\cup C} \in B$, $M \in P(X) \setminus C$ with $M\alpha_{B\cup C} \in C$ and all decompositions $X = X_1 \cup X_2$ with $X_1 < X_2$ such that X_1 has a maximum element or X_2 has a minimum element, there is $k' \in X$ such that $Y < k' < M$. Therefore, $k' \in [\alpha_B]_1$ or $k' \in [\alpha_B]_2$. If $k' \in [\alpha_B]_1$, then $Y\alpha_{B\cup C} \leq k'\alpha_{B\cup C}$ contradicting $Y\alpha_{B\cup C} \in B$. If $k' \in [\alpha_B]_2$, then $k'\alpha_{B\cup C} \leq M\alpha_{B\cup C}$ contradicting $M\alpha_{B\cup C} \in C$. As a result, $\alpha_{B\cup C} \notin \mathcal{OPR}(X) \setminus \mathcal{O}(X)$ which is a contradiction since $\alpha_{B\cup C} \in A \subseteq \mathcal{OPR}(X) \setminus \mathcal{O}(X)$.

Case 1.1.2. $\alpha_{B\cup C}|_Y$ is order-reversing. Since $Y \subseteq [\alpha_B]_1$, we have $Y\alpha_{B\cup C} < A_2\alpha_{B\cup C}$ contradicting $Y\alpha_{B\cup C} \in B$; that is, $\alpha_{B\cup C} \notin \mathcal{OPR}(X) \setminus \mathcal{O}(X)$ which is a contradiction because $\alpha_{B\cup C} \in A \subseteq \mathcal{OPR}(X) \setminus \mathcal{O}(X)$.

Case 1.2. $Y \subseteq [\alpha_B]_2$. We consider two subcases:

Case 1.2.1. $\alpha_{B\cup C}|_Y$ is order-preserving. Since $Y \subseteq [\alpha_B]_2$, $[\alpha_B]_1\alpha_{B\cup C} > Y\alpha_{B\cup C}$ contradicting $Y\alpha_{B\cup C} \in B$; that is, $\alpha_{B\cup C} \notin \mathcal{OPR}(X) \setminus \mathcal{O}(X)$ which is a contradiction because $\alpha_{B\cup C} \in A \subseteq \mathcal{OPR}(X) \setminus \mathcal{O}(X)$.

Case 1.2.2 $\alpha_{B\cup C}|_Y$ is order-reversing. So, we will consider again two possibilities:

(a) $M \subseteq [\alpha_B]_1$. Then $M < Y$. Since $Y \in P(X) \setminus (B \cup C)$ with $Y\alpha_{B\cup C} \in B$, $M \in P(X) \setminus C$ with $M\alpha_{B\cup C} \in C$ and all decompositions $X = X_1 \cup X_2$ with $X_1 < X_2$ such that X_1 has a maximum element or X_2 has a minimum element, there is $g \in X$ such that $M < g < Y$. Therefore, $g \in [\alpha_B]_1$ or $g \in [\alpha_B]_2$. If $g \in [\alpha_B]_1$, then $M\alpha_{B\cup C} \geq g\alpha_{B\cup C}$ contradicting $M\alpha_{B\cup C} \in C$. If $g \in [\alpha_B]_2$, then $g\alpha_{B\cup C} \geq Y\alpha_{B\cup C}$ contradicting $Y\alpha_{B\cup C} \in B$. Altogether, $\alpha_{B\cup C} \notin \mathcal{OPR}(X) \setminus \mathcal{O}(X)$ that is a contradiction since $\alpha_{B\cup C} \in A \subseteq \mathcal{OPR}(X) \setminus \mathcal{O}(X)$.

(b) $M \subseteq [\alpha_B]_2$. That is $Y \cup M \subseteq [\alpha_B]_2$ and $\alpha_{B\cup C}|_{Y\cup M}$ is order-reversing bcause $\alpha_{B\cup C}|_Y$ is order-reversing. Since $Y \in P(X) \setminus (B \cup C)$, there is $g' \in X$ such that $g' < Y$. Therefore, $g' \in [\alpha_B]_1$ or $g' \in [\alpha_B]_2$. If $g' \in [\alpha_B]_1$, then $g'\alpha_{B\cup C} < (Y \cup M)\alpha_{B\cup C}$; i.e., $g'\alpha_{B\cup C} < M\alpha_{B\cup C}$ contradicting $M\alpha_{B\cup C} \in C$. If $g' \in [\alpha_B]_2$, then $g'\alpha_{B\cup C} \geq Y\alpha_{B\cup C}$, contradicting $Y\alpha_{B\cup C} \in B$. Consequently, $\alpha_{B\cup C} \notin \mathcal{OPR}(X) \setminus \mathcal{O}(X)$ which is a contradiction since $\alpha_{B\cup C} \in A \subseteq \mathcal{OPR}(X) \setminus \mathcal{O}(X)$.

Case 1.3. $Y = Y_1 \cup Y_2$ such that $Y_1 \subseteq [\alpha_B]_1$ and $Y_2 \subseteq [\alpha_B]_2$ with $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$. So, we will consider the following subcases:

Case 1.3.1. $\alpha_{B\cup C}|_Y$ is order-preserving, i.e. $\alpha_{B\cup C}|_{Y_2}$ is order-preserving and $Y_2\alpha_{B\cup C} \in B$. Therefore, $[\alpha_B]_1\alpha_{B\cup C} > Y_2\alpha_{B\cup C}$ which contradicts $Y_2\alpha_{B\cup C} \in B$.

Case 1.3.2. $\alpha_{B\cup C}|_Y$ is order-reversing; i.e., $\alpha_{B\cup C}|_{Y_1}$ is order-reversing and $Y_1\alpha_{B\cup C} \in B$. Therefore, $[\alpha_B]_2\alpha_{B\cup C} > Y_1\alpha_{B\cup C}$ which contradicts $Y_1\alpha_{B\cup C} \in B$.

Case 2. $im(\alpha_{B\cup C}|_Y) \in C$, i.e. $Y\alpha_{B\cup C} \in C$. Since there is $N \in P(X) \setminus B$ such that $N\alpha_B \in B$ and $\alpha_B = \alpha_{B\cup C}$, we have $N\alpha_{B\cup C} \in B$. The rest of the proof of this case is similar to Case 1.

From both cases, we can conclude that $\alpha_B \neq \alpha_{B\cup C}$ or $\alpha_{B\cup C} \neq \gamma_C$, i.e. A contains at least two element. Therefore, $rank(\mathcal{OPR}(X) : \mathcal{O}(X)) \geq 2$. □

By Theorems 2.7 and 2.8, we obtain the following corollary.

Corollary 2.9. *$rank(\mathcal{OPR}(X) : \mathcal{O}(X)) = 2$.*

Example 2.10. *Let $X \in \{\mathbb{Q}, \mathbb{R}\}$. Since $\mathbb{Q}$ and $\mathbb{R}$ are infinite linearly densely ordered sets that have neither a minimal nor a maximal element and, for any decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$, X_1 has maximal element or X_2 has minimal element, we have $rank(\mathcal{OPR}(X) : \mathcal{O}(X)) = 2$.*

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